Gradient, Directional Derivative and Chain Rule

(1.) If $z = f(x, y)$ the *gradient* of f is $\nabla f = \langle f_x, f_y\rangle$ and if $w = f(x, y, z)$ the gradient of f is $\nabla f = \langle f_x, f_y, f_z\rangle$.

(2.) Properties of the gradient

(a) $\nabla(af + bg) = a\nabla f + b\nabla g$; (b) $\nabla(fg) = f\nabla g + g\nabla f$; (c) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$;
(d) $\nabla(f^n) = nf^{n-1}\nabla f$.

(3.) If S is a surface defined by $z = f(x, y)$, then the *equation of the tangent plane* at a point $P_0(x_0, y_0, z_0)$ on S can be written as

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) - (z - z_0) = 0$$

which can be written in the standard form $Ax + By + Cz + D = 0$.

(4.) If $z = f(x, y)$ the *total differential* of f is $df = f_x\,dx + f_y\,dy$ and if $w = f(x, y, z)$ the total differential of f is $df = f_x\,dx + f_y\,dy + f_z\,dz$.

(5.) Chain Rule II. If $z = f(x, y)$ and $x = x(u, v)$, $y = y(u, v)$ then

$$\frac{dz}{du} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{dz}{dv} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

and if $w = f(x, y, z)$ and $x = x(r, s, t)$, $y = y(r, s, t)$, $z = z(r, s, t)$ then

$$\frac{dw}{dr} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r},$$

$$\frac{dw}{ds} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s},$$

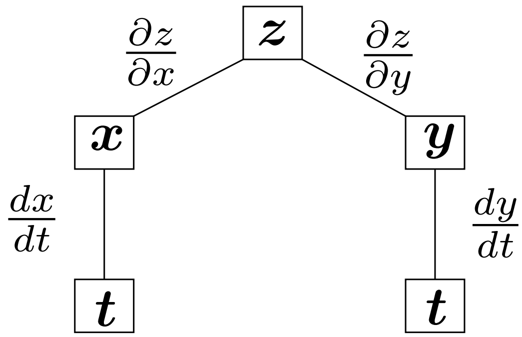
$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}.$$

(6.) If $z = f(x, y)$ and $\boldsymbol{u} = \langle u_1, u_2\rangle$ is a unit vector, then the *directional derivative* of f at the point $P_0(x_0, y_0)$ in the direction $\boldsymbol{u}$ is

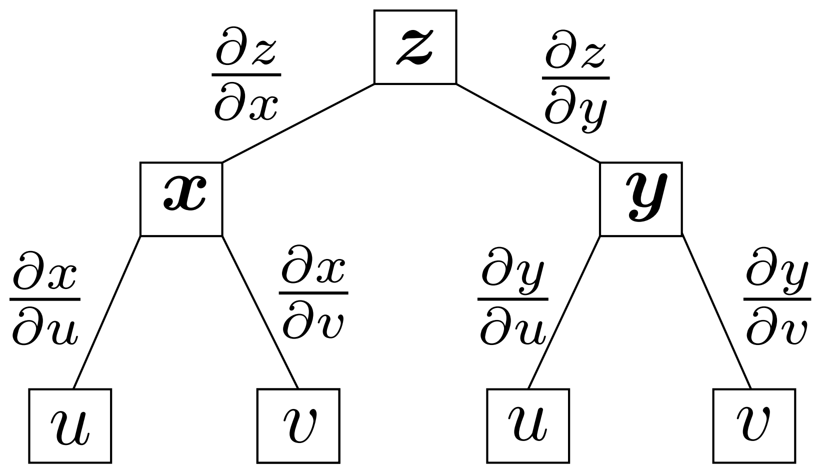
$$D_{\boldsymbol{u}}f(P_0) = (\nabla f)\big|_{P_0} \cdot \boldsymbol{u} = f_x(P_0)u_1 + f_y(P_0)u_2.$$

(7.) If $z = f(x, y)$ and $P_0(x_0, y_0)$ then P_0 is a *critical point* of f if $\nabla f(P_0) = 0$ or (a) $\nabla f(P_0)$ does not exist .

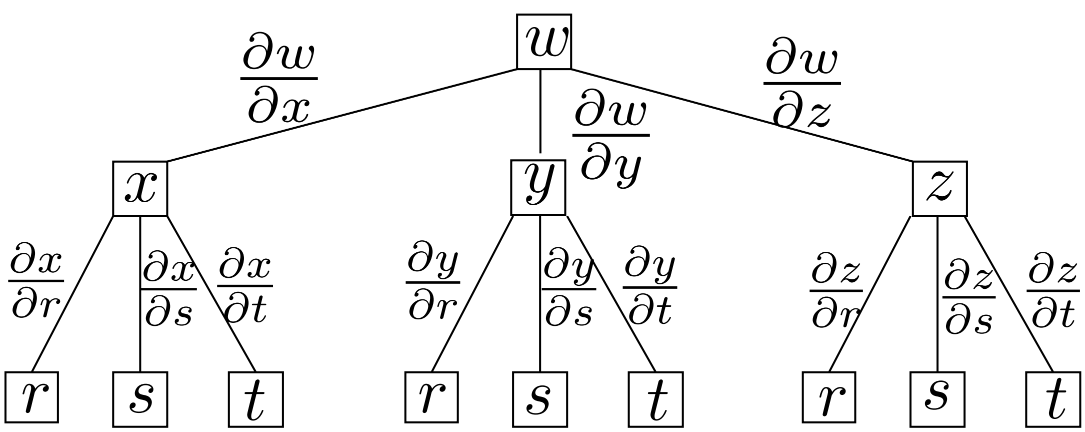
(8.) If $z = f(x, y)$ and $P_0(x_0, y_0)$ and all second order partial derivatives are continuous in a neighborhood of $P_0(x_0, y_0)$ we define the discriminant by $D = f_{xx}f_{yy} - f_{xy}^2$. If, in addition, P_0 is a critical point then P_0 is a (a) *Relative Maximum* if $D(P_0) > 0$ and $f_{xx}(P_0) < 0$ (or $f_{yy}(P_0) < 0$); (b) *Relative Minmum* if $D(P_0) > 0$ and $f_{xx}(P_0) > 0$ (or $f_{yy}(P_0) > 0$); (c) *Saddle* if $D(P_0) < 0$. .



$$z = f(x, y), \quad x = x(t), \quad y = y(t)$$



$$z = f(x, y), \quad x = x(u, v), \quad y = y(u, v)$$



$$w = f(x, y, z)$$

$$x = x(r, s, t), \;\; y = y(r, s, t), \;\; z = z(r, s, t)$$