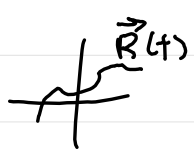
<u>13.1</u> Vector Fields

$\vec{F}(x,y,z)$ ★
↑
vector function of (x,y,z)

remember:  Ch. 9–10

$z = f(x,y)$ $\quad T(x,y,z)=c$ $\quad$ Ch. 11–12

$\nabla f = \langle \partial_x, \partial_y, \partial_z \rangle f = \langle f_x, f_y, f_z \rangle$
$= \langle f_1(x,y,z), f_2(x,y,z), f_3(x,y,z) \rangle$

At any point (x,y,z), we associate a vector from $\vec{F}(x,y,z) = \langle f_1(x,y,z), f_2(x,y,z), f_3(x,y,z) \rangle$

$D \subseteq \mathbb{R}^3 \quad R \subseteq \mathbb{R}^3$

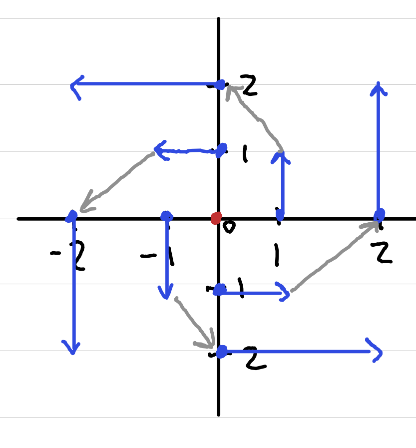
Also works in 2D:

$\vec{F}(x,y) = \langle f_1(x,y), f_2(x,y) \rangle$
$D \subseteq \mathbb{R}^2 \quad R \subseteq \mathbb{R}^2$

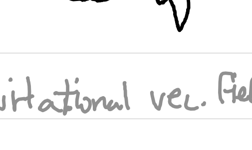
$F(x,y,z) = \langle f_1(x,y), f_2(x,y), 0 \rangle$
↑
from 2D → 3D

<u>Examples</u>
#1] $\vec{F}(x,y) = \langle -y, x \rangle$
$\forall (x,y \in \mathbb{R}^2)$

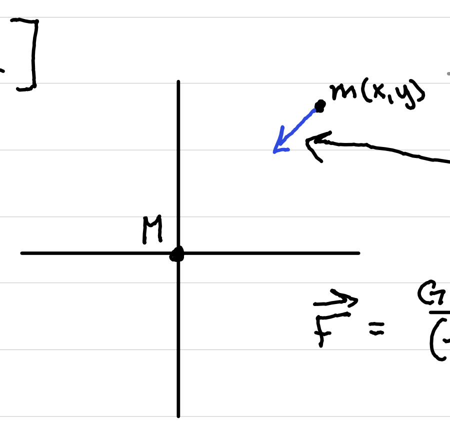
how do we visualize?
1. Sketch domain
2. choose points
3. Draw vectors



think of the vectors like the arrows on a weather report wind map



#2]



this is a gravitational vec. field example

$$\vec{F} = \frac{G(m)(M)}{(\sqrt{x^2+y^2})^2} \cdot \frac{-\langle x,y \rangle}{\sqrt{x^2+y^2}}$$
↑
$\hat{U}$

$\vec{F_1}(x,y,z)$ on D_1
$\vec{F_2}(x,y,z)$ on D_2
$g(x,y,z)$ on D_3

$(a\vec{F_1} + b\vec{F_2})(x,y,z) = a\vec{F_1}(x,y,z) + b\vec{F_2}(x,y,z)$ on $D_1 \cap D_2$

$(\vec{F_1}\, g)(x,y,z) = \vec{F_1}(x,y,z)\, g(x,y,z)$ on $D_1 \cap D_2$

$(\vec{F_1} \cdot \vec{F_2})(x,y,z) = \vec{F_1}(x,y,z) \cdot \vec{F_2}(x,y,z)$ on $D_1 \cap D_2$

Same for $\vec{F_1} \times \vec{F_2}$, no time to write

<u>Differentiation of Vector Fields</u> ★★★

divergence of $\vec{F}$
curl of $\vec{F}$

remember $\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$

<u>These are important</u>

$\text{div}\vec{F} = \nabla \cdot \langle f_1, f_2, f_3 \rangle = \langle f_{1_x}, f_{2_y}, f_{3_z} \rangle$

$\text{curl}\vec{F} = \nabla \times \langle f_1, f_2, f_3 \rangle$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial x & \partial y & \partial z \\ f_1 & f_2 & f_3 \end{vmatrix} = \hat{\imath}(f_{3_y} - f_{2_z}) - \hat{\jmath}(f_{3_x} - f_{1_z}) + \hat{k}(f_{2_x} - f_{1_y})$$

divergence → measure of compression / expansion
curl → measure of rotation

★ You can do divergence in 2D but not the curl

<u>Examples</u>
#1] $F = \langle \cos(xy), e^{z^2}, xyz \rangle$
find $\text{div}\vec{F}$ and $\text{curl}\vec{F}$

$\text{div}\vec{F} = \nabla \cdot \vec{F} = \langle \partial x, \partial y, \partial z \rangle \langle \cos(xy), e^{z^2}, xyz \rangle$
$= -\sin(xy)y + 0 + xy$
↑
returns a scalar function

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial x & \partial y & \partial z \\ \cos(xy) & e^{z^2} & xyz \end{vmatrix}$$

$= \hat{\imath}(xz - 2ze^{z^2}) - \hat{\jmath}(yz - 0) + \hat{k}(0 - (-\sin(xy)x))$
$= \vec{H}(x,y,z)$

<u>11-6-2024</u>

$\vec{F}(x,y,z) = \langle f_1(x,y,z), f_2(x,y,z), f_3(\ldots) \rangle$ in $D \subseteq \mathbb{R}^3$

$\text{div}\vec{F}(x,y,z) = \nabla \cdot \vec{F} = f_{1_x}\ f_{2_y}\ f_{3_z} = g(x,y,z)$

$\text{curl}\vec{F}(x,y,z) = \nabla \times \vec{F} = \hat{\imath}(f_{3_y} - f_{2_z}) - \hat{\jmath}(f_{3_x} - f_{1_z}) + \hat{k}(f_{2_x} - f_{1_y})$
$= \vec{H}(x,y,z)$ (another vec. field)

If $\vec{F} = \langle f_1(y,z), f_2(x,z), f_3(x,y) \rangle$,
then $\text{div}\vec{F} = 0$

If $\vec{F} = \langle f_1(x), f_2(y), f_3(z) \rangle$, then
$\text{curl}\vec{F} = \vec{0}$

<u>Examples</u>
#1] Find $\text{div}\vec{F}$ if $\vec{F} = y^2 z\hat{\imath} + xz^3\hat{\jmath} + \cos(xy)\hat{k}$

$= \boxed{0}$

#2] Find the $\text{curl}\vec{F}$ if $\vec{F} = \langle e^x, y^3, \ln(z) \rangle$

$= \boxed{\vec{0}}$

$\nabla \cdot \nabla f = \Delta f$ $\quad$ <u>Laplace operator</u>

Second order partial derivatives

$\nabla \cdot \langle \partial_x, \partial_y, \partial_z \rangle = \nabla \cdot \langle f_x, f_y, f_z \rangle$
$= \langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle f_x, f_y, f_z \rangle$
$= f_{xx} + f_{yy} + f_{zz}$

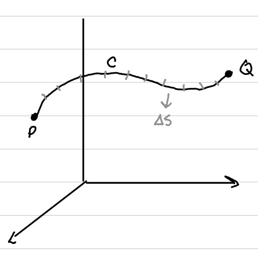
$\Delta f = f_{xx} + f_{yy} + f_{zz}$ ★

ex. Find the Laplace of $\cos(xy)z$

$\Delta f = \nabla \cdot \nabla f = \nabla \cdot \langle -y\sin(xy)z, -x\sin(xy)z, \cos(xy) \rangle$

$= -y^2\cos(xy)z - x^2\cos(xy)z + 0$
$= \boxed{-(x^2+y^2)\cos(xy)z}$

13.2 Line Integrals
(Throwback to 10.4)



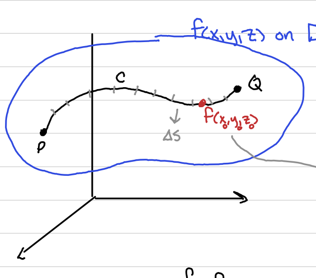
Review! :)

we parametrized the curve:

$C: \vec{R}(t) = \langle x(t), y(t), z(t) \rangle \quad t_1 \le t \le t_2$

and said Arclength $S = \int_C 1\, dS = \int_{t_1}^{t_2} \|\vec{R}'(t)\|\, dt$

— new stuff ↓ —



Line Integral $= \int_C f(x,y,z)\, dS$

this is a more generic version of 10.4

$= \lim_{n\to\infty}$ Riemann sum $= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_0, y_0, z_0)\, dS$

$= \int_{t_1}^{t_2} f(\vec{R}(t))\, \|\vec{R}'(t)\|\, dt$

$\|\vec{R}'(t)\|\,dt = dS$ (from arc length formula)

Examples

#1] Evaluate Line Int. on curve

$\int_C x^2 z\, ds$ where $C: \vec{R}(t) = \langle \cos(t), 2t, \sin(t) \rangle \quad 0 \le t \le \pi$

(given as parameterization)

1.

