Worked problems from Chapter 13, Part 2

Section 13.3

1. Problems 4 – 9. Decide if the vector field is conservative:

Ex. 5. Decide if the vector field $\boldsymbol{F}(x,y) = \langle P, Q\rangle = \langle 2xy^2, 3y^2x^2\rangle$ is conservative.

$$\frac{\partial P}{\partial y} = 6xy^2, \quad \frac{\partial Q}{\partial x} = 6xy^2, \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \Rightarrow \quad \text{conservative.}$$

Ex. 7. Decide if the vector field $\boldsymbol{F}(x,y) = \langle P, Q\rangle = \langle -y+e^x\sin(y), (x+2)e^x\cos(y)\rangle$ is conservative.

$$\frac{\partial P}{\partial y} = -1 + e^x\cos(y), \quad \frac{\partial Q}{\partial x} = (x+3)e^x\cos(y), \quad \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \quad \Rightarrow \quad \text{not conservative.}$$

2. Problems 10 – 13,

Ex. 11. Evaluate the line integral $\int_C [(3x+2y)\,dx - (2x+3y)\,dy]$:

(a) (See figure page 885) Let $x = \cos(t)$, $y = \sin(t)$, $0 \le t \le \pi \Rightarrow dx = -\sin(t)\,dt$ and $dy = \cos(t)\,dt$. Therefore

$$\begin{aligned}\int_C [(3x+2y)\,dx - (2x+3y)\,dy] &= \int_0^\pi [(3\cos(t) + 2\sin(t))(-\sin(t)\,dt) \\ &\qquad - (2\cos(t) + 3\sin(t))(\cos(t)\,dt)] \\ &= \int_0^\pi [-6\sin(t)\cos(t) - 2]\,dt \\ &= 2\pi\end{aligned}$$

(b) (See figure page 885) The curve $C = C_1 + C_2$ where C_1 is the line joining $(1,0)$ to $(0,1)$ which can be parameterized by $x = 1-t$, $y = t$ where $0 \le t \le 1$. Here we have $dx = -dt$ and $dy = dt$ so

$$\begin{aligned}\int_{C_1} [(3x+2y)\,dx - (2x+3y)\,dy] &= \int_0^1 [(3(1-t) + 2t)(-\,dt) - (2(1-t) + 3t)(\,dt)] \\ &= \int_0^1 [-5]\,dt \\ &= -5\end{aligned}$$

Next C_2 is the line joining $(0,1)$ to (-1.0) which can be parameterized by $x = -t$, $y = 1-t$ where $0 \le t \le 1$. Here we have $dx = -dt$ and $dy = -dt$ so

$$\begin{aligned}\int_{C_2} [(3x+2y)\,dx - (2x+3y)\,dy] &= \int_0^1 [(3(-t) + 2(1-t))(-\,dt) - (2(-t) + 3(1-t))(-\,dt)] \\ &= \int_0^1 [1]\,dt \\ &= 1\end{aligned}$$

Therefore

$$\int_C [(3x+2y)\,dx - (2x+3y)\,dy] = \int_{C_1} [(3x+2y)\,dx - (2x+3y)\,dy] + \int_{C_2} [(3x+2y)\,dx - (2x+3y)\,dy] = -4.$$

(c) (See figure page 885) $C = C_1 + C_2$ where C_1 is the curve in part (a) and therefore from part (a)

$$\int_{C_1} [(3x+2y)\,dx - (2x+3y)\,dy] = 2\pi.$$

The curve C_2 is the line joining the point $(-1,0)$ to $(1,0)$. For C_2 let $x=t$, $y=0$, $-1 \le t \le t \Rightarrow dx = dt$ and $dy = 0\,dt$.

$$\int_{C_2} [(3x+2y)\,dx - (2x+3y)\,dy] = \int_{-1}^{1} [3t]\,dt = \left.\frac{3t^2}{2}\right|_{-1}^{1} = 0.$$

$$\int_C [(3x+2y)\,dx - (2x+3y)\,dy] = 2\pi + 0 = 2\pi.$$

3. Problems 14 – 19 Show $\boldsymbol{F}(x,y)$ is conservative and find the potential f:

Ex. 15. $\boldsymbol{F} = \langle P, Q\rangle = \langle 2xy, x^2\rangle$. We have $\dfrac{\partial P}{\partial y} = 2x = \dfrac{\partial Q}{\partial x}$ so $\boldsymbol{F}$ is conservative. So there is a scalar function f so that $\langle f_x, f_y\rangle = \nabla f = \boldsymbol{F} = \langle 2xy, x^2\rangle$ or $f_x = 2xy$ and $f_y = x^2$.

Integrating $f_x = 2xy$ with respect to x we have $f(x,y) = x^2y + h(y)$ where h is an aribitray function of y. Thus we still need to find $h(y)$. Differentiating our expression for f and using the fact that $f_y = x^2$ we have $x^2 = f_y = x^2 + h'(y)$ which implies that $h'(y) = 0$ or $h(y) = C$. We always take the simplest case $C = 0$ and we have $f(x,y) = x^2y$.

