12.1

Double Integral $f(x,y)$ on R

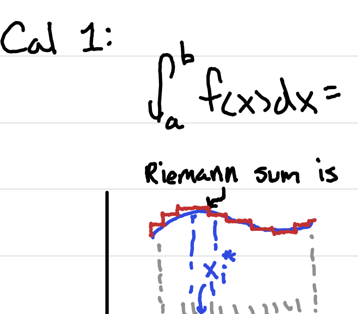
$R = \begin{cases} a \le x \le b \\ c \le y \le d \end{cases}$ ← Rectangle

$F'(x) = f(x)$

Cal 1:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \ (= F(b) - F(a))$$

Riemann sum is sum of all these rectangles (approximation)



$\Delta x = \frac{b-a}{n}$

Now Cal 3: (with the Rectangle)

double integral on R

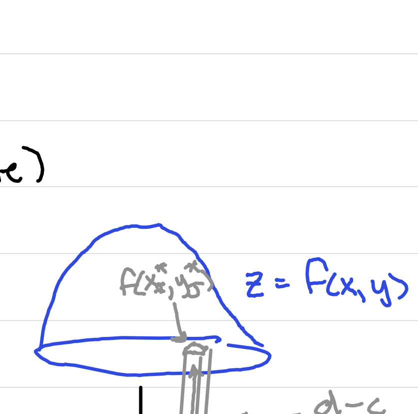
$$\iint_R f(x,y)\,dA$$

Riemann Sum

$$= \lim_{n\to\infty} R_n$$

double sum

$$= \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_I^*, y_J^*)\Delta A$$



3 Properties ★

Linearity Rules:

$$\iint_R a f(x,y) + b g(x,y)\,dA$$

$$= a\iint_R f(x,y)\,dA + b\iint_R g(x,y)\,dA$$

#7:

If $f(x,y) \ge g(x,y)$ for all (x,y) in R:

$$\iint_R f(x,y)\,dA \ge \iint_R g(x,y)\,dA$$

Subdivision Rule:

$$\iint_R f(x,y)\,dA = \iint_{R_1+R_2} f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

You can do any subdivision of a surface

Kinda like doing this:

a b c $\int_a^c = \int_a^b + \int_b^c$

Back 2 Cal 1

if $f(x) \ge 0 \ \forall x$ in (a,b), then:

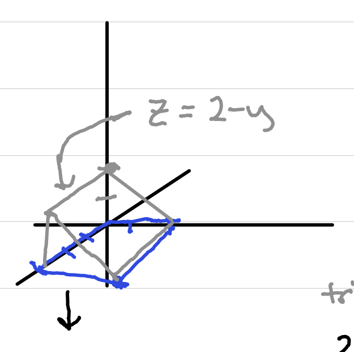
$\int_a^b f(x)dx =$ Area

Back 2 Cal 3

if $f(x,y) \ge 0 \ \forall (x,y)$ in R, then:

$\iint_R f(x,y)\,dA =$ Volume ★

Ex. #1 – Find $\iint_R 2-y\,dA$ where $R = \begin{cases} 0 \le x \le 3 \\ 0 \le y \le 2 \end{cases}$



so the $\iint_R$ is the volume because it's all ≥ 0

$= \text{Area} \cdot h = \underbrace{\frac{2 \cdot 2}{2}}_{\text{triangle geometry}} \cdot 3 = \boxed{6}$ ← volume

Definition: Iterated integral Type 1 (I_1)

$$I_1 = \int_a^b \left[\int_c^D f(x,y)\,dy\right] dx$$ ★

$F(x,y)$ such that $F_y(x,y) = f(x,y)$

$$I_1 = \int_a^b F(x,y)\Big|_c^D dx$$

$$= \int_a^b \underbrace{F(x,d) - F(x,c)}_{g(x)}\,dx = \int_a^b g(x)\,dx$$

Type 2 (I_2)

$$I_2 = \int_c^D \left[\int_a^b f(x,y)\,dx\right] dy$$ ★

$F(x,y)$ such that $F_x(x,y) = f(x,y)$

$$I_2 = \int_c^D \underbrace{F(b,y) - F(a,y)}_{h(y)}\,dy = \int_c^D h(y)\,dy$$

Theorem: Tonelli–Fubini Theorem

$$\iint_R f(x,y)\,dA \equiv I_1 \equiv I_2$$

Ex #1 w/ T-F theorem:

$$\iint_R 2-y\,dA$$

$$I_1 = \int_0^3 \left[\int_0^2 2-y\,dy\right] dx$$

$$= \int_0^3 2y - \frac{y^2}{2}\Big|_0^2 dx = \int_0^3 (4-2)-(0-0)\,dx$$

$$= \int_0^3 2dx = 2x\Big|_0^3 = 6-0 = \boxed{6}$$ Same as geometric volume

constant in dx

$$I_2 = \int_0^2 \left[\int_0^3 2-y\,dx\right] dy$$

$$= \int_0^2 x(2-y)\Big|_0^3 dy = 3\int_0^2 (2-y)dy$$

$$= 3\left(2y - \frac{y^2}{2}\right)\Big|_0^2 = 3(2) = \boxed{6}$$

Ex. #2 – $\iint_R y\sin(xy)\,dA$ on $R = \begin{cases} 1 \le x \le 2 \\ 0 \le y \le \pi \end{cases}$

0. Choose either I_1 or I_2

$$I_1 = \int_1^2 \left[\int_0^\pi y\sin(xy)\,dy\right] dx$$ ← NOT this because Int. by parts

$$I_2 = \int_0^\pi \left[\int_1^2 y\sin(xy)\,dx\right] dy$$

1. Do it

$$\int_0^\pi \left[\int_1^2 y\sin(xy)\,dx\right] dy$$

$\int \sin(xy)dx = \int \sin(u)\,du$ $= -\frac{\cos(xy)}{y}$

$u = xy$

$du = y\,dx$

$$= \int_0^\pi \cancel{y}\left(\frac{-\cos(xy)}{\cancel{y}}\right)\Big|_1^2 dy$$

$$= \int_0^\pi -\cos(2y) + \cos(y)\,dy$$

$$= \frac{-\sin(2y)}{2} + \sin(y)\Big|_0^\pi = (-0+0)-(-0+0) = \boxed{0}$$

Definition: this only applies to functions in this form ★

$$\iint_R g(x)h(y)\,dA$$

$$= \int_a^b \left[\int_c^D g(x)h(y)\,dy\right] dx$$

$$= \int_a^b g(x)\left[\int_c^D h(y)\,dy\right] dx$$ Because $g(x)$ is a constant initially

$$= \left[\int_c^D h(y)\,dy\right]\left[\int_a^b g(x)\,dx\right]$$ Because $\left[\int_c^D h(y)\,dy\right]$ is a constant in dx

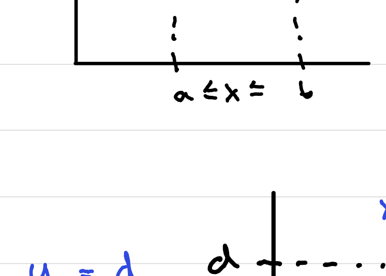
Ex. #3 – Based on #1 w/

$$\iint_R (2-y)\,dA = \left[\int_0^3 1\,dx\right]\left[\int_0^2 (2-y)\,dy\right]$$

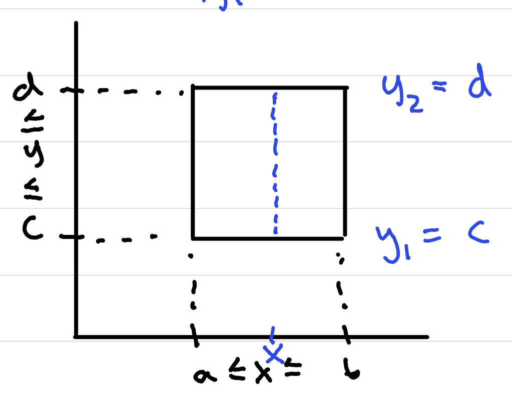
10-18-2024 Summary

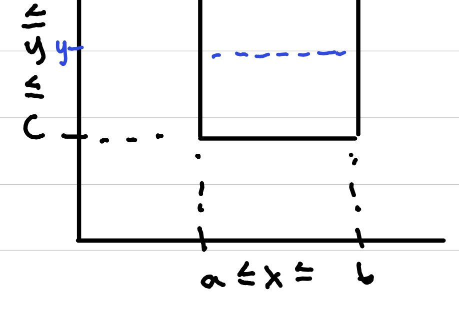
$$\iint_R f(x,y)\,dA = \underbrace{\left[\int_a^b\int_c^d f(x,y)\,dy\,dx\right]}_{\text{Type I}} = \underbrace{\left[\int_c^d\int_a^b f(x,y)\,dx\,dy\right]}_{\text{Type II}}$$