$\vec{R}'(t) = \langle -\sin(t), 2, \cos(t) \rangle$

$\|\vec{R}'(t)\| = \sqrt{(-\sin t)^2 + 2^2 + (\cos t)^2} = \sqrt{5}$

$\vec{R}(t) = \langle \cos(t), 2t, \sin(t) \rangle$ (x, y, z)

2.

$L.I. = \int_0^{\pi} (\cos(t))^2 \sin(t) \sqrt{5}\, dt$

$u = \cos(t) \quad du = -\sin(t)\, dt$

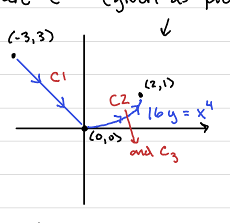
$-\int_0^{\pi} u^2 \sqrt{5}\, du$

$= -\sqrt{5} \frac{(\cos(t))^3}{3} \Big|_0^{\pi}$

$= \boxed{\frac{2}{3}\sqrt{5}}$

#2] Evaluate L.I. on $\int_C xy\, ds$

where C: (given as plot)



you choose what x = for these

1. Find line segments

$C_1: \vec{R}_1(t) = \langle t, -t \rangle \quad -3 \le t \le 0$ (x bounds)

$x = t$ because there are inf. parameterizations and this is easiest

$\vec{R}_1'(t) = \langle 1, -1 \rangle \quad \|\vec{R}_1'\| = \sqrt{2}$

$C_2: \vec{R}_2(t) = \langle t, \frac{t^4}{16} \rangle \quad 0 \le t \le 2$

↳ $x = t$ again

$C_3: \langle 2t, t^4 \rangle \quad 0 \le 2t \le 2 \rightarrow \boxed{0 \le t \le 1}$

↳ $x = 2t$

we use C_3 not C_2 because it is simpler

$\vec{R}_3' = \langle 2, 4t^3 \rangle = 2\langle 1, 2t^3 \rangle$

(using C_1 and C_3) $\quad \|\vec{R}_3'\| = 2\sqrt{1 + 4t^6}$

2.

$L.I. = \int_C xy\, dS = \int_{C_1 + C_3} xy\, dS$

$= \int_{C_1} xy\, dS + \int_{C_3} xy\, dS$

$= \int_{-3}^{0} t(-t)\sqrt{2}\, dt + \int_0^1 2t(t^4)(2\sqrt{1+4t^6})\, dt$

$4t^5(\sqrt{1+4t^6})$

$u = 1 + 4t^6$

$du = (4)(6)t^5\, dt$

$= -\sqrt{2} \frac{t^3}{3} \Big|_{-3}^{0} + \int_0^1 4t^5(\sqrt{1+4t^6})\, dt$

$+ \frac{1}{6} \int_0^1 \sqrt{u}\, du$

$+ \frac{1}{6}\left(\frac{2}{3}\right)(1+4t^6)^{3/2} \Big|_0^1$

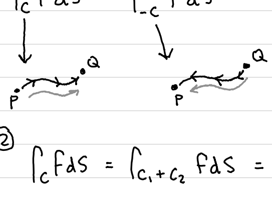
$= -\sqrt{2}(0 - (-9)) + \frac{1}{6}\left(\frac{2}{3}\right)(1 + 4t^6)^{3/2} \Big|_0^1$

$= \boxed{-9\sqrt{2} + \frac{1}{9}\left[5^{3/2} - 1\right]}$

Properties of Line Integrals

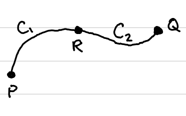
①

$\int_C f\, dS = \int_{-C} f\, dS$



②

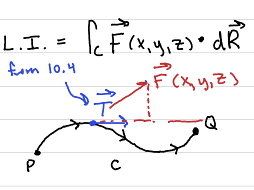
$\int_C f\, dS = \int_{C_1 + C_2} f\, dS = \int_{C_1} f\, dS + \int_{C_2} f\, dS$



Line Integral of Vector Field ★

$L.I. = \int_C \vec{F}(x,y,z) \cdot d\vec{R}$

from 10.4



$\vec{T} = \frac{\vec{R}'}{\|\vec{R}'\|}$

$\int_C \vec{F}(x,y,z) \cdot d\vec{R} = \int_C (\vec{F} \cdot \vec{T})\, dS$

we have a vector field $\vec{F}$

the dot product of $\vec{F}$ and $\vec{T}$ is the scalar projection of $\vec{F}$ in the direction of $\vec{T}$

$\int_C (\vec{F} \cdot \vec{T})\, dS = \int_{t_1}^{t_2} \vec{F}(\vec{R}(t)) \cdot \frac{\vec{R}'}{\|\vec{R}'\|} (\|\vec{R}'\|)\, dt$

$= \int_{t_1}^{t_2} \vec{F}(\vec{R}(t)) \cdot \vec{R}'(t)\, dt$ ★

focus on 2 things:

1. you are projecting in the tangential direction
2. $\int_{t_1}^{t_2} \vec{F}(\vec{R}(t)) \cdot \vec{R}'(t)\, dt$ is how you solve $\int_C \vec{F}(x,y,z) \cdot d\vec{R}$

what about $\int_{-C} \vec{F} \cdot d\vec{R}$?

that changes the orientation of $\vec{T}$

$\int_{-C} \vec{F} \cdot d\vec{R} = \int_{-C} \vec{F} \cdot (-\vec{T})\, dS$

$= \int_C (\vec{F} \cdot -\vec{T})\, dS = -\int_C \vec{F} \cdot \vec{T}\, dS$

this is all well related to physics :)

Examples

$\vec{F} \cdot d\vec{R} \longrightarrow \langle f_1, f_2, f_3 \rangle \cdot \langle dx, dy, dz \rangle$

#1] $\int_C [y\,dx - z\,dy + x\,dz]$ where

so $\vec{F}: \langle y, -z, x \rangle$

WeBWork does this notation

$C: \langle t^2, e^{-t}, e^t \rangle \quad 0 \le t \le 1$

$= \int_C \langle y, -z, x \rangle \cdot d\vec{R}$

1. $\int_{t_1}^{t_2} \vec{F}(\vec{R}(t)) \cdot \vec{R}'(t)\, dt$

$\vec{R}'(t) = \langle 2t, -e^{-t}, e^t \rangle$

$= \int_0^1 \langle e^{-t}, -e^{t}, t^2 \rangle \cdot \langle 2t, -e^{-t}, e^t \rangle\, dt$ (y, $-z$, x)

$-e^t(-e^{-t})$

$= \int_0^1 2te^{-t} + 1 + t^2 e^t\, dt$

... IBP :(

$= \left[-2e^{-t}(t+1) + t + e^t(t^2 - 2t + 2)\right] \Big|_0^1$

$= \boxed{e - \frac{4}{e} + 1}$

11-8-24 summary

$L.I. = \int_C f(x,y,z)\, dS = \boxed{\int_{t_1}^{t_2} f(\vec{R}(t))\, \|\vec{R}'(t)\|\, dt}$

$\vec{R}(t) = \langle x(t), y(t), z(t) \rangle \quad t_1 \le t \le t_2$

$L.I. = \int_C \vec{F}(x,y,z) \cdot d\vec{R}$

$= \int_C \vec{F}(x,y,z) \cdot \vec{T}\, dS \qquad \vec{T} = \frac{\vec{R}'(t)}{\|\vec{R}'(t)\|}$

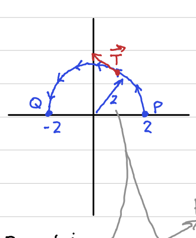
$= \boxed{\int_{t_1}^{t_2} \vec{F}(\vec{R}(t)) \cdot \vec{R}'(t)\, dt}$