4. Problems 20 – 25, Show $\boldsymbol{F}(x,y,z)$ is conservative and find the potential f:

Ex. 23. In this case we have $\boldsymbol{F}(x,y,z) = (x^2+y^2+z^2)\langle x,y,z\rangle$ and we show that $\nabla\times\boldsymbol{F} = 0$.

$$\nabla\times\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ x(x^2+y^2+z^2) & y(x^2+y^2+z^2) & z(x^2+y^2+z^2) \end{vmatrix}$$

$$= \langle (2yz - 2yz), -(2xz - 2xz), (2xy - 2xy)\rangle = \boldsymbol{0}$$

so $\boldsymbol{F}$ is conservative and, therefore, we know there exists a scalar f so that $\boldsymbol{F} = \nabla f$. Thus $f_x = x(x^2+y^2+z^2)$, $f_y = y(x^2+y^2+z^2)$ and $f_z = z(x^2+y^2+z^2)$. To find f we integrate the fist equation with respect to x to get

$$f(x,y,z) = \int f_x\,dx = \frac{x^4}{4} + \frac{x^2}{2}(y^2+z^2) + h(y,z),$$

where $h(y,z)$ is an arbitrary function of (y,z). Now we use this formula (differentiated with respect to y) and the second expression to get

$$y(x^2+y^2+z^2) = f_y = yx^2 + \frac{\partial h(y,z)}{\partial y}.$$

This implies $\dfrac{\partial h(y,z)}{\partial y} = y(y^2+z^2)$. We integrate this expression with respect to y to get

$$h(y,z) = \frac{y^4}{4} + \frac{y^2z^2}{2} + k(z)$$

where k is an arbitrary function of z. Finally we use this expression (differentiated with respect to z) and $f_z = z(x^2+y^2+z^2)$ to get

$$z(x^2+y^2+z^2) = f_z = x^2z + y^2z + \frac{dk(z)}{dz}.$$

From this we see that $dk(z)/dz = z^3$ or $k(z) = z^4/4$. Finally then we have

$$f(x,y,z) = \frac{x^4+y^4+z^4}{4} + \frac{x^2y^2+x^2z^2+y^2z^2}{2}.$$

5. Problems 26 – 29, Evaluate the integral of the conservative v.f. from $A(1,0,-1)$ to $B(0,-1,1)$:

Ex. 27. We are told that the v.f. $\boldsymbol{F} = \langle \sin(z), -z\sin(y), x\cos(z)+\cos(y)\rangle$ is conservative (but you can check this by showing $\nabla\times\boldsymbol{F} = 0$) so, by the Fundamental Theorem of Line Integrals, we need only find the potential function f and then $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{R} = f(B) - f(A)$ where C is any curve from A to B and where $\boldsymbol{R}$ is a parametric representation of C.

We have

$$f_x = \sin(z), \quad f_y = -z\sin(y), \quad f_z = x\cos(z)+\cos(y).$$

Proceeding as in the last examples we find f by integrating the first equation with respect to x to get $f = \displaystyle\int f_x\,dx = x\sin(z) + h(y,z)$. We now differentiate this expression with respect to y and use the second condition above to obtain

$$-z\sin(y) = f_y = \frac{\partial h(y,z)}{\partial y}.$$

Integrating with respect to y we get $h(y,z) = z\cos(y) + k(z)$. So we now have

$$f = x\sin(z) + z\cos(y) + k(z).$$

Finally we differentiate this with respect to z and use the third condition to obtain

$$x\cos(z)+\cos(y) = f_z = x\cos(z)+\cos(y) + \frac{dk(z)}{dz}$$

which implies that $dk(z)/dz = 0$ or $k(z) = 0$. Therefore

$$f = x\sin(z) + z\cos(y)$$

and for any curve C from A to B we have

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{R} = (0\sin(1) + (1)\cos(1)) - (\sin(-1) + (-1)\cos(0)) = \cos(1) + \sin(1) + 1.$$

6. Problems 31 – 36, Show path independent and evaluate:

Ex. 31. $\displaystyle\int_C [(3x^2 + 2x + y^2)\,dx + (2xy + y^3)\,dy]$

To show path independence we note that $\boldsymbol{F} = \langle P, Q\rangle = \langle (3x^2 + 2x + y^2), (2xy + y^3)\rangle$ and we check that $P_y = Q_x$:

$$P_y = 2y \stackrel{\checkmark}{=} 2y = Q_x.$$

So we have

$$f_x = (3x^2 + 2x + y^2), \quad f_y = (2xy + y^3)$$

and (as above)

$$f = \int f_x\,dx = x^3 + x^2 + xy^2 + h(y).$$

Differentiating with respect to y and using the formula for f_y we have

$$(2xy + y^3) = f_y = 2xy + \frac{dh(y)}{dy}$$

so that $dh(y)/dy = y^3$ and, integrating, we have $h(y) = y^4/4$. Thus we have

$$f(x, y) = x^3 + x^2 + xy^2 + \frac{y^4}{4}.$$

Thus we obtain

$$\begin{aligned}\int_C [(3x^2 + 2x + y^2)\,dx + (2xy + y^3)\,dy] &= f(1,1) - f(0,0)\\ &= (1 + 1 + 1 + 1/4) - (0) = \frac{13}{4}.\end{aligned}$$