$R: \begin{cases} a \le x \le b \\ c \le y \le d \end{cases}$

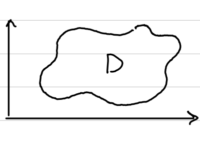


Type 1

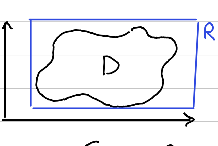




12.2 Double integral on D



we're looking at $\iint_D f(x,y)\,dA$



$g(x,y) \begin{cases} f(x,y) & \text{for all } (x,y) \text{ in } D \\ 0 & \text{otherwise} \end{cases}$ ★

$\iint_D f(x,y)\,dA = \iint_R g(x,y)\,dA = \lim_{n\to\infty}$ Riemann Sum

---

★ **RULES!**

Linearity:

$\iint_D (af(x,y) + bh(x,y))\,dA$

$= a\iint_D f(x,y)\,dA + b\iint_D h(x,y)\,dA$

Dominance:
if $f(x,y) \geq h(x,y)$ for all (x,y) in D,

$\iint_D f(x,y)\,dA \geq \iint_D h(x,y)\,dA$

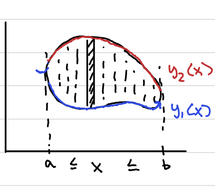
★★ Subdivision Rule:

$\iint_D f(x,y)\,dA = \iint_{D_1+D_2} f(x,y)\,dA$ where $D = D_1 + D_2$

$= \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$

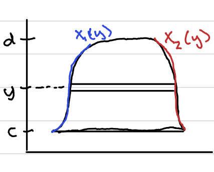
We say D is of type I IF:

$\forall (x,y) \in D \begin{cases} a \leq x \leq b \\ y_1(x) \leq y \leq y_2(x) \end{cases}$ ★



of Type II IF:

$\forall (x,y) \in D \begin{cases} c \leq y \leq d \\ x_1(y) \leq x \leq x_2(y) \end{cases}$ ★



If D = Type 1, then

$\iint_D f(x,y)\,dA = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx$ ← functions of x

$F_y(x,y) = f(x,y)$

$= \int_a^b F(x,y)\Big|_{y_1(x)}^{y_2(x)} = \int_a^b \underbrace{F(x, y_2(x)) - F(x, y_1(x))}_{g(x)}\,dx$

$= \int_a^b g(x)\,dx$

Type I → "y-simple integrable region"

If D = Type 2 (or x-simple), then

$\iint_D f(x,y)\,dA = \int_c^d \int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy$

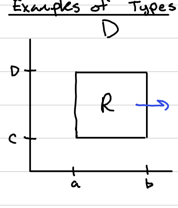
$F_x(x,y) = f(x,y)$

$= \int_c^d F(x,y)\Big|_{x_1(y)}^{x_2(y)} = \int_c^d \underbrace{F(x_2(y), y) - F(x_1(y), y)}_{h(y)}\,dy$

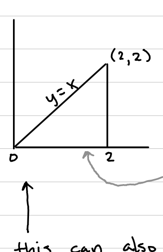
$= \int_c^d h(y)\,dy$

★ If your D is both T I and T II, we will use subdivision rule to split them up (later)

Examples of Types

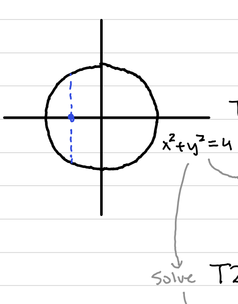


→ Both T1 and T2



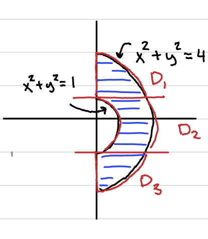
T1: $\begin{cases} 0 \leq x \leq 2 \\ 0 \leq y \leq x \end{cases}$ ($y=0$, $y=x$)

this can also be T2: $\begin{cases} 0 \leq y \leq 2 \\ y \leq x \leq 2 \end{cases}$ ($x=y$, $x=2$)



T1: $\begin{cases} -2 \leq x \leq 2 \\ -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2} \end{cases}$ (solve)

(Solve) T2: $\begin{cases} -2 \leq y \leq 2 \\ -\sqrt{4-y^2} \leq x \leq \sqrt{4-y^2} \end{cases}$



this can only be T2

we will use subdiv. rule

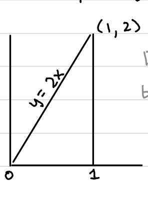
D_1 $\begin{cases} 1 \leq y \leq 2 \\ 0 \leq x \leq \sqrt{4-y^2} \end{cases}$ (y-axis)

D_2: $\begin{cases} -1 \leq y \leq 1 \\ \sqrt{1-y^2} \leq x \leq \sqrt{4-y^2} \end{cases}$

D_3: $\begin{cases} -2 \leq y \leq -1 \\ 0 \leq x \leq \sqrt{4-y^2} \end{cases}$

Problem from the book

$\iint_T (x+y)\,dA$



Doing this as T1 is better because a function will be zero

1. Set up

T1: $\begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 2x \end{cases}$

2. Make Integrated Integral

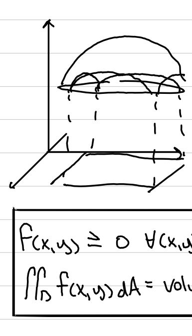
$I = \int_0^1 \int_0^{2x} (x+y)\,dy\,dx$

3. Solve

$= \int_0^1 xy + \frac{y^2}{2}\Big|_0^{2x}\,dx$ Be sure to plug in for y

$= \int_0^1 \left[x(2x) + \frac{(2x)^2}{2}\right] - 0\,dx$

$= \frac{4}{3}x^3\Big|_0^1 = \boxed{\frac{4}{3}}$



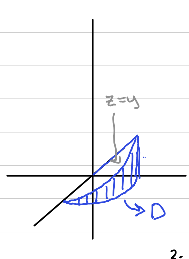
$\iint_R f(x,y)\,dA =$ Volume for a function that is never negative

$f(x,y) \geq 0 \;\forall (x,y) \in D$:
$\iint_D f(x,y)\,dA =$ volume

$\iint_D 1\,dA =$ Area of D

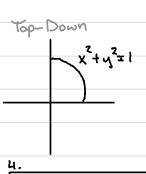
More Examples

#1) Find volume of body bounded above by $z=y$, below by $z=0$ (xy-plane), and inside $x^2+y^2 \leq 1$ for $x \geq 0$ and $y \geq 0$ (In Q1)



1. $\iint y\,dA =$ Volume

because $z=y$ is the roof, this is what we integrate since it's all ≥ 0



2. T1: $\begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq \sqrt{1-x^2} \end{cases}$

3. $I = \int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx$

4. $= \int_0^1 \frac{y^2}{2}\Big|_0^{\sqrt{1-x^2}}\,dx = \frac{1}{2}\int_0^1 1-x^2\,dx$

$= \frac{1}{2}\left(x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{2}\left(1-\frac{1}{3}\right) = \boxed{\frac{1}{3}}$

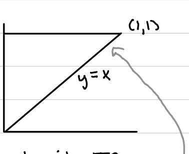
#2) Solve the Int. Integral:

$\int_0^1 \int_x^1 e^{y^2}\,dy\,dx = \iint_D e^{y^2}\,dA$

by reversing the order of integration
THAT DOES NOT JUST MEAN FLIP DX and DY!

1. construct D and the picture

$D = \begin{cases} 0 \leq x \leq 1 \\ x \leq y \leq 1 \end{cases}$



2. make it T2

$T2 = \begin{cases} 0 \leq y \leq 1 \\ 0 \leq x \leq y \end{cases}$

3. solve

$= \int_0^1 \int_0^y e^{y^2}\,dx\,dy$

$= \int_0^1 e^{y^2}\; x\Big|_0^y\,dy$ (pull out as const.)