More Examples

#1] $\int_C \vec{F} \cdot d\vec{R}$ where $\vec{F}: y\hat{i} + x\hat{j}$

and

$C: x^2 + y^2 = 4$ traversed counter clockwise from $(2,0) \longrightarrow (-2,0)$



because circle :)

1. Parameterize

$\vec{R}(t) = \langle 2\cos(t), 2\sin(t) \rangle \quad 0 \le t \le \pi$

$\vec{R}'(t) = \langle -2\sin(t), 2\cos(t) \rangle$

2. Build integral

$\int_0^{\pi} \langle 2\sin t, 2\cos t \rangle \cdot \langle -2\sin t, 2\cos t \rangle\, dt$ ($\vec{F}(\vec{R}(t))$, $\vec{R}'(t)$)

$= \int_0^{\pi} -4\sin^2(t) + 4\cos^2(t)\, dt$

$= 4(\cos^2 - \sin^2)$

$= 4(\cos(2t))$ ← Trig. Identity

$= 4 \frac{\sin(2t)}{2} \Big|_0^{\pi} = 4[0 - 0] = \boxed{0}$

OR

$y = \sqrt{4 - x^2}, \quad x = -t \qquad -2 \le t \le 2$

this order because $x = -t$

$y = \sqrt{4 - t^2}$

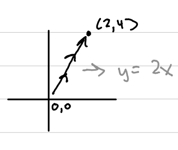
this is another parameterization you could do, but would be harder

If there is a circle, go w/ trig, probably. ★

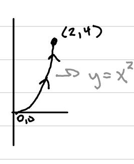
#2] Find L.I. on $\int_C \vec{F} \cdot d\vec{R}$ where

$\vec{F} = \langle xy^2, x^2 y \rangle$ and

C_1:



C_2:



1. Parameterize

$\vec{R}_1 = \langle t, 2t \rangle \quad 0 \le t \le 2$

$\vec{R}_1' = \langle 1, 2 \rangle$

from graphs

$\vec{R}_2 = \langle t, t^2 \rangle \quad 0 \le t \le 2$

$\vec{R}_2' = \langle 1, 2t \rangle$

2. Do the integrals

$\int_0^2 \langle t(2t)^2, t^2(2t) \rangle \cdot \langle 1, 2 \rangle\, dt$ ($4t^3$, $2t^3$)

$= \int_0^2 4t^3 + 4t^3\, dt = \int_0^2 8t^3\, dt$

$= 8 \frac{2^4}{4} = \boxed{32}$ for C_1

$\int_0^2 \langle t(t^2)^2, t^2(t^2) \rangle \cdot \langle 1, 2t \rangle\, dt$ (t^5, t^4)

$= \int_0^2 t^5 + 2t^5 = \int_0^2 3t^5$

$= 3 \frac{t^6}{6} \Big|_0^2 = 3 \frac{64}{6} = \frac{64}{2} = \boxed{32}$

With these 2 points, you will always get 32

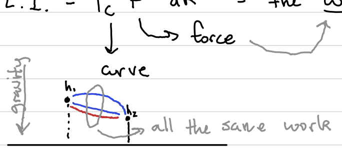
This is a conservative vector field

(not to get political or anything...)

Gravity is also an ex. of a conservative vector field

$L.I. = \int_C \vec{F} \cdot d\vec{R}$ = the work done

(C ↓ curve; $\vec{F}$ ↳ force)



13.3 Scalar Potential

Def. in 2D:

$\vec{F}(x,y)$ is conservative on D if
$\exists (f(x,y))$ such that $\vec{F}(x,y) = \nabla f(x,y)$ $\forall (x,y) \in D$

(↑ $\exists$: there exists; $\forall$: for all x,y; $\in$: in)

In this case, $f(x,y)$ is called the <u>scalar potential</u> of $\vec{F}(x,y)$

Ex. 1. $\vec{F} = \langle xy^2, x^2y \rangle$ is conservative because

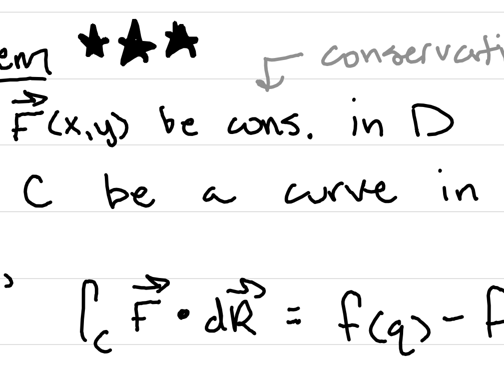
$f(x,y) = \frac{x^2y^2}{2}$ 2. $\vec{F} \stackrel{?}{=} \nabla f$

3. $\nabla f = \langle \partial_x, \partial_y \rangle \frac{x^2y^2}{2}$ = 4. $\langle xy^2, x^2y \rangle$ □

Theorem ★★★

Let $\vec{F}(x,y)$ be cons. in D (cons. = conservative)
Let C be a curve in D
Then,

$$\int_C \vec{F} \cdot d\vec{R} = f(q) - f(p)$$



this is fundamental for Line Ints.

Proving that theorem

$$\int_C \vec{F} \cdot d\vec{R} = \int_{t_0}^{t_1} \vec{F}(\vec{R}(t)) \cdot \vec{R}' \, dt$$

since the V.F. is cons.:

$$= \int_{t_0}^{t_1} \nabla f(\vec{R}(t)) \cdot \langle x', y' \rangle (t) \, dt$$

($\nabla f \to \langle f_x, f_y \rangle (\vec{R}(t))$)

$$= \int_{t_0}^{t_1} (f_x x' + f_y y')(\vec{R}(t)) dt$$

($f_x x' + f_y y'$: What is this? It's)

$$= \int_{t_0}^{t_1} \frac{df}{dt}(\vec{R}(t)) dt$$

$$= f(\vec{R}(t_1)) - f(\vec{R}(t_0)) = f(Q) - f(P)$$ □ proven

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Show the theorem w/ earlier example

$(0,0) \to (2,4)$ ★ ($f = \frac{x^2y^2}{2}$)

$= f(2,4) - f(0,0) = \frac{2^2(4^2)}{2} - 0 = \boxed{32}$

dope. pretty cool. pretty neat.

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1. Recognize if $\vec{F}$ is conservative
2. If conservative, find f

Test for Cons. ★★★

we know if $\vec{F}$ is cons., $\vec{F} = \nabla f$

$\langle f_1, f_2 \rangle = \langle f_x, f_y \rangle$

$(f_1 = f_x)_y$ $(f_2 = f_y)_x$

$f_{1y} = f_{xy}$ $f_{2x} = f_{yx}$

(the same)

this implies $\boxed{f_{1y} = f_{2x}}$

ex. $\vec{F} = \langle xy^2, x^2y \rangle$

$f_{1y} = 2xy$ $f_{2x} = 2xy$

$f_{1y} = f_{2x}$ $\boxed{\vec{F} \text{ is conservative}}$

Finding little f ★★

we know $\nabla f = \vec{F}$

$f_x = f_1$ $f_y = f_2$

$\int f_x dx = \int f_1 dx + g(y) = f$
$\int f_y dy = \int f_2 dy + h(x) = f$ (should be both)

$\int f_1 dx = \ell(x,y) + h(x)$
$\int f_2 dy = \ell(x,y) + g(y)$

we want this + that, avoiding this again

$f = \ell(x,y) + h(x) + g(y) + C$ ($C \to 0$)

ex. $\vec{F} = \langle xy^2, x^2y \rangle$

$f = \int xy^2 dx + \ldots = \frac{x^2y^2}{2} + \ldots$ ($\ldots$: $g(y)$)

$f = \int x^2y \, dy + \ldots = \frac{x^2y^2}{2} + \ldots$ ($\ldots$: $h(x)$)

$\Downarrow$

$f = \boxed{\frac{x^2y^2}{2} + c}$ → no $g(y)$ or $h(x)$ because there is nothing that depends only on x or only on y

---

Examples

#1] Test for cons. If it's cons., find scalar potential

$\vec{F} = \langle e^x\sin(y) - y, e^x\cos(y) - x - 2 \rangle$

① Is $\vec{F}$ conservative? ($f_{1y} \stackrel{?}{=} f_{2x}$)

$f_{1y} = e^x\cos(y) - 1$ $f_{2x} = e^x\cos(y) - 1 - 0$

✓✓ $f_{1y} = f_{2x}$ $\boxed{\vec{F} \text{ is cons.}}$

② get the scalar potential

$f = \int e^x\sin(y) - y \, dx = e^x\sin(y) - xy + \ldots$ ($e^x\sin(y) - xy$: $\ell(x,y)$; $h(x) = 0$)

$f = \int e^x\cos(y) - x - 2 \, dy = e^x\sin(y) - xy - 2y + \ldots$ ($e^x\sin(y) - xy$: $\ell(x,y)$; $-2y$: $g(y)$)

$f = \boxed{e^x\sin(y) - xy - 2y} + C$ ($C \to 0$)

$\stackrel{?}{=}$ means "test"

to check:
if $\nabla f = \vec{F}$, you have the right one

13.1 — 13.3 Review (NOT comprehensive of those sections.)

