Vector Formulas

In these notes we use notation like $\boldsymbol{F}$ for vector valued functions and we use either

$$\boldsymbol{F}(t) = \langle f_1(t), f_2(t), f_3(t)\rangle = f_1(t)\boldsymbol{i} + f_2(t)\boldsymbol{j} + f_3(t)\boldsymbol{k}$$

for vector valued functions in $\mathbb{R}^3$ or $\boldsymbol{F}(t) = \langle f_1(t), f_2(t)\rangle = f_1(t)\boldsymbol{i} + f_2(t)\boldsymbol{j}$ for vector valued functions in $\mathbb{R}^2$. In what follows we will usually give the formulas for $\mathbb{R}^3$. If a formula is only valid in $\mathbb{R}^3$ (such as the cross product we will note this).

Consider vector valued function $\boldsymbol{F}$ as above and $\boldsymbol{G} = \langle g_1(t), g_2(t), g_3(t)\rangle = g_1(t)\boldsymbol{i}+g_2(t)\boldsymbol{j}+g_3(t)\boldsymbol{k}$ and scalar function $f(t)$, $g(t)$.

1. Scalar Times Vector: $(f\boldsymbol{G})(t) = f(t)f_1(t)\boldsymbol{i} + f(t)f_2(t)\boldsymbol{i} + f(t)f_3(t)\boldsymbol{k}$
2. Dot Product: $(\boldsymbol{F}\cdot\boldsymbol{G})(t) = f_1(t)g_1(t) + f_2(t)g_2(t) + f_3(t)g_3(t)$
3. Cross Product (only in $\mathbb{R}^3$):

$$(\boldsymbol{F}\times\boldsymbol{G})(t) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ f_1(t) & f_2(t) & f_3(t) \\ g_1(t) & g_2(t) & g_3(t) \end{vmatrix}$$

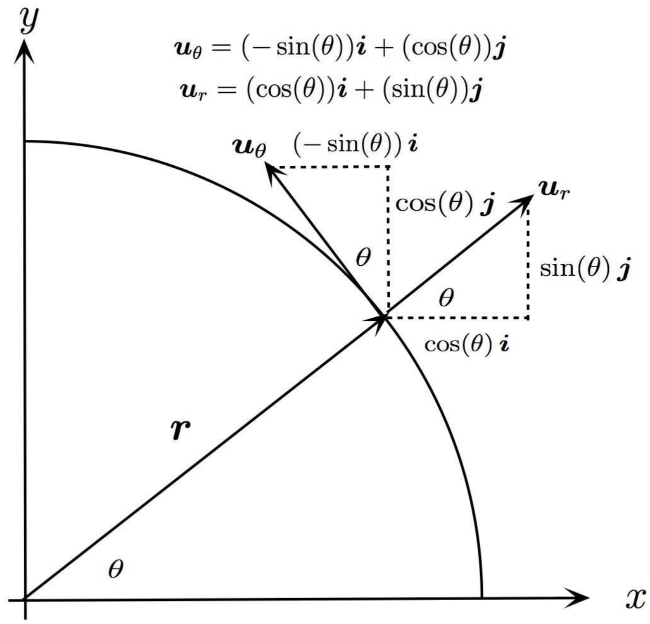
$$= (f_2g_3 - f_3g_2)(t)\boldsymbol{i} - (f_1g_3 - f_3g_1)(t)\boldsymbol{j} + (f_1g_2 - f_2g_1)(t)\boldsymbol{k}$$

4. $\lim\limits_{t\to c}\boldsymbol{F}(t) = \langle \lim\limits_{t\to c} f_1(t), \lim\limits_{t\to c} f_2(t), \lim\limits_{t\to c} f_3(t)\rangle$
5. $\dfrac{d\boldsymbol{F}}{dt}(t) = \left\langle \dfrac{df_1}{dt}(t), \dfrac{df_2}{dt}(t), \dfrac{df_3}{dt}(t)\right\rangle.$
6. Given a curve $\boldsymbol{F}(t)$ the derivative $\boldsymbol{T}(t) = \boldsymbol{F}'(t)$ is a tangent vector to the curve at $\boldsymbol{F}(t)$ and $\boldsymbol{T}(t)/\|\boldsymbol{T}(t)\|$ is a unit tangent vector.
7. $\displaystyle\int \boldsymbol{F}(t)\,dt = \left\langle \int f_1(t)\,dt, \int f_2(t)\,dt, \int f_3(t)\,dt\right\rangle + \boldsymbol{C}$ where $\boldsymbol{C} = \langle C_1, C_2, C_3\rangle$.
8. $\displaystyle\int_a^b \boldsymbol{F}(t)\,dt = \left\langle \int_a^b f_1(t)\,dt, \int_a^b f_2(t)\,dt, \int_a^b f_3(t)\,dt\right\rangle.$
9. $(f\boldsymbol{F})' = f'\boldsymbol{F} + f\boldsymbol{F}', \quad (\boldsymbol{F}\cdot\boldsymbol{G})' = \boldsymbol{F}'\cdot\boldsymbol{G} + \boldsymbol{F}\cdot\boldsymbol{G}',$

