# Worked problems from Chapter 13

## Section 13.1

3. Find the divergence and curl of $\boldsymbol{F}(x,y,z)=\langle x^2, xy, z^3\rangle$

$$\nabla\cdot\boldsymbol{F}=(2x)+(x)+(3z^2)=3x+3z^2,\quad \nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\ \partial_x&\partial_y&\partial_z\\ x^2&xy&z^3\end{vmatrix}=(0)\boldsymbol{i}-0\boldsymbol{j}+(y-0)\boldsymbol{k}=y\boldsymbol{k}$$

21. Find the divergence and curl of $\boldsymbol{F}(x,y,z)=x^2\boldsymbol{i}+y^2\boldsymbol{j}+z^2\boldsymbol{k}$.

$$\mathrm{div}\boldsymbol{F}=\left(\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial x}\boldsymbol{j}+\frac{\partial}{\partial x}\boldsymbol{k}\right)\cdot\left(x^2\boldsymbol{i}+y^2\boldsymbol{j}+z^2\boldsymbol{k}\right)=2x+2y+2z$$

$$\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\ \partial_x&\partial_y&\partial_z\\ x^2&y^2&z^2\end{vmatrix}=\left(\frac{\partial}{\partial y}(z^2)-\frac{\partial}{\partial z}(y^2)\right)\boldsymbol{i}-\left(\frac{\partial}{\partial x}(z^2)-\frac{\partial}{\partial z}(x^2)\right)\boldsymbol{j}+\left(\frac{\partial}{\partial x}(y^2)-\frac{\partial}{\partial y}(x^2)\right)\boldsymbol{k}=0$$

29. Determine whether the scalar function $u=e^{-x}(\cos(y)-\sin(y))$ is harmonic, i.e. $\Delta u=\mathrm{div}(\nabla u)=u_{xx}+u_{yy}+u_{zz}=0$. We have $u_x=-e^{-x}(\cos(y)-\sin(y))$, $u_{xx}=e^{-x}(\cos(y)-\sin(y))$, $u_y=e^{-x}(-\sin(y)-\cos(y))$, $u_{yy}=e^{-x}(-\cos(y)+\sin(y))$, $u_{zz}=0$. Therefore

$$\Delta u=e^{-x}(\cos(y)-\sin(y))+e^{-x}(-\cos(y)+\sin(y))+0=0$$

and $u$ is harmonic.

39. If $\boldsymbol{F}=\langle xy, yz, z^2\rangle$, $\boldsymbol{G}=\langle x,y,-z\rangle$, find $\mathrm{div}(\boldsymbol{F}\times\boldsymbol{G})$.

$$(\boldsymbol{F}\times\boldsymbol{G})=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\ xy&yz&z^2\\ x&y&-z\end{vmatrix}=(-2yx^2)\boldsymbol{i}+(xyz+xz^2)\boldsymbol{j}+(xy^2-xyz)\boldsymbol{k}$$

so

$$\mathrm{div}(\boldsymbol{F}\times\boldsymbol{G})=\left(\frac{\partial(-2yx^2)}{\partial x}+\frac{\partial(xyz+xz^2)}{\partial x}+\frac{\partial(xy^2-xyz)}{\partial x}\right)=x(z-y).$$

## Section 13.1

3. Evaluate the line integral $\int_C \frac{1}{3+y}\,ds$ for $C:\ x=2t^{3/2},\ y=3t,\ 0\le t\le 1$.
We have $ds=\sqrt{x'(t)^2+y'(t)^2}\,dt=\sqrt{(3t^{1/2})^2+(3)^2}\,dt=3\sqrt{t+1}\,dt$ so

$$\int_C\frac{1}{3+y}\,ds=\int_0^1\frac{3\sqrt{t+1}}{3+3t}\,dt=\int_0^1\frac{1}{\sqrt{t+1}}\,dt=2\sqrt{t+1}\Big|_0^1=2(\sqrt{2}-1).$$

7. Evaluate the line integral $\int_C (-y\,dx + x\,dy)$ for $C:\ y = 4x^2$ from $(-1, 4)$ to $(0, 0)$.
Let $x = t$, $y = 4t^2$, $-1 \leq t \leq 0$. Then

$$\int_C (-y\,dx + x\,dy) = \int_{-1}^{0} \left[(-4t^2)\,dt + (t)(8t\,dt)\right] = \int_{-1}^{0} 4t^2\,dt = \frac{4}{3}t^3\Big|_{-1}^{0} = \frac{4}{3}.$$

13. Evaluate $\int_C \left[(x^2 + y^2)\,dx + 2xy\,dy\right]$ for

(a) $C$ the quarter circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$.

Let $x = \cos(\theta)$, $y = \sin(\theta)$ for $0 \leq \theta \leq \pi/2$. Then we have

$$\begin{aligned}\left[(x^2 + y^2)\,dx + 2xy\,dy\right] &= (1)(-\sin(\theta)\,d\theta) + (2\cos(\theta)\sin(\theta))(\cos(\theta)\,d\theta)\\ &= (-\sin(\theta) + 2\sin(\theta)\cos^2(\theta))\,d\theta.\end{aligned}$$

So we have

$$\int_C \left[(x^2 + y^2)\,dx + 2xy\,dy\right] = \int_0^{\pi/2} (-1 + 2\cos^2(\theta))\sin(\theta)\,d\theta = -\frac{1}{3}$$

Where to integrate this we set $u = \cos(\theta)$ so that $du = -\sin(\theta)\,d\theta$ and changed variables.

(b) $C$ is the straight line $y = 1 - x$ from $(1, 0)$ to $(0, 1)$.

Let $x = 1 - t$, $y = 1$, $0 \leq t \leq 1$ and we have

$$\begin{aligned}\left[(x^2 + y^2)\,dx + 2xy\,dy\right] &= \left[(1 - t)^2 + t^2\right](-dt) + 2(1 - t)(t)\,dt\\ &= (-1 + 4t - 4t^2)\,dt.\end{aligned}$$

So we have

$$\int_C \left[(x^2 + y^2)\,dx + 2xy\,dy\right] = \int_0^1 (-1 + 4t - 4t^2)\,dt = -\frac{1}{3}.$$