$\int_C \vec{F} \cdot d\vec{R} = \int_C \vec{F} \cdot \vec{T} ds = \int_{t_0}^{t_1} \vec{F}(\vec{R}(t)) \cdot \vec{R}'(t) dt$

C: $\vec{R}(t)$ $t_0 \le t \le t_1$

$\vec{F}$ is conservative on D if $\vec{F} = \nabla f$ — f is called scalar potential of $\vec{F}$

if $\vec{F}$ is conservative, $\boxed{\int_C \vec{F} \cdot d\vec{R} = f(Q) - f(P)}$ ★

(C from P to Q)

(2D) $\vec{F} = \langle f_1, f_2 \rangle \rightarrow \vec{F}$ is conservative if:

$f_{1y} = f_{2x}$ ★ ← this is the test

$f = \int f_1(xy)dx + g(y)$
$f = \int f_2(xy)dy + h(x)$

f is given by merging the 2 integrals without repetitions

(3D) $\vec{F} = \langle f_1, f_2, f_3 \rangle$

$\vec{F}$ is conservative if:
$\vec{F} = \langle f_1, f_2, f_3 \rangle = \langle f_x, f_y, f_z \rangle$

Test:

$f_{1y} = f_{xy}$ $f_{1z} = f_{xz}$

$f_{2x} = f_{yx}$ $f_{2z} = f_{yz}$

$f_{3x} = f_{zx}$ $f_{3y} = f_{zy}$

simplified:

$f_{1y} = f_{2x}$

$f_{1z} = f_{3x}$

$f_{2z} = f_{3y}$

simplified again:

$\boxed{\nabla \times \vec{F} \overset{?}{=} \vec{0}}$ $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x & \partial y & \partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$ ★

$= \boxed{\hat{i}(f_{3y} - f_{2z}) - \hat{j}(f_{3x} - f_{1z}) + \hat{k}(f_{2x} - f_{1y}) = \vec{0}}$

If this is true, $\vec{F}$ is conservative

Test:

$f = \int f_1 dx + g(y,z)$
$f = \int f_2 dy + h(x,z)$
$f = \int f_3 dz + \ell(x,y)$

Examples

#1] $\vec{F} = \langle 20x^3z + 2y^2, 4xy, 5x^4 + 3z^2 \rangle$

Test $\nabla \times \vec{F} = \vec{0}$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x & \partial y & \partial z \\ 20x^3z + 2y^2 & 4xy & 5x^4 + 3z^2 \end{vmatrix}$

$= \hat{i}(0-0) - \hat{j}(20x^3 - 20x^3) + \hat{k}(4y - 4y)$

$= \boxed{\vec{0}}$ it is conservative

Now find scalar potential

$f = \int(20x^3z + 2y^2)dx + \ldots = \frac{20x^4}{4}z + 2y^2x + \ldots$
$f = \int(4xy)dy + \ldots = 4x\frac{y^2}{2} + \ldots$
$f = \int(5x^4 + 3z^2)dz + \ldots = 5x^4z + \frac{3z^3}{3} + \ldots$

the same
the same

just add at the end

Scal. Pot. = $\boxed{f = 5x^4z + 2y^2x + z^3 + C}$

#2] For $\vec{F}$ from #1, find $\int_C \vec{F} \cdot d\vec{R}$
where $\vec{R} = \langle 3\cos(t), 3\sin(t), 4t \rangle$ $0 \le t \le 2\pi$
$\oint_C \vec{F} \cdot d\vec{R}$ on C: $\{ x^2+y^2=4, z=1 \}$ is the same

$\vec{F} = \langle 20x^3z + 2y^2, 4xy, 5x^4 + 3z^2 \rangle$

since $\vec{F}$ is conservative:
$\int_C \vec{F} \cdot d\vec{R} = f(Q) - f(P)$
$f = 5x^4z + 2y^2x + z^3 + 0$ — we choose c

1. Find endpoints p and q from $\vec{R}(t)$

$P = \vec{R}(0) = (3, 0, 0)$ — $(3\cos(0), 3\sin(0), 4(0))$
$Q = \vec{R}(2\pi) = (3, 0, 8\pi)$ — ...

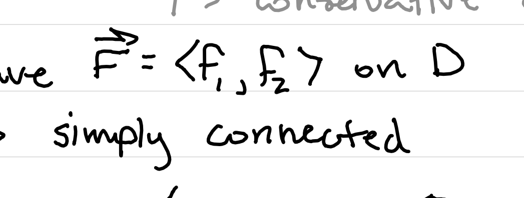
Line Integral (LI) =

$\left[5(3)^4 8\pi + 0 + (8\pi)^3\right] - 0$
f(Q) f(P)

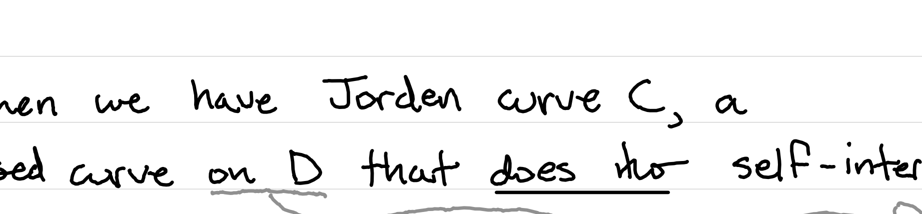
Note: If $P \equiv Q$, $f(Q) - f(P) = 0$
(duh) but important

13.4 Green's Theorem (2D)

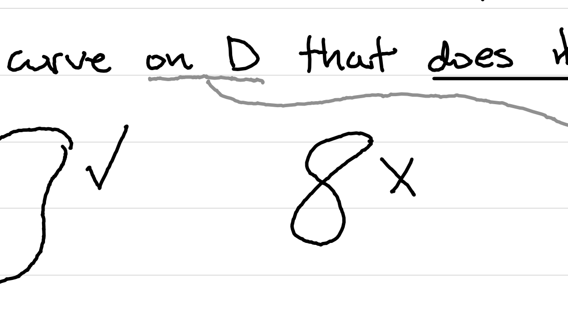
Say we have $\vec{F} = \langle f_1, f_2 \rangle$ on D (→ conservative or not)
when D is simply connected



Then we have Jorden curve C, a closed curve on D that does not self-intersect



Positively-oriented vs. Negatively oriented



Green's Theorem:

↦ L.I. of positive Jorden curve (arrow direction matters)

$$\oint_C \vec{F} \cdot d\vec{R} = \iint_{D_1} (f_{2_x} - f_{1_y})\, dA$$

↳ area inside C

think of a vector field $\vec{F}$ that can decompose to conservative and non-cons. parts

$$\vec{F} = \vec{F}_C + \vec{F}_{NC}$$

$\vec{F}_C$ would immediately go to $\vec{0}$

Green's Theorem is very powerful, because we can now relate a $\oint_C$ to a $\iint$

Strategy for L.I. on a curve C

NC → non-cons.

