Some Multiple Integral Formulas

1. The change of variables $x = x(u, v)$, $y = y(u, v)$ for a double integral is

$$\iint f(x, y)\, dA = \iint f(u, v)\, J(u, v)\, dudv, \quad J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

For polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ and $J = r$ so

$$\iint f(x, y)\, dA = \iint f(r, \theta)\, r\, drd\theta = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\, drd\theta.$$

2. The area of the surface A given by $z = f(x, y)$ for $(x, y) \in D \subset \mathbb{R}^2$ for a continuous function f is

$$A(S) = \iint_D \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dA$$

3. Let S be a surface defined parametrically by $\boldsymbol{R}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ on a region $D \subset \mathbb{R}^2$ (in the uv-plane). Assume that $\boldsymbol{R}_u$ and $\boldsymbol{R}_v$ are continuous in D and $\boldsymbol{R}_u \times \boldsymbol{R}_v \neq 0$ in D. Then the surface area of S is

$$A(S) = \iint_D \|\boldsymbol{R}_u \times \boldsymbol{R}_v\|\, dudv$$

4. Conversion formulas rectangular - cylindrical coordinates - spherical coordinates

(x, y, z), (r, θ, z):
$x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$

$r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $z = z$

(x, y, z), (r, θ, φ): $x = \rho\sin(\varphi)\cos(\theta)$,
$y = \rho\sin(\varphi)\sin(\theta)$, $z = \rho\cos(\varphi)$

$\rho = \sqrt{x^2 + y^2 + z^2}$, $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$,

$\varphi = \cos^{-1}\left(\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$

5. If $R = \{(r, \theta, z) \,:\, u(r, \theta) \le z \le v(r, \theta),\ (r, \theta) \in D \subset \mathbb{R}^2\}$ then

$$\iint_R f(x, y, z)\, dV = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} \int_{u(r,\theta)}^{v(r,\theta)} f(r\cos(\theta), r\sin(\theta), z)\, r\, dzdrr\theta$$

6. If R is a region in $\mathbb{R}^3$ then

$$\iiint_R f(x, y, z)\, dV = \iiint_{R'} f(\rho\sin(\varphi)\cos(\theta), \rho\sin(\varphi)\sin(\theta), \rho\cos(\varphi))\, \rho^2\sin(\varphi)\, d\rho d\theta d\varphi$$

where R' is R in spherical coordinates.