   $(\boldsymbol{F}\times\boldsymbol{G})' = \boldsymbol{F}'\times\boldsymbol{G} + \boldsymbol{F}\times\boldsymbol{G}', \quad (\boldsymbol{F}(f(t)))' = \boldsymbol{F}'(f(t))f'(t).$
10. Position: $\boldsymbol{R}(t) = f_1(t)\boldsymbol{i} + f_2(t)\boldsymbol{j} + f_3(t)\boldsymbol{k}$; Velocity: $\boldsymbol{V}(t) = \boldsymbol{R}'(t)$; Speed: $\|\boldsymbol{V}(t)\|$; Direction $\boldsymbol{V}(t)/\|\boldsymbol{V}(t)\|$; Acceleration: $\boldsymbol{A}(t) = \boldsymbol{V}'(t)$.
11. Projectile in $\mathbb{R}^2$: $\boldsymbol{A} = -g\boldsymbol{j}$ ($g = 32ft/s^2$, $g = 9.8m/s^2$), $\boldsymbol{V}(0) = v_0\cos(\alpha)\boldsymbol{i} + v_0\sin(\alpha)\boldsymbol{j}$, $v_0 = \|V(0)\|$, $\boldsymbol{R}(0) = s_0\boldsymbol{j}$. Then $\boldsymbol{R}(t) = [(v_0\cos(\alpha))\,t]\boldsymbol{i} + \left[(v_0\sin(\alpha))\,t - \frac{1}{2}gt^2 + s_0\right]\boldsymbol{j}$

In order to write vector $\boldsymbol{r}$ with respect to orthogonal components in polar coordinates we use the unit vectors $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$ which are defines via

$$\boldsymbol{u}_r = (\cos(\theta))\boldsymbol{i} + (\sin(\theta))\boldsymbol{j}, \quad \boldsymbol{u}_\theta = (-\sin(\theta))\boldsymbol{i} + (\cos(\theta))\boldsymbol{j}$$



We can readily compute

$$\frac{d\boldsymbol{u}_r}{d\theta} = (-\sin(\theta))\boldsymbol{i} + (\cos(\theta))\boldsymbol{j} = \boldsymbol{u}_\theta$$

and

$$\frac{d\boldsymbol{u}_\theta}{d\theta} = (-\cos(\theta))\boldsymbol{i} + (-\sin(\theta))\boldsymbol{j} = -\boldsymbol{u}_r$$

If we have a radial vector field, i.e., $\boldsymbol{R}(t) = r\boldsymbol{u}_r = (r\cos(\theta))\boldsymbol{i} + (r\sin(\theta))\boldsymbol{j}$ where $r = \|\boldsymbol{R}\|$, we can compute $\boldsymbol{V}$ and $\boldsymbol{A}$. It is important to notice that r and θ depend on t.

$$\boldsymbol{V} = \frac{d\boldsymbol{R}}{dt} = \frac{dr}{dt}\boldsymbol{u}_r + r\frac{d\boldsymbol{u}_r}{dt} = \frac{dr}{dt}\boldsymbol{u}_r + r\frac{d\boldsymbol{u}_r}{d\theta}\frac{d\theta}{dt}$$

$$= \frac{dr}{dt}\boldsymbol{u}_r + r\frac{d\theta}{dt}\boldsymbol{u}_\theta$$

Similarly we can compute

$$\boldsymbol{A} = \frac{d\boldsymbol{V}}{dt} = \frac{d^2\boldsymbol{R}}{dt^2}$$

$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\boldsymbol{u}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\boldsymbol{u}_\theta$$