$= \frac{1}{2}\int e^{y^2} 2y\,dy = \frac{1}{2}e^{y^2}\Big|_0^1 = \frac{1}{2}(e^1 - e^0)$
$= \boxed{\frac{1}{2}(e-1)}$

because $u = y^2$, $du = 2y\,dy$

$\int e^u\,du = e^u$

Summary - 10/21/2024

if $f(x,y) \geq 0$ for all (x,y) in D,

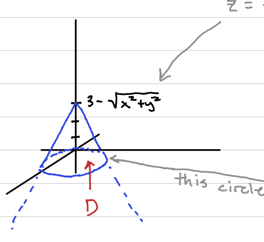
$\iint_D f(x,y) = \underline{\text{volume}}$ bottom = xy plane, top = function

$\iint_D 1\,dA = \underline{\text{Area}}\ (D)$

Ex. $f(x,y) = 3 - \sqrt{x^2+y^2}$ D: domain where $f \geq 0$

find volume

$z = \sqrt{x^2+y^2}$
$x^2+y^2-z^2=0$



1. make system of equations to find D

$\begin{cases} z = 3 - \sqrt{x^2+y^2} \\ z = 0 \end{cases}$ to find D, we find the intersection of these two

↓ Plug in

$0 = 3 - \sqrt{x^2+y^2} \xrightarrow{\dots} x^2+y^2 = 9$

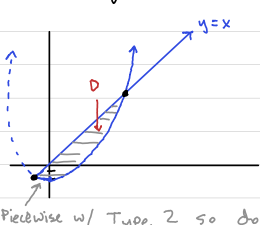
$\boxed{D: x^2+y^2 \leq 9}$

2. get volume

$\iint_D f(x,y)\,dA = \text{volume} = \frac{1}{3}Ah = \frac{1}{3}\pi 3^2(3) = \boxed{9\pi}$ by formula for cone volume

---

Ex. 2 Find Area of D where D is the region bounded by the parabola $y = x^2 - 2$ and the line $y = x$



Piecewise w/ Type 2 so do Type 1

Area D = $\iint_D 1\,dA$

1. Find intersections

$\begin{cases} y = x \\ y = x^2 - 2 \end{cases}$

$x^2 - 2 = x \rightarrow x^2 - x - 2 = 0$
$(x+1)(x-2)$

$D: \begin{cases} -1 \leq x \leq 2 \\ x^2 - 2 \leq y \leq x \end{cases}$

2. Set up integral Type 1

$\int_{-1}^2 \int_{x^2-2}^x 1\,dy\,dx = \int_{-1}^2 y\Big|_{x^2-2}^{x}\,dx = \int_{-1}^2 x - (x^2-2)\,dx$

$= \int_{-1}^2 x - x^2 + 2\,dx$

Area $= \frac{x^2}{2} - \frac{x^3}{3} + 2x\Big|_{-1}^2$

$= \left(\frac{4}{2} - \frac{8}{3} + 4\right) - \left(\frac{1}{2} + \frac{1}{3} - 2\right) = \boxed{4.5}$

12.3 - Double Integrals in Polar Coordinates

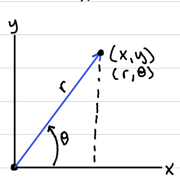
$$\iint_D f(x,y)\,dA = \iint_D f(r,\theta)\,dA$$

dA = $dx\,dy$; dA cannot be $dr\,d\theta$ because θ is an angle, so it is $r\,dr\,d\theta$

replace the missing dimension

$$= \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)} \dots$$

In polar coordinates, we have to re-describe D to match if it is defined in x and y

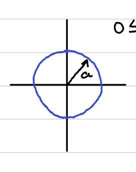


$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r^2 = x^2+y^2 \\ \tan\theta = \frac{y}{x} \end{cases}$$

this is a polar coords crash course section

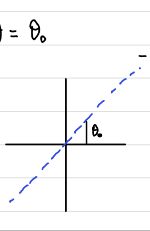
$r = \pm\sqrt{x^2+y^2}$

Start with simplest function: $r = a$
(circle w/ radius A centered at origin)



$0 \le \theta \le 2\pi$ ← limit to 1 rotation

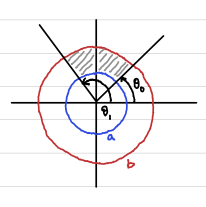
$\theta = \theta_0$



$-\infty \le r \le \infty$ ← line is of ∞ length

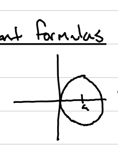
Rectangle in polar coordinates

$$R\begin{cases} a \le r \le b \\ \theta_0 \le \theta \le \theta_1 \end{cases}$$



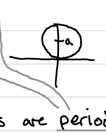
Other Important formulas

$r = 2a\cos\theta$
$0 \le \theta \le \pi$



radius of a

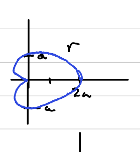
$r = 2a\sin\theta$
$0 \le \theta \le \pi$



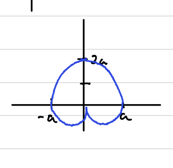
these circles are periodic every pi ★

Cardiods

$r = a(1+\cos\theta)$
$0 \le \theta \le 2\pi$



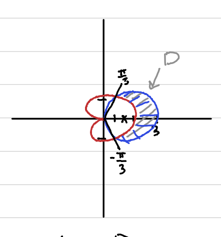
$r = a(1+\sin\theta)$
$0 \le \theta \le 2\pi$



He wants us to know: ★
- Cardiod formula
- $2a\cos\theta$ and $2a\sin\theta$ circles
- regular circles

review over.

Ex. $r = 3\cos\theta$ $\quad r = 1+\cos\theta$
want region D inside $3\cos\theta$ and outside $1+\cos\theta$



1. Set up D

$$\begin{cases} \theta_1 \le \theta \le \theta_2 \\ r_1(\theta) \le r \le r_2(\theta) \end{cases}$$

$1+\cos\theta \le r \le 3\cos\theta$

2. Find intersection

$$\begin{cases} r = 3\cos\theta \\ r = 1+\cos\theta \end{cases}$$

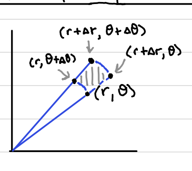
$1+\cos\theta = 3\cos\theta \longrightarrow \cos\theta = \frac{1}{2}$

$\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

meaning $D = \begin{cases} -\frac{\pi}{3} \le \theta \le \frac{\pi}{3} \\ 1+\cos\theta \le r \le 3\cos\theta \end{cases}$

$$\iint_{D(x,y)} f(x,y)\,dA = \iint_{D(R,\theta)} f(r,\theta)\,dA$$

mid-example explanation



imagine doing $(r+\Delta r)$ and $(\theta+\Delta\theta)$

$$\Delta A = (r+\Delta r)^2\left(\frac{\Delta\theta}{2}\right) - (r^2)\left(\frac{\Delta\theta}{2}\right)$$
$$= (r^2 + 2r\Delta r + \Delta r^2 - r^2)\left(\frac{\Delta\theta}{2}\right)$$
$$= r\Delta\theta\Delta r + \left(\frac{\Delta r^2\Delta\theta}{2}\right)$$

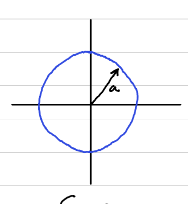
$$\lim_{(\Delta r,\Delta\theta)\to(0,0)} \Delta A = r\,d\theta\,dr + \cancel{\left(\frac{dr^2\,d\theta}{2}\right)}$$

this goes to zero

explanation over

3. Inexplicably don't finish the example

Ex. Find the area of a circle of radius a
(we know it is πa^2)

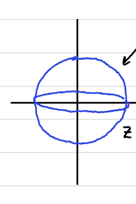


$\text{Area} = \iint_D 1\,dA$ ← rect. coordinates. we want polar

$$R: \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le a \end{cases}$$ ← Region in polar coords

$$\text{Area} = \iint 1\, r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^a 1\, r\,dr\,d\theta$$
$$= \int_0^{2\pi} \frac{r^2}{2}\Big|_0^a d\theta = \frac{a^2}{2}\int_0^{2\pi} d\theta = \frac{a^2}{2}\theta\Big|_0^{2\pi} = \boxed{a^2\pi}$$

Ex. Find volume of a sphere of radius a
($V = \frac{4}{3}(\pi)(a^3)$)



we will take volume of top half and double it

$x^2+y^2+z^2 = a^2$

$z = \sqrt{a^2-x^2-y^2}$

$$\text{Volume} = 2\iint_D \sqrt{a^2-x^2-y^2}\,dA$$

1. Set up integral

$$2\int_0^{2\pi}\int_0^a \sqrt{a^2-r^2}\; r\,dr\,d\theta$$

r^2 ← x^2-y^2

(u-sub)
$a^2 - r^2 = u$
$-2r\,dr = du$

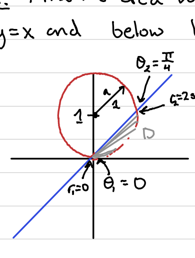
$\int \sqrt{u}\,du = \frac{2}{3}u^{3/2}$

$$= -\int_0^{2\pi} \frac{2}{3}(a^2-r^2)^{3/2}\Big|_0^a d\theta$$
$$= -\frac{2}{3}\left[0 - a^3\right]\int_0^{2\pi} d\theta \to 2\pi$$
$$= \boxed{\frac{4}{3}\pi a^3}$$

