

EMMY NOETHER MIDDLE SCHOOL MATHEMATICS DAY
Texas Tech University
May 14, 2014

SOLUTIONS.

1.) Any such number must have at least two digits. Otherwise it is the sum of its digits, not three times the sum. If the number has three digits, then the sum of its digits is at most 27 ($9 + 9 + 9$) and three times the sum of its digits is at most 81, not a three digit number. Similarly, the number cannot have more than three digits. If the number has two digits and the first digit is 4, the sum of the digits is at most 13 and three times the sum of its digits is at most 39, smaller than a two digit number beginning with 4. Similarly, the number cannot have two digits with the first digit greater than 4. Since the number is three times the sum of its digits, it must be a multiple of 3. A check of the multiples of 3 from 12 through 39 shows that $\boxed{27}$ is the only number which is three times the sum of its digits.

2.) There are seven distinct letters in T E X A S T E C H without repetition. If the word has four distinct letters without repetition, there are seven choices for the first letter, six choices for the second letter, five choices for the third letter and four choices for the last letter. Thus there are $7 \cdot 6 \cdot 5 \cdot 4 = 840$ possible such words. If the word has one letter repeated and the other two letters distinct, then the repeated letter can be either T or E. The repetition can occur at six possible places in the order of letters (first and second, first and third, first and fourth, second and third, second and fourth or third and fourth). There are then six choices for the first nonrepeated letter and five choices for the second nonrepeated letter. There are thus $2 \cdot 6 \cdot 6 \cdot 5 = 360$ possible such words. The only other possibilities are words that have both E and T repeated. There are six possibilities for which places in the order of letters the E's occur. There are thus a total of $840 + 360 + 6 = \boxed{1206}$ possible four letter words.

3.) Consider a triangle with one vertex at the center of the circle filling the hole and with the other two vertices at the centers of two adjacent smaller circles arranged around the inside of the larger circle. The angle at the vertex of this triangle at the center of the hole 60° since six such triangles could be formed at this vertex. The legs of this triangle going from the center of the hole to the centers of the outer circles are of equal length. Thus, this is an equilateral triangle. The edge of this triangle connecting the centers of two adjacent outer circles is bisected by each of the two circles. Thus, each of the legs from the center of the hole to the center of an outer circle is also bisected by the circle. The inner circle is tangent to the outer circle at this point of bisection. Thus the radius of the inner circle is the same as the radius of each of the outer circles. The larger circle has radius 1. A radius of the larger circle is formed by a radius of the circle in the hole and by the diameter of an outer circle. Thus the radius of each of the smaller circles, including the one in the hole, is $\boxed{\frac{1}{3}}$.

4.) For the Earth and Mars to again line up on the same side of the Sun, the Earth must complete one more revolution around the Sun than Mars does. Thus Mars will go around the Sun n times while the Earth goes around the Sun $n + 1$ times. Since these occur over the same period, $n \cdot 687 = (n + 1) \cdot 365$. Solving this for n gives $n = \frac{365}{322}$. Here n is the number

of times that Mars goes around the Sun. The length of time for this is thus $(687) \cdot \left(\frac{365}{322}\right) = \boxed{\frac{250755}{322} \approx 778.74 \text{ days}}$. (This period is known as the *synodic period* of Mars. In reality neither the orbit of Mars nor Earth is circular and their rates of motion vary slightly.)

5.) If the hour displayed is 1, there will be a second 1 in the minutes display for 1:01, 1:10 through 1:19, 1:21, 1:31, 1:41 and 1:51. The minutes display will be two identical digits different from 1 at 1:00, 1:22, 1:33, 1:44 and 1:55. This is a total of 20 times. There will be a similar 20 times with two identical digits when the hour display is either of 2, 3, 4 or 5. If the hour display is 6, 7, 8 or 9, the total display will have two identical digits only at times such as 6:00, 6:11, 6:22, 6:33, 6:44 or 6:55, six times for each hour. If the hour display is 10, there will be two identical digits when the hour display contains at least one 0 or 1, which will occur 28 times. There will also be the four times 10:22, 10:33, 10:44 and 10:55. If the hour display is 11, then every time has at least two identical digits, for 60 times. If the hour display is 12 the situation is similar to that for an hour display of 10 with $28 + 4$ times when at least two digits are identical. There are thus a total of $20 + 20 + 20 + 20 + 20 + 6 + 6 + 6 + 6 + 32 + 60 + 32 = \boxed{248}$ times when the display has at least two identical digits. Since there are 720 distinct hour-minute displays in a 12-hour period, more than one-third of the time the display will have at least two identical digits.

6.) Each nonleap year has $365 = (52) \cdot 7 + 1$ days. Thus the day of the week corresponding to a particular date shifts one day each nonleap year. In a leap year it shifts two days. Although 1924 was a leap year, December 7 that year was after the extra leap day. There were 22 other leap years from 1924 through 2014. Thus the day of the week corresponding to December 7 shifted $90 + 22 = 112 = (16) \cdot 7$ days during this period. This is an exact number of 16 weeks with no extra days. Hence December 7, 1924 was on the same day of the week that December 7, 2014 will be, i.e. on $\boxed{\text{Sunday}}$.