Divergence, Curl, Line Integrals, Green's Theorem, etc

1. A vector field (v.f.) mapping a domain D in $\mathbb{R}^2$ (or $\mathbb{R}^3$) to $\mathbb{R}^2$ (or $\mathbb{R}^3$) is given by

$$\boldsymbol{F}(x,y) = \langle P(x,y), Q(x,y)\rangle, \quad \boldsymbol{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle.$$

2. For convenience we often identify a point (x,y,z) with the position vector $\boldsymbol{x} = \langle x,y,z\rangle$ and write $\boldsymbol{F}(x,y,z) = \boldsymbol{F}(\boldsymbol{x})$.

3. Recall the gradient of a scalar function f given by $\nabla f = \langle f_x, f_y, f_z\rangle$. We also define the divergence of a v.f. $\boldsymbol{F}$ by $\text{div}\boldsymbol{F} = P_x + Q_y + R_z = \nabla\cdot\boldsymbol{F}$ and the curl by

$$\nabla\!\times\!\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = \nabla\times\boldsymbol{F}.$$

For a scalar function f we define the Laplacian by $\Delta f = f_{xx} + f_{yy} + f_{zz}$.

4. A v.f. $\boldsymbol{F}$ is called **conservative** if it is a gradient v.f., i.e., $\boldsymbol{F} = \nabla f$ for some scalar function f.

5. f is a scalar function in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and $\boldsymbol{r}(t) = \langle x(t), y(t)\rangle$ (or $\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$) for $a \le t \le b$ is a curve C in $\mathbb{R}^2$ (or $\mathbb{R}^3$), then we define the **Line Integral** of f over C by (we only give the formula in $\mathbb{R}^3$)

$$\int_C f(x,y,z)\,ds$$

where ds is incremental arc length, $ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$. So we have

$$\int_C f(x,y,z)\,ds = \int_a^b f(\boldsymbol{r}(t))\,\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt.$$

6. We also consider line integrals with respect to x or y:

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt, \quad \int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt.$$

7. **Line integral of a Vector Field**

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t))\cdot r'(t)\,dt = \int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds$$

where we have $\boldsymbol{T} = \dfrac{r'(t)}{\|r'(t)\|}$ unit tangent vector, $ds = \|r'(t)\|\,dt$. If $\boldsymbol{F} = \langle P, Q, R\rangle$ and $\boldsymbol{r}(t) = \langle (x(t), y(t), z(t)\rangle$ we can also write

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_a^b \left[Px'(t) + Qy'(t) + Rz'(t)\right]\,dt = \int_C [P\,dx + Q\,dy + R\,dz].$$

8. A curve C is **orientable** if direction can be described along the curve.

9. [**Fundamental Theorem of Line Integrals**] C a smooth curve $\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \leq t \leq b$, f differentiable and ∇f continuous implies $\displaystyle\int_C \nabla f \cdot d\boldsymbol{r} = f(\boldsymbol{r}(b)) - f(\boldsymbol{r}(a))$.

10. [**Equivalent conditions for Path Independence**] If $\boldsymbol{F}$ is a continuous v.f. on an open connected region D, then the following are equivalent:

    (a) $\boldsymbol{F}$ is conservative, i.e., $F = \nabla f$ for some scalar function f.

    (b) $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ for every closed curve C in D.

    (c) $\displaystyle\int_c \boldsymbol{F} \cdot d\boldsymbol{r}$ is path independent

11. If $\boldsymbol{F} = \langle P, Q\rangle$ with P and Q having continuous first order partials in D (a simply connected domain). Then $\boldsymbol{F}$ is conservative, if and only if, $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

12. [**Green's theorem**] Let C be a positively oriented, piecewise-smooth simple closed curve in the region D bounded by C. If P and Q have continuous partials on D then

$$\int\limits_C [P\,dx + Q\,dy] = \int\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

13. [**Divergence Theorem in the Plane**] Let $\boldsymbol{F} = \langle P, Q\rangle$ and let D be a domain with piecewise smooth boundary C, then

$$\oint_C \boldsymbol{F} \cdot \boldsymbol{N}\,ds = \iint\limits_D \operatorname{div} \boldsymbol{F}\,dA \quad \text{where } \boldsymbol{N} \text{ is the unit normal vector.}$$

14. Let S be a surface in $\mathbb{R}^3$ given by $z = f(x, y)$ and R is the projection on the xy-plane. If f, f_x and f_y are continuous in R and g is a continuous function on S, then the **surface integral** of g over S is

$$\iint\limits_S g(x, y, z)\,dS = \iint\limits_R g(x, y, f(x, y))\sqrt{f_x(x, y)^2 + f_y(x, y)^2}\,dA.$$

15. [**Stoke's Theorem**] Let S be an oriented surface with unit normal vector $\boldsymbol{N}$, assume the boundary C is a piecewise smooth Jordan curve with compatible orientation. Let $\boldsymbol{F}$ be a continuously differentiable v.f. on S, then

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{R} = \iint\limits_S (\nabla \times \boldsymbol{F} \cdot \boldsymbol{N})\,dS.$$

16. [**Divergence Theorem**] Let S be a smooth orientable surface enclosing a solid region D in $\mathbb{R}^3$. Let $\boldsymbol{F}$ be a continuously differentiable v.f. in an open set containing D, then

$$\iint\limits_S \boldsymbol{F} \cdot \boldsymbol{N}\,dS = \iiint\limits_D \operatorname{div} \boldsymbol{F}\,dV.$$