10-23-24

Ex. Find the area bounded above by $y = x$ and below by $x^2+y^2-2y = 0$

$x^2 + (y^2-2y+1) = 1$
$x^2 + (y-1)^2 = 1^2$
complete the square



$\theta_2 = \frac{\pi}{4}$, $r_2 = 2\sin\theta$, $r_1 = 0$, $\theta_1 = 0$

$$R: \begin{cases} 0 \le \theta \le \frac{\pi}{4} \\ 0 \le r \le 2\sin\theta \end{cases}$$

$\text{Area} = \iint_D 1\,dA$

$r = 2a\sin\theta$
$r = 2\sin\theta \to$ the circle

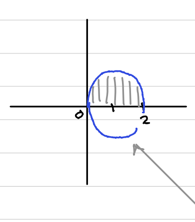
★ $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$
★ $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$\text{Area} = \int_0^{\frac{\pi}{4}}\int_0^{2\sin\theta} 1\, r\,dr\,d\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{r^2}{2}\Big|_0^{2\sin\theta} d\theta$$
$$= \int_0^{\frac{\pi}{4}} 2\sin^2\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{4}} \cancel{2}\left(\cancel{\tfrac{1}{2}}(1-\cos 2\theta)\right)d\theta$$
$$= \theta - \frac{\sin 2\theta}{2}\Big|_0^{\frac{\pi}{4}} = \boxed{\left[\frac{\pi}{4} - \frac{1}{2}\right] - 0}$$

Ex. $\int_0^2\int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\,dy\,dx$
convert this to polar coordinates and get the integral

1. convert back to sketch

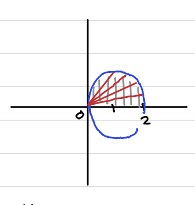
$$R: \begin{cases} 0 \le x \le 2 \\ 0 \le y \le \sqrt{2x-x^2} \end{cases}$$



$y^2 = 2x - x^2$
$x^2 - 2x + y^2 = 0$
$+1$ complete the square $+1$
$= (x-1)^2 + y^2 = 1$

2. Describe in polar coords

$r = 2a\cos\theta \longrightarrow$ since radius $= 1$, $r = 2\cos\theta$
(a = radius)



3. Do it

$$R: \begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 2\cos\theta \end{cases}$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} r\sin\theta(r)\; r\,dr\,d\theta$$

$y = r\sin\theta$, $\sqrt{x^2+y^2} = \sqrt{r^2} = r$, dA

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} \sin\theta\, r^3\,dr\,d\theta$$
$$= \int_0^{\frac{\pi}{2}} \sin\theta \frac{r^4}{4}\Big|_0^{2\cos\theta} d\theta$$
$$= \int_0^{\frac{\pi}{2}} \sin\theta\left[\frac{16\cos^4\theta}{4}\right]d\theta$$

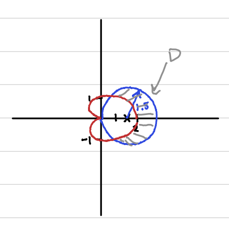
u-sub $\quad u = \cos\theta \quad du = -\sin\theta\,d\theta$

$$= -4\int_0^{\frac{\pi}{2}} \underbrace{\cos^4\theta(-\sin\theta)\,d\theta}_{u^4\,du}$$

$\int u^4 = \frac{u^5}{5}$, $u = \cos\theta$

$$= -4\,\frac{(\cos\theta)^5}{5}\Big|_0^{\frac{\pi}{2}} = -\frac{4}{5}(0-1) = \boxed{\frac{4}{5}}$$

Ex. Find area of D inside $r = 3\cos\theta$, outside $r = 1+\cos\theta$



1.

$$D: \begin{cases} \\ 1+\cos\theta \le r \le 3\cos\theta \end{cases}$$

Find θ limits by equating the formulas

$$\begin{cases} r = 3\cos\theta \\ r = 1+\cos\theta \end{cases}$$

$3\cos\theta = 1+\cos\theta$ ... Already done earlier in notes
$\theta = \frac{\pi}{3}$

$$D: \begin{cases} -\frac{\pi}{3} \le \theta \le \frac{\pi}{3} \\ 1+\cos\theta \le r \le 3\cos\theta \end{cases}$$

2. Do it

$$\text{Area} = \iint_D 1\,dA$$
$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_{1+\cos\theta}^{3\cos\theta} r\,dr\,d\theta$$
$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{r^2}{2}\Big|_{1+\cos\theta}^{3\cos\theta} d\theta$$
$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 9\cos^2\theta - (1+\cos\theta)^2\,d\theta$$

$(1+\cos\theta)^2 \to 1 + 2\cos\theta + \cos^2\theta$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 8\cos^2\theta - 2\cos\theta - 1\,d\theta$$

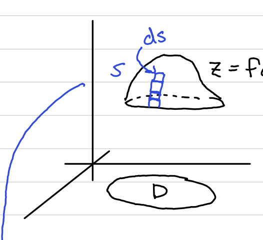
$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cancel{8}^{\,4}\left(\cancel{\tfrac{1}{2}}(1+\cos 2\theta)\right) - 2\cos\theta - 1\,d\theta$$
$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 3 + 4\cos 2\theta - 2\cos\theta\,d\theta$$
$$= \frac{1}{2}\left[3\theta + \frac{\cancel{4}^{\,2}\sin 2\theta}{\cancel{2}} - 2\sin\theta\right]\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$
$$= \frac{1}{2}\left[\left(\frac{3\pi}{3} + 2\sin\frac{2\pi}{3} - 2\sin\frac{\pi}{3}\right) - \left(3\left(-\frac{\pi}{3}\right) + 2\sin\left(-\frac{2\pi}{3}\right) - 2\sin\left(-\frac{\pi}{3}\right)\right)\right]$$

120°, 60°, 0, 0

$$= \frac{1}{2}(2\pi) = \boxed{\pi}$$

12.4 - Surface Area



Find surface area

$$S.A. = \iint_S ds$$

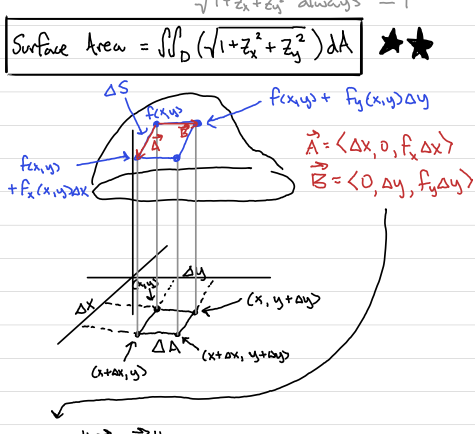
↓ how does this relate to dA?

$$= \iint_D (\ \ ) \, dA$$ (something)

ds is always ≥ dA because dA is a proj. of dS

$\sqrt{1+z_x^2+z_y^2}$ always ≥ 1

Surface Area $= \iint_D \left(\sqrt{1+z_x^2+z_y^2}\right) dA$ ★★



$\vec{A} = \langle \Delta x, 0, f_x \Delta x \rangle$

$\vec{B} = \langle 0, \Delta y, f_y \Delta y \rangle$

$\Delta S \simeq \|\vec{A} \times \vec{B}\|$ as seen in 9.4

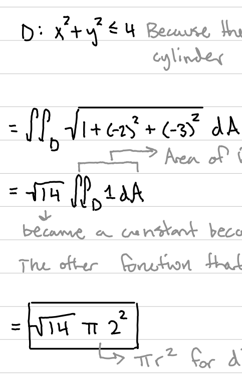
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x & 0 & f_x \Delta x \\ 0 & \Delta y & f_y \Delta y \end{vmatrix} = \hat{i}(0 - f_x \Delta x \Delta y) - \hat{j}(f_y \Delta x \Delta y - 0) + \hat{k}(\Delta x \Delta y - 0)$$

$$\| \ \ \| = \sqrt{f_x^2(\Delta x \Delta y)^2 + f_y^2(\Delta x \Delta y)^2 + (\Delta x \Delta y)^2}$$

$$= \sqrt{1 + f_x^2 + f_y^2} \ \Delta x \Delta y$$

↑ Equation for surface area □

Ex. Find the S.A. of the portion of the plane $2x + 3y + z = 6$ inside the cylinder $x^2 + y^2 = 4$



$$S.A. = \iint_S ds = \iint_D \sqrt{1+z_x^2+z_y^2} \, dA$$

$D: x^2 + y^2 \leq 4$ Because the plane goes all through the cylinder

$$= \iint_D \sqrt{1 + (-2)^2 + (-3)^2} \, dA$$

→ Area of D

$$= \sqrt{14} \iint_D 1 \, dA$$

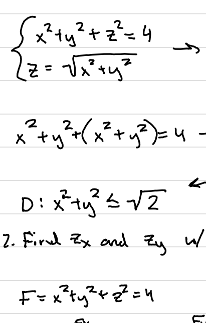
became a constant because our surface is a plane. The other function that does this is the cone.

