Smoothness of a Vector Field in $\mathbb{R}^2$ and $\mathbb{R}^3$

We have seen that a vector field $\boldsymbol{F}(t) = \langle f_1(t), f_2(t), f_3(t)\rangle$ in $\mathbb{R}^3$ (or $\boldsymbol{F}(t) = \langle f_1(t), f_2(t)\rangle$ in $\mathbb{R}^2$) has derivative given by

$$\boldsymbol{F}'(t) = \langle f_1'(t), f_2'(t), f_3'(t)\rangle.$$

We define a vector field F to be smooth at a point t_0 if $\boldsymbol{F}'(t_0)$ exists and $\boldsymbol{F}'(t_0) \neq 0$. The vector field is smooth on an interval $a < x < b$ if it is smooth at each point of the interval and a vector field is said to be piecewise smooth on this interval if it is smooth except at a finite number of points in the interval.

Note that this may look a bit strange to require that for the function to smooth at t_0 we need $\boldsymbol{F}'(t_0) \neq 0$. Recall that a zero derivative for a scalar function only meant that it had a horizontal tangent line in the plane. But for a vector field if the derivative has a zero it indicates the presence of a so-called *cusp* (a sharpe change in direction of the tangent line at that point). Therefore it is actually quite reasonable for smoothness of a vector field to assume that the vector field not be zero.

Consider a simple example in $\mathbb{R}^2$. We claim that there is a cusp for the vector field $\boldsymbol{F}(t) = \langle t^3, t^2\rangle$ when $t = 0$. We can easily compute

$$\boldsymbol{F}'(t) = \langle 3t^2, 2t\rangle$$

and we see that $\boldsymbol{F}'(0) = 0$. Notice that the direction vector is given by

$$d(t) = \frac{F'(t)}{\|F'(t)\|} = \frac{\langle 3t^2, 2t\rangle}{\sqrt{9t^4 + 4t^2}} = \frac{\langle 3|t|, 2\text{sgn}(t)\rangle}{\sqrt{9t^2 + 4t}}.$$

$$\lim_{t\to 0^+} d(t) = \langle 0, 1\rangle, \qquad \lim_{t\to 0^-} d(t) = \langle 0, -1\rangle$$

so that $d(t)$ is discontinuous at $t = 0$.

To see what happens at $t = 0$ let us convert from parametric to rectangular form using $x = t^3$ and $y = t^2$ which implies $t = x^{1/3}$ and

$$y = f(x) = \left(x^{1/3}\right)^2 = x^{2/3}.$$

We then compute

$$f'(x) = \frac{2}{3}x^{-1/3}$$

and we see that $f'(0)$ does not exist.