★★★

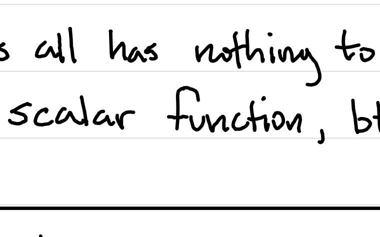
$\int_C \vec{F} \cdot d\vec{R}$
- C is open
  - $\vec{F}$ is NC → $\int_{t_0}^{t_1} \vec{F}(\vec{R}(t)) \cdot \vec{R}'(t)\, dt$ 13.2
  - $\vec{F}$ is cons. → Find f → $f(Q) - f(P)$ 13.3
- C is closed → $\iint_{D_1} f_{2_x} - f_{1_y}\, dA$ 13.4
  - ↓ from now on, this region inside C will be written D, not D_1

this all has nothing to do w/ the L.I. of a scalar function, btw. ($\int_C f\, ds$)

Examples

#1] $\int_C -y\,dx + x\,dy$ on

$= \int_C \vec{F} \cdot d\vec{R}$ $\vec{F} = \langle -y, x \rangle$



0\.
Since we can see it is a pos. (+) Jorden curve,

$= \oint_C -y\,dx + x\,d$

1\. Solve w/ Green's Theorem

$= \iint_D 1 - (-1)\, dA = 2\iint 1\, dA = 2\left(\frac{\pi(1)^2}{2}\right) = \boxed{\pi}$

($f_{2_x} - f_{1_y}$)

Bonus: Doing it the other way:

$\vec{R}_1(t) = \langle \cos(t), \sin(t) \rangle \quad 0 \le t \le \pi$
$\vec{R}_2(t) = \langle t, 0 \rangle \quad -1 \le t \le 1$

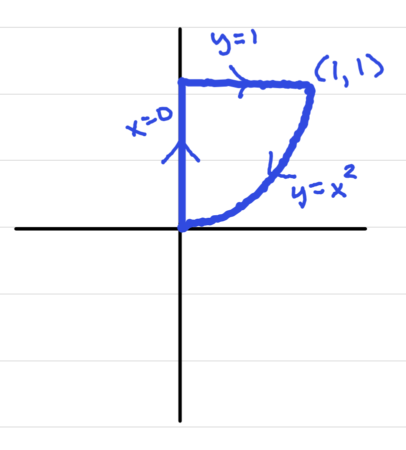
$\vec{R}_1'(t) = \langle -\sin(t), \cos(t) \rangle$
$\vec{R}_2'(t) = \langle 1, 0 \rangle$

$\int_{C_1} \vec{F} \cdot d\vec{R} = \int_0^{\pi} \langle -\sin(t), \cos(t) \rangle \langle -\sin(t), \cos(t) \rangle\, dt = \pi$

$\int_{C_2} \cdots = 0$

↦ $W = \int_C \vec{F} \cdot d\vec{R}$

#2] Find work done by
$\vec{F} = \langle x + xy^2, 2(x^2y - y^2\sin(y)) \rangle$ where
c = neg. Jorden curve described by
$y = 1$ and $y = x^2$



★★ pos. Jorden's = $F_{2_x} - F_{1_y}$
neg. Jorden's = $F_{1_y} - F_{2_x}$

1\.
$W = \oint_C \vec{F} \cdot d\vec{R}$

$= \iint_D 2xy - 4xy\, dA$ (F_{1_y}, F_{2_x})
$-2xy\, dA$

observe graph (like ch. 12)

$= \int_0^1 \int_{x^2}^1 -2xy\, dy\, dx$

$= \int_0^1 -2x \frac{y^2}{2}\Big|_{x^2}^1 dx = \int_0^1 -x(1 - x^4)\, dx$

$= \int_0^1 x^5 - x\, dx = \frac{x^6}{6} - \frac{x^2}{2}\Big|_0^1 = \frac{1}{6} - \frac{1}{2} = \boxed{-\frac{1}{3}}$

#3] (Reverse) Assume you want to find the area in C, $\iint_D 1\, dA$ and you know the $f_{2_x} - f_{1_y}$ that this needs is some

$\oint_C \langle \ , \ \rangle \cdot d\vec{R}$

This can be represented by many different combos:

$\oint_C \frac{1}{2}\langle -y, x \rangle \cdots$

$\oint_C \langle 0, x \rangle \cdots$

$\oint_C \langle -y, 0 \rangle \cdots$

Find the area for the ellipse of semi-axis a, b

$\vec{R}(t) = \langle a\cos(t), b\sin(t) \rangle \quad 0 \le t \le 2\pi$

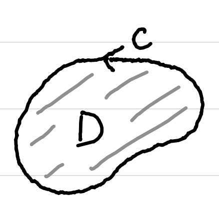
$\vec{R}'(t) = \langle -a\sin(t), b\cos(t) \rangle$

$L.I. = \int_0^{2\pi} \frac{1}{2}\langle -b\sin(t), a\cos(t) \rangle \cdot \langle -a\sin(t), b\cos(t) \rangle\, d$

$= \int_0^{2\pi} \frac{1}{2}(ab\sin^2(t) + ab\cos^2(t))\, dt$

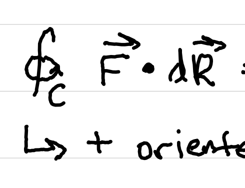
$= \frac{1}{2} ab\, 2\pi = \boxed{\pi ab}$

11-18-24 Summary



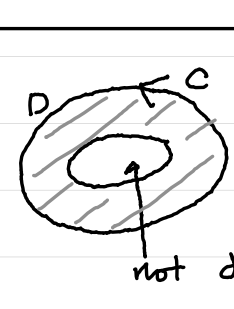
$\vec{F} = \langle f_1, f_2 \rangle$

$\oint_C \vec{F} \cdot d\vec{R} = \iint_D f_{2_x} - f_{1_y}\, dA$
↳ + oriented



$\vec{F} = \langle f_1, f_2 \rangle$ in D

not defined because hole, however



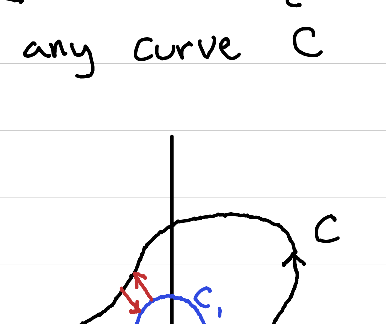
$\oint \vec{F} \cdot d\vec{R} = \iint_{D_2} f_{2_x} - f_{1_y}\, dA$
($C + C_2 + C_1 - C_2$)

$= \oint_C \vec{F} \cdot d\vec{R} + \oint_{C_1} \vec{F} \cdot d\vec{R}$

$\oint_C \vec{F} \cdot d\vec{R} = \iint_{D_2} f_{2_x} - f_{1_y}\, dA + \oint_{-C_1} \vec{F} \cdot d\vec{R}$

Examples

#1] Show that $\oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2} = 2\pi$
for any curve C



$= \iint_D f_{2_x} - f_{1_y}\, dA + \oint_{-C_1} \vec{F} \cdot d\vec{R}$

1\. See if the vector field is conservative

$\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

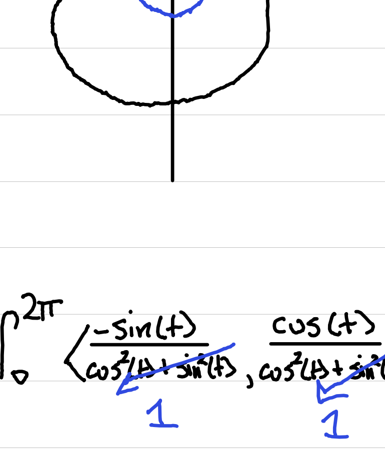
$f_{2_x} = \frac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$

$f_{1_y} = \frac{-1(x^2+y^2) + y(2y)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$

conservative because these match

2\.
no more $\iint_D$ because

$\oint_C \vec{F} \cdot d\vec{R} = \oint_{-C_1} \vec{F} \cdot d\vec{R}$



$-C_1$: $\vec{R} = \langle \cos(t), \sin(t) \rangle$
$\vec{R}' = \langle -\sin(t), \cos(t) \rangle$
$0 \le t \le 2\pi$

3\.
$= \int_0^{2\pi} \left\langle \frac{-\sin(t)}{\cos^2(t) + \sin^2(t)}, \frac{\cos(t)}{\cos^2(t) + \sin^2(t)} \right\rangle \cdot \langle -\sin(t), \cos(t) \rangle\, dt$ (denominators = 1)