$$= \boxed{\sqrt{14} \, \pi \, 2^2}$$

↳ πr^2 for disk $x^2 + y^2 \leq 4$ (Area of D)

Ex. Find S.A. of the portion of the sphere $x^2 + y^2 + z^2 = 4$ inside the cone $z = \sqrt{x^2 + y^2}$

THIS IS A TOUGH PROBLEM. STUDY ME



$$S.A. = \iint_S dS = \iint_D \sqrt{1+z_x^2+z_y^2} \, dA$$

1. Find intersection

$$\begin{cases} x^2+y^2+z^2=4 \\ z = \sqrt{x^2+y^2} \end{cases} \rightarrow \begin{cases} x^2+y^2+z^2=4 \\ z^2 = x^2+y^2 \end{cases}$$ sub

$$x^2+y^2+(x^2+y^2) = 4 \rightarrow x^2+y^2 = 2 = (\sqrt{2})^2$$

$D: x^2 + y^2 \leq \sqrt{2}$

2. Find z_x and z_y w/ impl. diff.

$F = x^2 + y^2 + z^2 = 4$

$z_x = -\frac{F_x}{F_z} \quad z_y = -\frac{F_y}{F_z}$

$= \frac{-x}{z} \quad = -\frac{y}{z}$

$$\sqrt{1 + \left(\frac{-x}{z}\right)^2 + \left(\frac{-y}{z}\right)^2} \rightarrow \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}$$

$$= \sqrt{\frac{x^2+y^2+z^2}{z^2}}$$ ($x^2+y^2+z^2=4$ → replace w/ 4; ↳ solve for z)

$$= \frac{\sqrt{4}}{\sqrt{4-x^2-y^2}} = \frac{2}{\sqrt{4-x^2-y^2}}$$

$$S.A. = \iint_D \frac{2}{\sqrt{4-x^2-y^2}} \, dA$$

10-25-2024 problem continued

3. go to polar coords because it is easier

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \frac{2}{\sqrt{4-r^2}} \, r \, dr \, d\theta$$

u-sub $\quad u = 4 - r^2 \quad du = -2r \, dr$

$$= -\int_0^{2\pi} \int_0^{\sqrt{2}} \frac{1}{\sqrt{u}} \, du \, d\theta$$

↓ $2u^{1/2}$

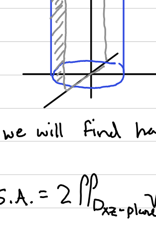
$$= -\int_0^{2\pi} 2(4-r^2)^{1/2} \Big|_0^{\sqrt{2}} \, d\theta$$

$$= -\int_0^{2\pi} 2\sqrt{2} - 4 \, d\theta$$

$$= -\int_0^{2\pi} 2(2 - \sqrt{2})$$

$$\ldots = \boxed{4\pi(2 - \sqrt{2})}$$

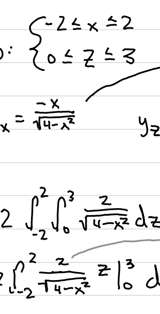
Ex. Find S.A. of portion of $x^2 + y^2 = 4$ on $0 \leq z \leq 3$



$$S.A. = \iint_D \sqrt{1 + z_x^2 + z_y^2} \, dA$$

that is an issue because we have no z_x, z_y

1. Shuffle stuff around (use projection on xz-plane)



$y(x, z) = \sqrt{4 - x^2}$

y is our dependent in this problem

we will find half the S.A., so x2

$$S.A. = 2 \iint_{D_{xz\text{-plane}}} \sqrt{1 + y_x^2 + y_z^2} \, dA$$

2. Do it

$$D: \begin{cases} -2 \leq x \leq 2 \\ 0 \leq z \leq 3 \end{cases}$$

$y_x = \frac{-x}{\sqrt{4-x^2}} \quad y_z = 0$

$$\sqrt{1 + y_x^2 + y_z^2} = \sqrt{1 + \frac{x^2}{4-x^2}} = \sqrt{\frac{4-x^2}{4-x^2} + \frac{x^2}{4-x^2}} \cdot \frac{2}{\sqrt{4-x^2}}$$

$$= 2 \int_{-2}^{2} \int_0^3 \frac{2}{\sqrt{4-x^2}} \, dz \, dx$$

$$= 2 \int_{-2}^{2} \frac{2}{\sqrt{4-x^2}} \, z \Big|_0^3 \, dx$$

$$= 2 \cdot 3 \int_{-2}^{2} \frac{1}{\frac{\sqrt{4-x^2}}{2}} \, dx$$

$\int \frac{1}{\sqrt{1-u^2}} \, du = \arcsin(u)$

$$= 6 \int_{-2}^{2} \frac{1}{\sqrt{\frac{4-x^2}{4}}} = 6 \int_{-2}^{2} \frac{1}{\sqrt{1 - \left(\frac{x}{2}\right)^2}} \, dx$$

u-sub $\quad u = \frac{x}{2}, \quad du = \frac{dx}{2}$

$$= 12 \left[\arcsin\left(\frac{x}{2}\right) \Big|_{-2}^{2} \right]$$

$$= 12 \left[\arcsin(1) - \arcsin(-1) \right]$$

$\quad \frac{\pi}{2} \quad - \quad -\frac{\pi}{2}$

$= \boxed{12\pi}$ check w/ geometry

$S.A. = 2\pi(r) \cdot h$

$= 2\pi(2) \cdot 3 = \boxed{12\pi}$

Surface Parameterization ★★★

$\vec{R}(u,v) \rightarrow$ vector function with foot at origin & head on our surface

$$\vec{R}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle \quad \begin{matrix} u_1 \leq u \leq u_2 \\ v_1 \leq v \leq v_2 \end{matrix}$$

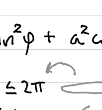
Do it w/ above ex.'s cylinder

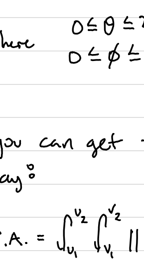
$$\vec{R}(u,v) = \langle 2\cos(u), 2\sin(u), v \rangle \quad \begin{cases} 0 \leq u \leq 2\pi \\ 0 \leq z \leq 3 \end{cases}$$

$x^2 + y^2 = 4\cos^2(u) + 4\sin^2(u) = 4$

This puts you on the circle

this makes the wall





$$\langle x, y, z \rangle = \langle a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi \rangle$$

$$x^2 + y^2 + z^2 = \left(a^2\sin^2\varphi\cos^2\theta + a^2\sin^2\varphi\sin^2\theta\right) + a^2\cos^2\varphi$$

$$= a^2\sin^2\varphi + a^2\cos^2\varphi = a^2$$

where $\quad 0 \leq \theta \leq 2\pi$
$\quad\quad\quad 0 \leq \phi \leq \pi$

you can get the S.A. in a more straightforward way:

$$S.A. = \int_{u_1}^{u_2} \int_{v_1}^{v_2} \|\vec{R}_u \times \vec{R}_v\| \, dv \, du$$ ★★

w/ above ex:

$\vec{R}_u = \langle -2\sin u, 2\cos u, 0 \rangle$

$\vec{R}_v = \langle 0, 0, 1 \rangle$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \hat{i}(2\cos u) - \hat{j}(-2\sin u) + \hat{k}(0)$$

$$\vec{R}_u \times \vec{R}_v = \langle 2\cos u, 2\sin u, 0 \rangle$$

$$\|\vec{R}_u \times \vec{R}_v\| = \sqrt{4\cos^2 u + 4\sin^2 u} = 2$$

$$S.A. = \int_0^{2\pi} \int_0^3 2 \, dv \, du$$

$\ldots$

$$= 2(3)(2\pi) = \boxed{12\pi}$$

12.5 Triple Integrals in Rectangular Coords

$\iiint_V g(x,y,z)\,dV$

(x,y,z): Domain in 3D

V: Some region in 3D

Box: (B) $\begin{cases} a \le x \le b \\ c \le y \le d \\ e \le z \le f \end{cases}$

$= \iiint_B g(x,y,z)\,dV$

$= \left[\int_a^b \left[\int_c^d \left[\int_e^f g(x,y,z)\,dz\right] dy\right] dx\right]$, where $H(x,y) = \int_e^f g(x,y,z)\,dz$

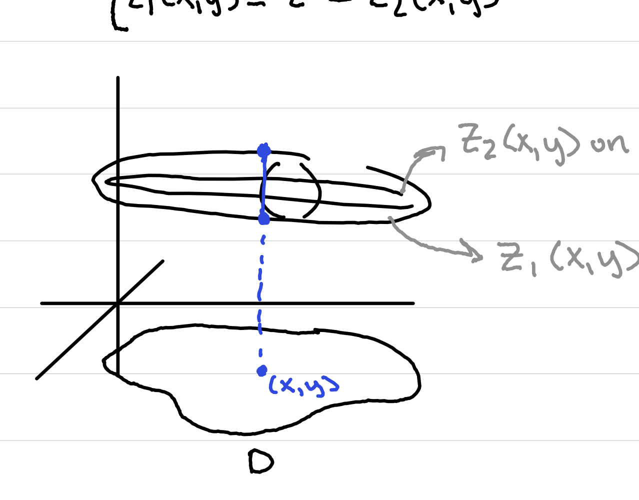
This has 6 possible permutations of the order of integration

More interesting sub-regions

z-simple integral:

V is z-simple if $\forall (x,y \in D)$, z is bounded below by $z_1(x,y)$ and above by $z_2(x,y)$

V: $\begin{cases} (x,y) \in D \\ z_1(x,y) \le z \le z_2(x,y) \end{cases}$ ★

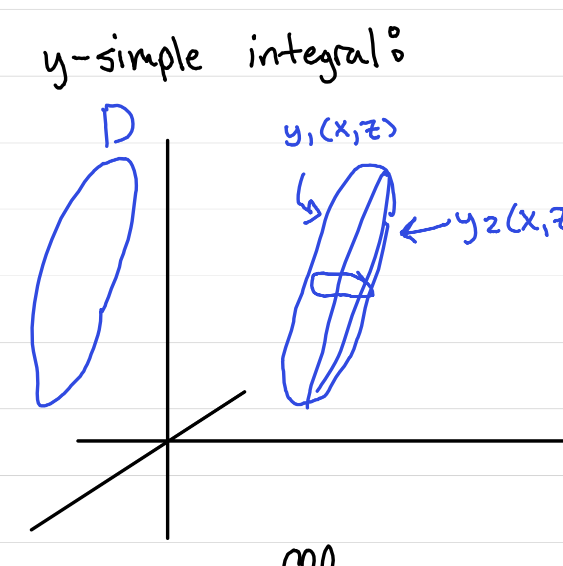


Theorem: If V is z-simple, then

$\iiint_V g(x,y,z)\,dV = \iint_D \left[\int_{z_1(x,y)}^{z_2(x,y)} g(x,y,z)\,dz\right] dA$, where $H(x,y) = \int_{z_1(x,y)}^{z_2(x,y)} g(x,y,z)\,dz$

y-simple integral:

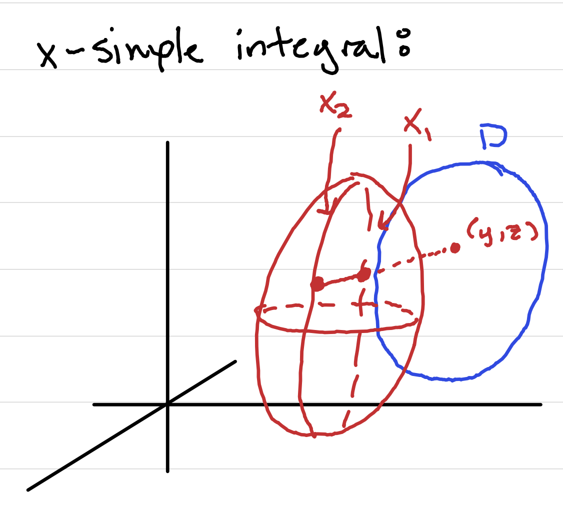
V: $\begin{cases} (x,z) \in D \\ y_1(x,z) \le y \le y_2(x,z) \end{cases}$



Theorem: $\iiint_V g(x,y,z)\,dV = \iint_D \int_{y_1}^{y_2} g(x,y,z)\,dy\,dA$

x-simple integral:

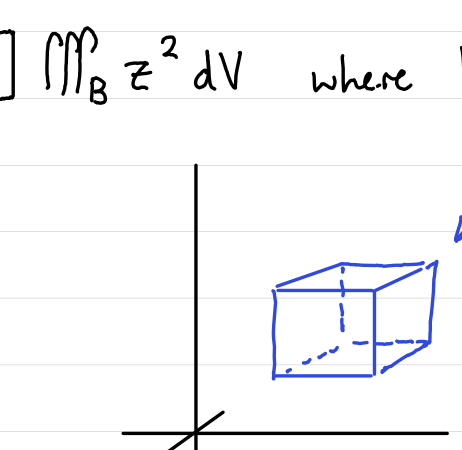
V: $\begin{cases} (y,z) \in D \\ x_1(y,z) \le x \le x_2(y,z) \end{cases}$



Theorem: $\iiint_V g(x,y,z) = \iint_D \int_{x_1}^{x_2} g(x,y,z)\,dx\,dA$

Examples

#1] $\iiint_B z^2\,dV$ where B: $\begin{cases} 0 \le x \le 1 \\ 1 \le y \le 2 \\ -1 \le z \le 1 \end{cases}$



$I = \iint_D \int_{-1}^{1} z^2\,dz\,dA$

$= \iint_D \frac{z^3}{3}\Big|_{-1}^{1} = \iint_D \frac{2}{3}\,dA$

$= \frac{2}{3}\iint_D 1\,dA = \frac{2}{3}\text{Area}(D) = \boxed{\frac{2}{3}}$ bcuz D: $\begin{cases} 0 \le x \le 1 \\ 1 \le y \le 2 \end{cases}$ has sides = 1

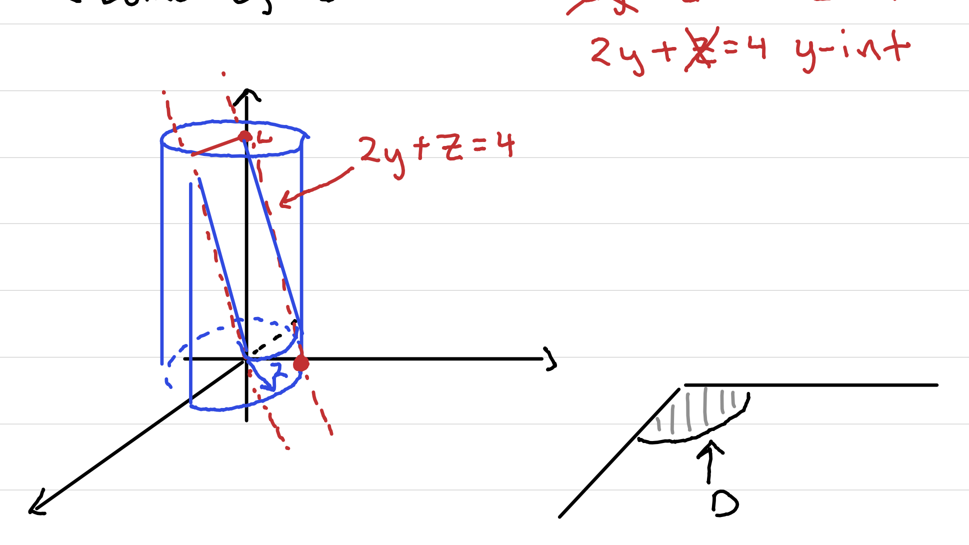
($\iint_D 1\,dA$: area of D)

#2] $\iiint_V x\,dV$ where

V: $\begin{cases} \text{First octant } (x \ge 0, y \ge 0, z \ge 0) \\ \text{inside } x^2+y^2=4 \\ \text{below } 2y+z=4 \end{cases}$

~~2y~~ + z = 4 z-int

2y + ~~z~~ = 4 y-int



1. Determine x-y-or-z-simple

<u>z-simple</u>

$\begin{cases} x^2+y^2 \le 4,\ x \ge 0,\ y \ge 0 \\ 0 \le z \le 4-2y \end{cases}$

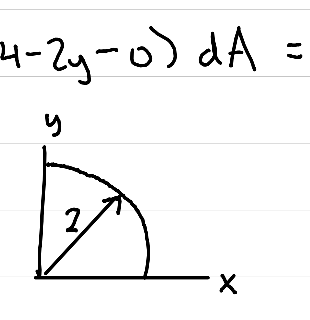
↳ solve plane eq. for z

y-simp $\begin{cases} D\begin{cases} 0 \le x \le 2 \\ 0 \le z \le 4 \end{cases} \\ 0 \le y \le \frac{4-z}{2} \end{cases}$ ⟶ solve plane eq. for y

x-simp $\begin{cases} D = \text{triangle} \\ 0 \le x \le \sqrt{4-y^2} \end{cases}$