$= \int_0^{2\pi} 1\, dt = \boxed{2\pi}$ □

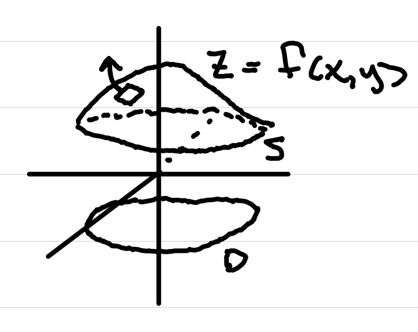
this is an advanced example

13.5 Surface Integral

$SI = \iint_S g(x,y,z)\, dS$

much like surface area, which is $\iint_S 1\, dS$

$dS = \sqrt{1 + z_x^2 + z_y^2}\, dA$



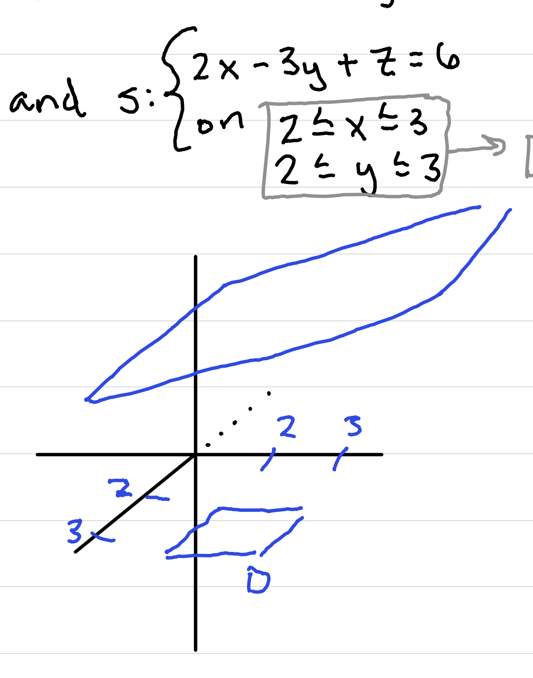
$g(x, y, f(x,y))$

$S.I. = \boxed{\iint_D g(x,y,f(x,y)) \sqrt{1+z_x^2+z_y^2}\, dA}$ ★★

Examples

#1] Integrate $\iint_S g\, ds$ where
$g = xz + 2x^2 - 3xy$

and $S: \begin{cases} 2x - 3y + z = 6 \\ \text{on } \boxed{\begin{matrix} 2 \le x \le 3 \\ 2 \le y \le 3 \end{matrix}} \to D \end{cases}$



1. Find Jacobian
$z = f(x,y) = 6 - 2x + 3y$
$z_x = -2$
$z_y = 3$ $\quad \sqrt{1+4+9} = \sqrt{14}$

2. Substitute z

$= \iint_D \Big[\overbrace{x(6-2x+3y) + 2x^2 - 3y}^{g(x,y,f(x,y))}\Big] \overbrace{\sqrt{14}}^{\sqrt{1+z_x^2+z_y^2}}\, dA$

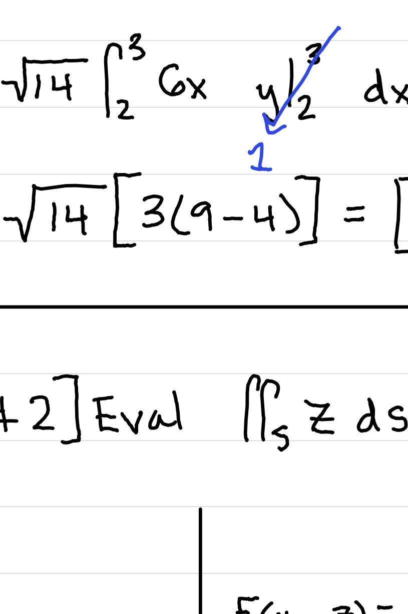
...

$= \sqrt{14} \int_2^3 \int_2^3 6x\, dy\, dx$

$= \sqrt{14} \int_2^3 6x\, \underbrace{y\Big|_2^3}_{1}\, dx$

$= \sqrt{14}\,[3(9-4)] = \boxed{15\sqrt{14}}$

#2] Eval $\iint_S z\, ds$ where $S: z = \sqrt{1-x^2-y^2}$



$F(x,y,z) = x^2+y^2+z^2 = 1 \to$ the whole sphere

$z_x = \frac{-F_x}{F_z} = \frac{-2x}{2z} = -\frac{x}{z}$

$z_y = \dots = -\frac{y}{z}$

$\sqrt{1+z_x^2+z_y^2} = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}} = \sqrt{\frac{x^2+y^2+z^2}{z^2}}$

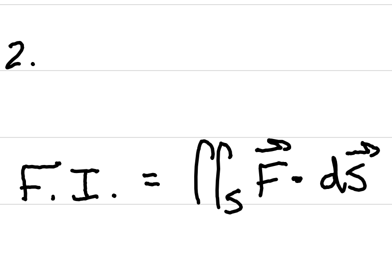
$= \frac{1}{z}$

$SI = \iint_D \cancel{z\left(\tfrac{1}{z}\right)} dA = \iint_D 1\, dA$

this is the area of the unit disk, so
$= \pi(1)^2 = \boxed{\pi}$

Flux Integrals

$\iint_S \vec{F} \cdot d\vec{S}$ = Integral of a vector field on a surface



the surface has to be orientable (you can be on one side or the other)

$= \iint_S \underbrace{\vec{F} \cdot \vec{N}}_{g(x,y,z)}\, dS$ $\quad$ vector field projected in normal direction

1. Get the upward normal $\quad \hat{N} = \frac{\nabla H}{\|\nabla H\|} = \frac{\langle -F_x, -F_y, 1\rangle}{\sqrt{1+F_x^2+F_y^2}}$

downward $= \langle F_x, F_y, -1\rangle$ ...

2.

$F.I. = \iint_S \vec{F} \cdot d\vec{S}$

$= \iint_D \vec{F}(x,y,f(x,y)) \cdot \frac{\langle -F_x, -F_y, 1\rangle)}{\cancel{\sqrt{1+F_x^2+F_y^2}}} \left(\cancel{\sqrt{1+F_x^2+F_y^2}}\right) dA$

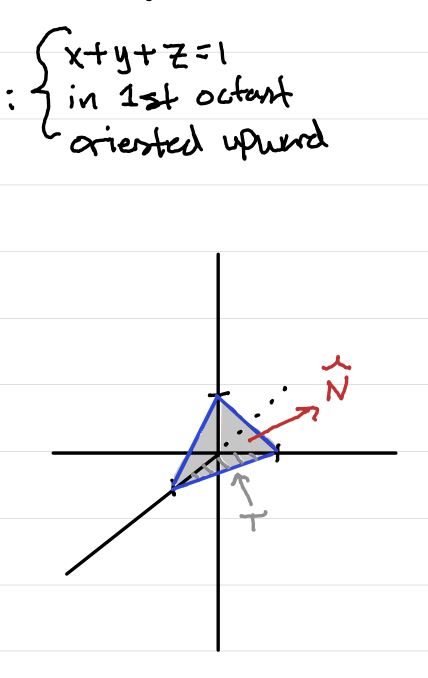
$= \iint_D \vec{F}(x,y,f(x,y)) \cdot \underbrace{\langle -F_x, -F_y, 1\rangle}_{(\vec{N})} dA$ $\quad$ this is where we get orientation

Reminder: scalar projection $\vec{a}$ into $\vec{b} = \vec{a} \cdot \frac{\vec{b}}{\|\vec{b}\|} = \vec{a} \cdot \hat{b}$ $\quad$ projection of a in direction of $\vec{b}$

Examples

#1] $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle xy, z, x+y\rangle$ and

$S: \begin{cases} x+y+z=1 \\ \text{in 1st octant} \\ \text{oriented upward} \end{cases}$



0.
$z = 1-x-y$
$\vec{N} = \langle \overset{-F_x}{1}, \overset{-F_y}{1}, \overset{\text{upward}}{1} \rangle$

1.
$= \iint_T \langle xy, 1-x-y, x+y\rangle \cdot \langle 1,1,1\rangle\, dA$ $\quad$ Replace z

$= \iint_T 1 + xy\, dA$ $\quad$ (separate by linearity rule)

$= \frac{1}{2} + \int_0^1 \int_0^{1-x} xy\, dy\, dx$

$= \frac{1}{2} + \int_0^1 x\, \frac{y^2}{2}\Big|_0^{1-x} dx$

$= \frac{1}{2} + \frac{1}{2}\int_0^1 x(1-2x+x^2)\, dx$

$= \frac{1}{2} + \frac{1}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]\Big|_0^1$

$= \boxed{\frac{1}{2} + \frac{1}{2}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right]} \dots = \boxed{\frac{13}{24}}$

11-20-2024 Review

Line.I. $= \iint_S g(x,y,z)\, dS$ $\quad S: z = f(x,y)$ on D

$= \iint_D g(x,y,f(x,y)) \overbrace{\sqrt{1+f_x^2+f_y^2}}^{dS}\, dA$

$= \iint_D h(x,y)\, dA$

Flux.I. $= \iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{N}\, dS$ $\quad$ normal of orientable surface S