$I = \iint_D \int_0^{4-2y} x\,dz\,dA$

$= \iint_D x(4-2y-0)\,dA = \iint_D 4x - 2xy\,dA$

D ⟶ 

2. Polar coords time

D: $\begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 2 \end{cases}$

$= \int_0^{\frac{\pi}{2}} \int_0^2 4r\cos\theta - 2r^2\cos\theta\sin\theta \; r\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}} 4\frac{r^3}{3}\cos\theta - \sin(2\theta)\frac{r^4}{4}\Big|_0^2$ ($\sin(2\theta) = 2\cos\theta\sin\theta$)

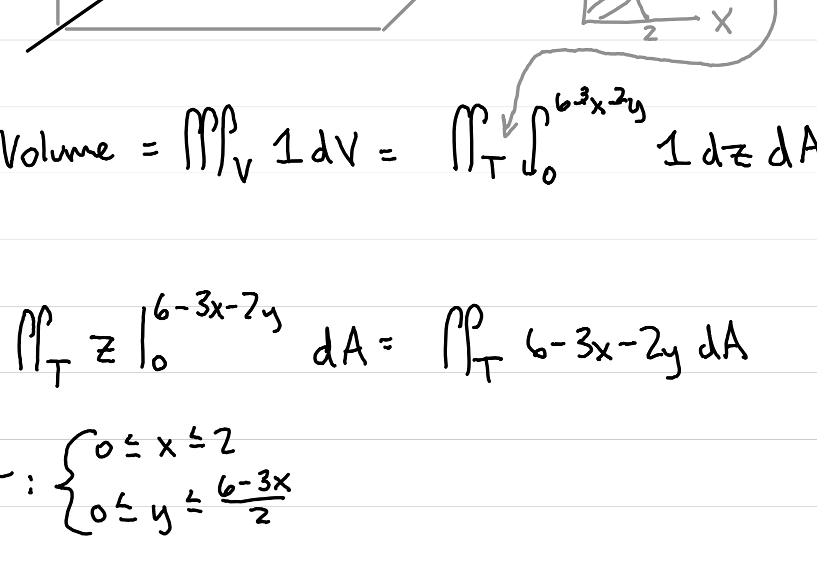
... Plug in then int again

$= \frac{32}{3}\sin\theta + 4\frac{\cos(2\theta)}{2}\Big|_0^{\frac{\pi}{2}} = \frac{32}{3} + 2(-1-1) = \frac{32}{3} - 4 = \boxed{\frac{20}{3}}$

$\iint_D 1\,dA = \text{Area}(D)$ ★

$\iiint_V 1\,dV = \text{Volume}(V)$

#3] Find vol. of tetrahedron bounded by the reference planes and the plane $3x+2y+z=6$



$\text{Volume} = \iiint_V 1\,dV = \iint_T \int_0^{6-3x-2y} 1\,dz\,dA$

$= \iint_T z\Big|_0^{6-3x-2y} dA = \iint_T 6-3x-2y\,dA$

T: $\begin{cases} 0 \le x \le 2 \\ 0 \le y \le \frac{6-3x}{2} \end{cases}$

$= \int_0^2 \int_0^{\frac{6-3x}{2}} (6-3x)-2y\,dy\,dx$

$= \int_0^2 (6-3x)y - y^2\Big|_0^{\frac{6-3x}{2}} dx$

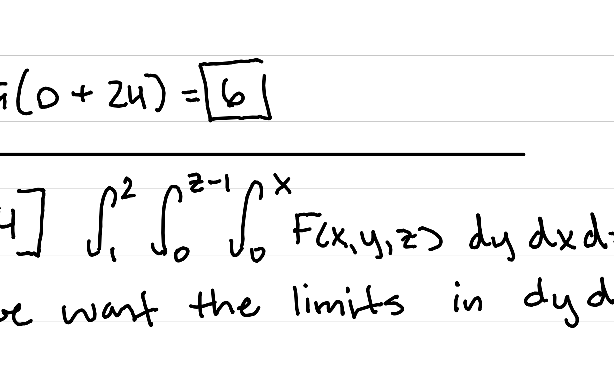
$= \int_0^2 \frac{(6-3x)(6-3x)}{2} - \frac{(6-3x)^2}{4}dx = \int_0^2 \frac{(6-3x)^2}{4}dx$

$= \frac{1}{4}\int_0^2 36 - 36x + 9x^2\,dx = \frac{1}{4}\left(36x - 36\frac{x^2}{2} + 9\frac{x^3}{3}\right)\Big|_0^2$

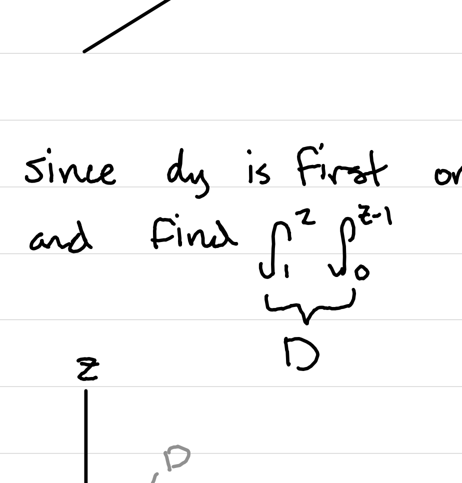
$= \frac{1}{4}(0+24) = \boxed{6}$

#4] $\int_1^2 \int_0^{z-1} \int_0^x F(x,y,z)\,dy\,dx\,dz$

we want the limits in dydzdx order



since dy is first on both, keep $0 \le y \le x$ and find $\underbrace{\int_1^2 \int_0^{z-1}}_{D}$ ★



Re-write:

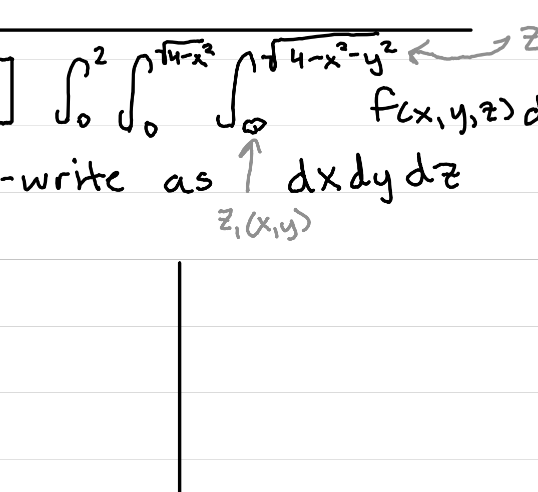
$\begin{cases} 0 \le x \le 1 \\ x+1 \le z \le 2 \end{cases}$

$= \boxed{\int_0^1 \int_{x+1}^2 \int_0^x F\,dy\,dz\,dx}$

#5] $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$ (upper: $z_2(x,y)$, lower: $z_1(x,y)$)

re-write as dxdydz

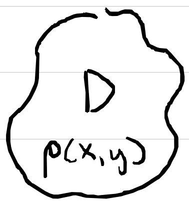
$\int_0^{\sqrt{4-x^2}}$ (upper: $y_2(x)$, lower: $y_1(x)$)



$= \int_e^f \int_{y_1(z)}^{y_2(z)} \int_{x_1(y,z)}^{x_2(y,z)} F(x,y,z)\,dx\,dy\,dz = \text{a number}$

12.6 Geometry n' physics stuff

(we will work in 2D)



Mass ★

density of $\rho(x,y) = \lim_{\Delta V \to 0} \frac{\Delta m(x,y)}{\Delta V(x,y)}$ (mass)

$$\text{mass} = \iint_D \rho \, dS \qquad \text{mass} = \iiint_V \rho \, dV$$

Center of Mass ★

(3D examples)

$$X_G = \frac{\iiint x\rho \, dV}{\iiint \rho \, dV} \quad \iiint \rho \, dV = m$$

$$Y_G = \frac{\iiint y\rho \, dV}{m}$$

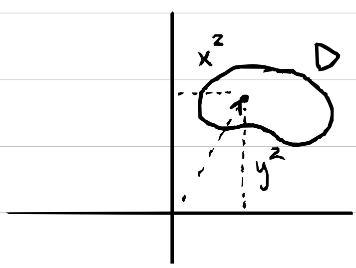
$$Z_G = \frac{\iiint z\rho \, dV}{m}$$

Moment of Inertia ★

$$I_x = \iint_D y^2 \rho \, dV$$

$$I_y = \iint_D x^2 \rho \, dV$$

$$I_z = \iint_D (x^2 + y^2) \rho \, dV = I_x + I_y$$



12.7 Cylindrical + Spherical Coords

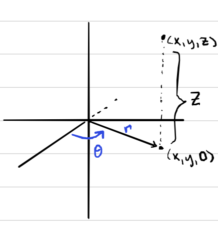
Cylindrical

is polar coords + z

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$\begin{cases} r^2 = x^2+y^2 \\ \tan\theta = \frac{y}{x} \\ z = z \end{cases}$

$dV = dA\,dz = r\,dr\,d\theta\,dz$ ★

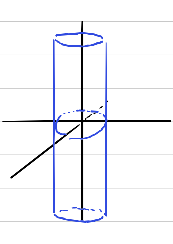


★

$r = a,\ \theta = \theta_0,\ z = c$

circle — a plane that shows a line on xy-plane with inclination to θ_0 — horizontal plane

these make a cylinder w/ radius a



Box in Cylindrical Coords

$\begin{cases} \theta_1 \le \theta \le \theta_2 \\ a \le r \le b \\ c \le z \le d \end{cases}$

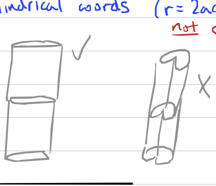
★

Circular in cylindrical coords (x^2+y^2)

$z = 4x^2 + 4y^2 = 4(x^2+y^2)$
↓
$z = 4r^2$ these are easy to do in cyl. coords.