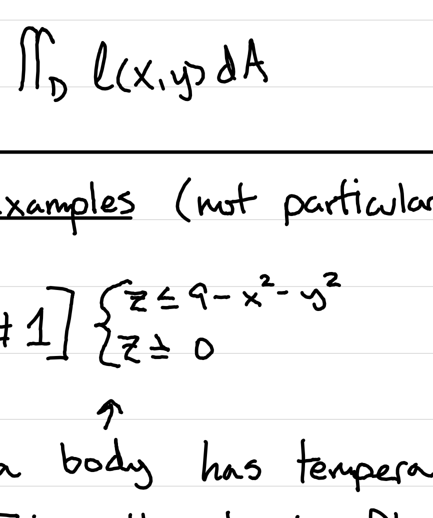
$= \iint_D \vec{F}(x,y,f(x,y)) \cdot \underline{\langle -f_x, -f_y, 1\rangle}\, dA$ $\quad$ If upward

$= \iint_D \ell(x,y)\, dA$

Examples (not particularly for 13.5)

#1] $\begin{cases} z \le 9 - x^2 - y^2 \\ z \ge 0 \end{cases}$

a body has temperature $T(x,y,z) = 2x + y - 3z^2$
Find the heat flux coming out of the surface $\quad$ upward normal



Heat Flux $= \iint_S \overbrace{-K\nabla T}^{\vec{q}} \cdot d\vec{S}$

0.
$\vec{q} = -K\nabla(2x+y-3z^2)$
$= -K\langle 2, 1, -6z\rangle = K\langle -2,-1,6z\rangle$

1.
$= \iint_{S_1} K\langle -2,-1,6z\rangle \cdot d\vec{S} + \iint_{S_2} K\langle -2,-1,6z\rangle \cdot d\vec{S}$

$S_2 \to -\hat{k} = \langle 0,0,-1\rangle$

$S_1 \to H = z - 9 + x^2 + y^2$
$\vec{N} = \langle 2x, 2y, 1\rangle$

2.
$\iint_D K\langle -2,-1, 6(9-x^2-y^2)\rangle \cdot \langle 2x, 2y, 1\rangle\, dA$ $\quad$ z =

$+ \underbrace{\iint_D K\langle -2,-1,0\rangle \cdot \langle 0,0,-1\rangle\, dA}_{=0}$ $\quad$ z =

$= \iint_D K\langle -2,-1,6(9-x^2-y^2)\rangle \cdot \langle 2x, 2y, 1\rangle\, dA$

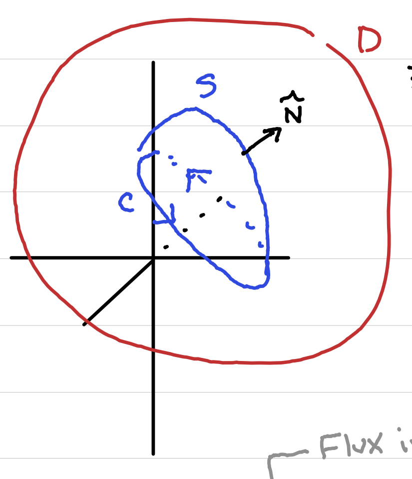
3. $D: \begin{cases} z=0 \\ z = 9-x^2-y^2 \end{cases} \to \underline{x^2+y^2=9}$

4.
$H.F. = K\iint_D \underset{0}{-4x} - \underset{0}{2y} + 6(9-x^2-y^2)\, dA$
(ask prof., IDK)

5.?
$= 6K\left(81\pi - \int_0^{2\pi}\int_0^3 r^3\, dr\, d\theta\right.$ $\quad$ (moved to polar coords)
...

$= 6K\left(81\pi - \frac{81\pi}{2}\right) = \dots = \boxed{3(81)\pi K}$

13.6 Stoke's Theorem (Green's in 3D)



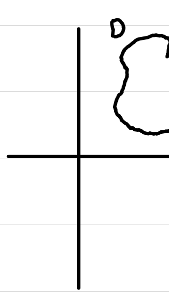
$\vec{F}(x,y,z)$

* under 2D restrictions, Stoke's theorem is Green's theorem

Flux integral ↓ ; ← curl of $\vec{F}$

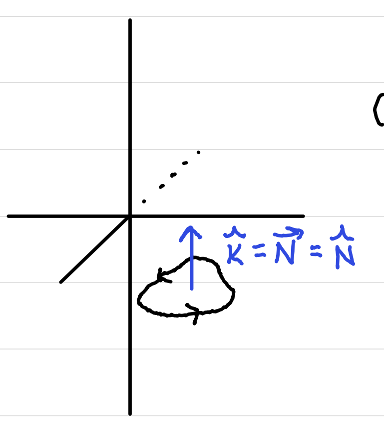
$$\oint_C \vec{F}\cdot d\vec{R} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S}$$

Proving this



$\vec{F}(x,y) = \langle F_1, F_2\rangle$

$$\oint_C \vec{F}\cdot d\vec{R} = \iint_D (F_{2_x} - F_{1_y})\,dA$$



(in 3D)

$$\iint_S (\nabla\times\vec{F})\cdot d\vec{S}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial x & \partial y & \partial z \\ F_1(x,y) & F_2(x,y) & 0 \end{vmatrix} = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(F_{2_x} - F_{1_y})$$

$$\iint_S (\nabla\times\vec{F})\cdot d\vec{S} = \iint_D \langle 0, 0, F_{2_x} - F_{1_y}\rangle \cdot \langle 0,0,1\rangle\,dA$$

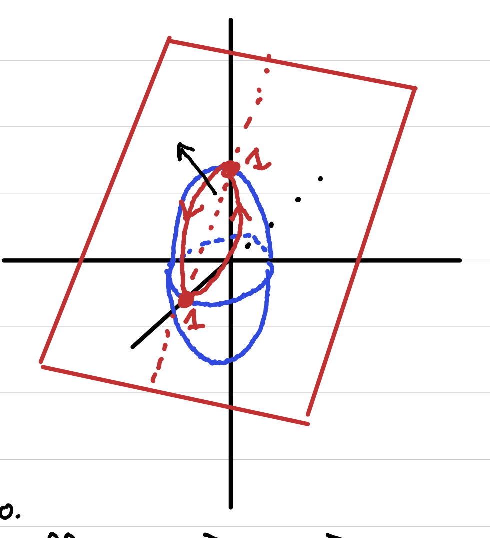
$$= \boxed{\iint_D F_{2_x} - F_{1_y}\,dA}$$

☐ proved

Examples

#1] Eval. $\oint_C \frac{1}{2}y^2dx + z\,dy + x\,dz$ on

$C: \begin{cases} x+z=1 \\ x^2+2y^2+z^2=1 \end{cases}$ (counterclockwise from above)



I need to start drawing bigger (a little late, I know)

0.
$$= \iint_S (\nabla\times\vec{F})\cdot d\vec{S}$$

1. curl →
$$\nabla\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial x & \partial y & \partial z \\ \frac{1}{2}y^2 & z & x \end{vmatrix} = \hat{\imath}(0-1) - \hat{\jmath}(1-0) + \hat{k}(0-y) = \langle -1,-1,-y\rangle$$

$$= \iint_S \langle -1,-1,-y\rangle\cdot d\vec{S}$$

2. $\vec{N}$
$z = 1-x = f(x,y)$

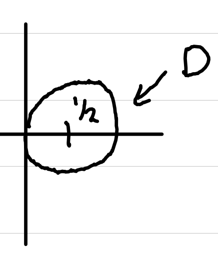
$$\vec{N} = \langle -f_x, -f_y, 1\rangle = \langle 1, 0, 1\rangle$$

$$= \iint_S \langle -1,-1,-y\rangle \cdot \langle 1,0,1\rangle$$

3.
$D: \begin{cases} z = 1-x \\ x^2 + 2y^2 + (1-x)^2 = 1 \end{cases}$, $(1-x)^2 \to 1-2x+x^2$

$$\Rightarrow 2x^2 - 2x + 2y^2 = 0 \Rightarrow x^2 - x + y^2 = 0$$

complete the square:
$$(x^2 - x + \tfrac{1}{4}) + y^2 = \tfrac{1}{4}$$
$$(x-\tfrac{1}{2})^2 + y^2 = (\tfrac{1}{2})^2$$



4.
$$\oint_C \vec{F}\cdot d\vec{R} = \iint_D \langle -1,-1,-y\rangle\cdot\langle 1,0,1\rangle\,dA$$

$$= \iint_D -1-y\,dA \to \iint_D -1\,dA + \iint_D -y\,dA$$

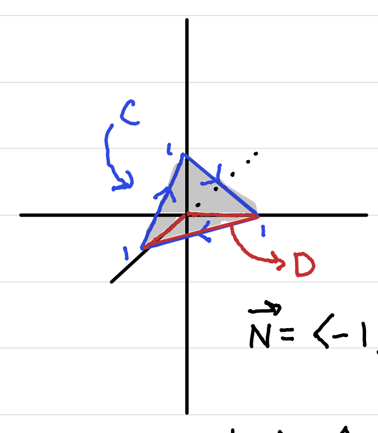
$\iint_D -y\,dA \to 0$ because odd function on symmetric domain

$$= \boxed{-\pi(\tfrac{1}{2})^2}$$

#2]
$C: \begin{cases} x+y+z=1 \\ x=0 \end{cases} + \begin{cases} \\ y=0 \end{cases} + \begin{cases} \\ z=0 \end{cases}$ clockwise from above

Eval $\oint_C \vec{F}\cdot d\vec{S}$ where $\vec{F} = \langle -\frac{3}{2}y^2, -2xy, yz\rangle$

$$= \iint_S (\nabla\times\vec{F})\cdot d\vec{S}$$



$\vec{N} = \langle -1,-1,-1\rangle$

1.
$$\nabla\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial x & \partial y & \partial z \\ -\frac{3}{2}y^2 & -2xy & yz \end{vmatrix} = \hat{\imath}(z+0) - \hat{\jmath}(0-0) + \hat{k}(-2y+3y) = \langle z, 0, y\rangle$$ nks, scoob.

2.
$$\oint_C \vec{F}\cdot d\vec{R} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S} = \iint_D \langle 1-x-y, 0, y\rangle\cdot\langle -1,-1,-1\rangle$$
($z = 1-x-y$)

$$= \iint_D -1+x+y-y = \iint_D x-1\,dA$$

$$= -\iint_D 1\,dA + \iint_D x\,dA$$

$$= -\tfrac{1}{2} + \int_0^1\int_0^{1-x} x\,dA$$

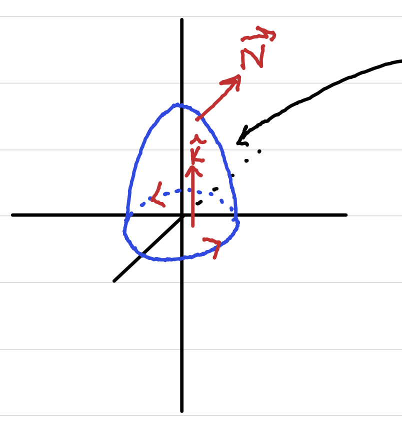
$$= -\tfrac{1}{2} + \int_0^1 x - x^2\,dx$$

$$= -\tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{3}\,dx = \boxed{-\tfrac{1}{3}}$$

#3] $\iint_S (\nabla\times\vec{F})\cdot d\vec{S}$ where (Flux int.)

$S: \begin{cases} z = 1-x^2-2y^2 \\ z \geq 0 \end{cases}$ oriented upward

$\vec{F} = \langle x, y^2, ze^{xy}\rangle$



instead of solving on this we will solve on $z=0$

$$= \oint_C \vec{F}\cdot d\vec{R} = \iint_{S_1} (\nabla\times\vec{F})\cdot d\vec{S}$$
$S_1 = D$

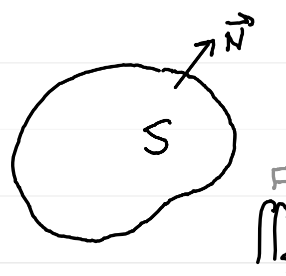
$$= \iint_D (\nabla\times\vec{F})\big|_{z=0}\cdot\langle 0,0,1\rangle\,dA$$

this means we just need $F_{2_x} - F_{1_y}$

$$= \iint_D (0-0)\,dA = \boxed{0}$$
(F_{2_x}, F_{1_y})

we got this by applying stoke's theorem twice

13.7 Divergence Theorem



when S oriented outward

$\vec{F} = \langle f_1, f_2, f_3 \rangle$ on D (if inward, just change a −)

Flux Int.

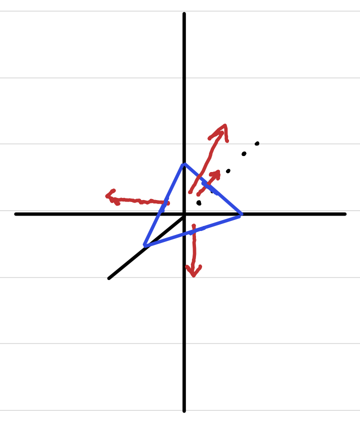
$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V \operatorname{div} \vec{F} \, dV$$

($\operatorname{div} \vec{F} \to \nabla \cdot \vec{F}$)

Examples

#1] $\iint_S \vec{F} \cdot d\vec{S}$ where S is the surface of the tetrahedron bounded by $x=0$, $y=0$, $z=0$ and $x+y+z=1$ oriented outward.

$\vec{F} = \langle x^2, xy, x^3y^3 \rangle$

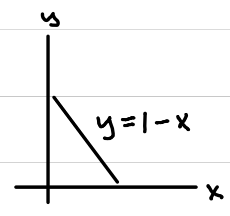


0. Use Div. Theorem

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V \nabla \cdot \vec{F} \, dV$$

$$\nabla \cdot \vec{F} = \partial_x(x^2) + \partial_y(xy) + \partial_z(x^3y^3)$$

$$= 2x + x + 0 = \boxed{3x}$$



$= \iiint_V 3x \, dV$

1. Solve (z-simple)

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 3x \, dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} 3x(1-x-y) \, dy\,dx$$

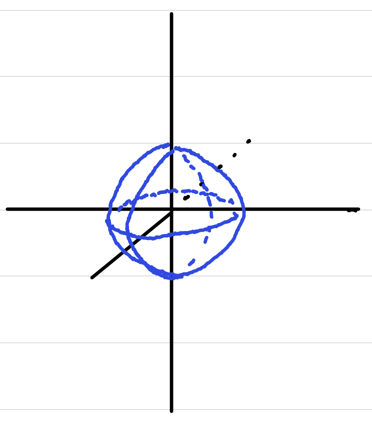
$$= \int_0^1 3x(1-x)^2 - \frac{3x}{2}(1-x)^2 \, dx$$

$$= \int_0^1 \frac{3}{2}(x - 2x^2 + x^3) \, dx$$

...

$$= \frac{3}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{3}{2}\left(\frac{6-8+3}{12}\right) = \dots = \boxed{\frac{1}{8}}$$

#2] $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle xy^2, yz^2, zx^2 \rangle$ and $S = x^2+y^2+z^2 = 4$ oriented outward



0.

$\operatorname{div} \vec{F} = y^2 + z^2 + x^2$

1.

$\iiint_V y^2 + z^2 + x^2 \, dx$ (spherical coords)

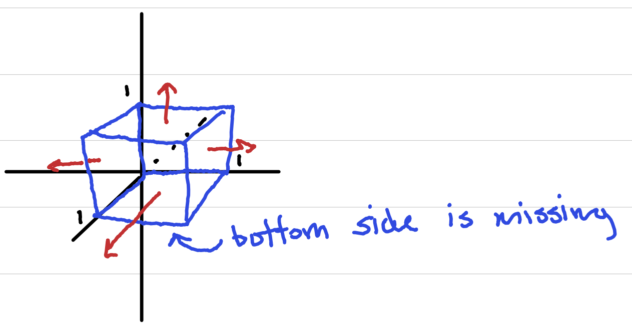
$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{2} \rho^2 \, \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$= \boxed{2\pi(2)\left(\frac{2^5}{5}\right)}$ he solved it in like a second

---

#3] $\vec{F} = \langle xy, 0, -z^2 \rangle$

S:



instead of solving 5 F.Is, we will add and subtract the F.I. for the bottom side, S_6

0.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S+S_6} \vec{F} \cdot d\vec{S} - \iint_{S_6} \vec{F} \cdot d\vec{S}$$

$$= \iiint_V \operatorname{div} \vec{F} \, dV - \iint_{S_6} \vec{F} \cdot d\vec{S}$$

$\operatorname{div} \vec{F} = y - 2z$

1.

$\vec{N}$ of $S_6 = \langle 0, 0, -1 \rangle$ ($z=0$ at S_6)

$$\iint_{S_6} \langle xy, 0, 0 \rangle \cdot \langle 0, 0, -1 \rangle = 0$$

because of symmetric space, $\iiint_V y - 2z \, dV = -\iiint x \, dV \dots$

$$-\int_0^1 \int_0^1 \int_0^1 x \, dz\,dy\,dx = \dots = \boxed{\frac{1}{2}}$$

Splendid.