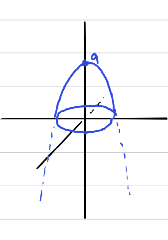
$z = \sqrt{x^2+y^2}$ $\quad z = \sqrt{r^2} = r$

* when everything is symmetric about the z-axis, cylindrical coords. is easy
* things that are easy in polar are easy in cylindrical coords ($r = 2a\cos\theta$, $r = 2a\sin\theta$, etc.)
  not cardioids rip



Cylindrical Coord. Examples

#1] $\iiint_V 1\,dV$ where $V: \begin{cases} z \ge 0 \\ z \le 9 - x^2 - y^2 \end{cases}$



1. solve w/ rectangular then go polar
OR
2. cylindrical coords (we're doing #2)

$= \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} r\,dz\,dr\,d\theta$

1. Get coords ⟶ $\begin{cases} z \ge 0 \\ z \le 9 - r^2 \end{cases}$

2. Find the projection

$\begin{cases} z = 0 \\ z = 9 - r^2 \end{cases}$ $\quad 0 = 9 - r^2 \quad r^2 = 9 \quad \boxed{r = 3}$

3. Do it

$I = \int_0^{2\pi} \int_0^3 \int_0^{9-r^2} r\,dz\,dr\,d\theta$

$= \int_0^{2\pi} \int_0^3 rz \Big|_0^{9-r^2} dr\,d\theta$

$= \int_0^{2\pi} \int_0^3 r(9 - r^2 - 0)\,dr\,d\theta = \int_0^{2\pi} \int_0^3 9r - r^3\,dr\,d\theta$

$= \int_0^{2\pi} \frac{9r^2}{2} - \frac{r^4}{4} \Big|_0^3 d\theta$

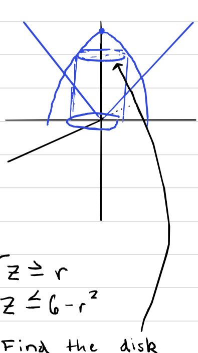
$= \int_0^{2\pi} \left[\frac{81}{2} - \frac{81}{4}\right] - 0\ d\theta$ $\quad \left(\frac{81}{4}\right)$

$= \frac{81}{4}\theta \Big|_0^{2\pi} = \boxed{\frac{81}{2}\pi}$

#1 but more complicated

#2] $\iiint_V z(x^2+y^2)\,z\,dV$

$\begin{cases} z \ge \sqrt{x^2+y^2} \\ z \le 6 - x^2 - y^2 \end{cases}$



bad drawing, my bad

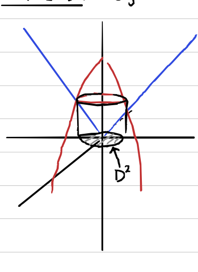
$\begin{cases} z \ge r \\ z \le 6 - r^2 \end{cases}$

1. Find the disk

$\begin{cases} z = r \\ z = 6 - r^2 \end{cases} = r^2 + r - 6 = 0$

never finished

11-1-2024 Cyl. coord. ex. $\iiint_V z(x^2+y^2)\,dV$



$V: \begin{cases} z \le 6 - x^2 - y^2 \\ z \ge \sqrt{x^2+y^2} \end{cases}$

$\to \begin{cases} z \le 6 - r^2 \\ z \ge \sqrt{r^2} = r \end{cases}$

$\begin{cases} z = 6 - r^2 \\ z = r \end{cases}$ $\quad r^2 + r - 6 = 0$

$r = \frac{-1 \pm \sqrt{1^2 + 24}}{2}$ ⟶ 2, −3

$I = \int_0^{2\pi} \int_0^2 \int_{z=r}^{z=6-r^2} zr^2\ r\,dz\,dr\,d\theta$

$= \int_0^{2\pi} \int_0^2 \frac{r^3 z^2}{2} \Big|_r^{6-r^2} dr\,d\theta$

$= \frac{1}{2} \int_0^{2\pi} \int_0^2 r^3\left[(6-r^2)^2 - r^2\right] dr\,d\theta$ $\quad (36 - 12r^2 + r^4)$

$= \frac{1}{2} \int_0^{2\pi} \int_0^2 36r^3 - 13r^5 + r^7\,dr\,d\theta$

$= \frac{1}{2} \int_0^{2\pi} 9r^4 - \frac{13}{6}r^6 + \frac{1}{8}r^8 \Big|_0^2 d\theta$ $\quad (2^{-3},\ 2^5)$

$= \frac{1}{2} \int_0^{2\pi} 9(16) - \frac{13}{6}(64) + (32)\ d\theta$

$= \boxed{\frac{1}{2}\left[9(16) - \frac{13}{6}(64) + 32\right] 2\pi}$

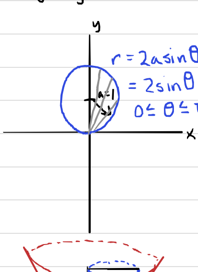
Cyl. coord. Ex. #2] Find volume of V satisfying

$V: \begin{cases} x \ge 0,\ y \ge 0,\ z \ge 0 & \text{first octant} \\ x^2 + y^2 - 2y \le 0 & \text{cylinder} \\ z \le \sqrt{x^2+y^2} & \text{cone} \end{cases}$

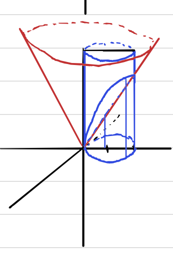
$\text{Vol}(V) = \iiint_V 1\,dV$

complete the square

$x^2 + (y^2 - 2y + 1) = 1 \to x^2 + (y-1)^2 = 1$



$r = 2a\sin\theta$
$= 2\sin\theta$
$0 \le \theta \le \pi$



$\begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 2\sin\theta \\ 0 \le z \le r \end{cases}$

$\text{Volume}(V) = \int_0^{\frac{\pi}{2}} \int_0^{2\sin\theta} \int_0^r r\,dz\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}} \int_0^{2\sin\theta} r^2\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{r^3}{3} \Big|_0^{2\sin\theta} d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{8}{3}\sin^3\theta\,d\theta$

$\sin\theta\sin^2\theta$
$\sin\theta(1 - \cos^2\theta)$

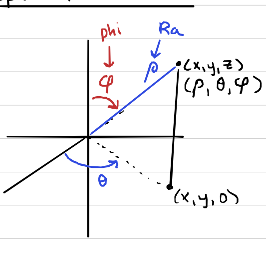
$\left[\ u = \cos\theta \quad du = -\sin\theta\,d\right.$

$= \frac{8}{3} \int_0^{\frac{\pi}{2}} \sin\theta - \sin\theta\cos^2\theta\,d\theta$ $\quad u^2$, $\frac{u^3}{3}$

$= \frac{8}{3}\left[-\cos\theta + \frac{\cos^3\theta}{3} \Big|_0^{\frac{\pi}{2}}\right]$

$= -\frac{8}{3}\left[-1 + \frac{1}{3}\right] = \boxed{\frac{16}{9}}$

Spherical Coordinates



φ a.k.a azimuth

both are phi
$\phi = \varphi$

uh oh

$\begin{cases} x = r\sin\theta = \rho\cos\varphi\cos\theta \\ y = r\sin\theta = \rho\sin\varphi\sin\theta \\ z = \rho\cos\theta \end{cases}$

$r = \rho\sin\varphi$ ★

↓

$\begin{cases} \rho^2 = r^2 + z^2 = x^2 + y^2 + z^2 \\ \tan\theta = \frac{y}{x} \\ \cos\varphi = \frac{z}{\rho} = \frac{z}{\pm\sqrt{x^2+y^2+z^2}} \end{cases}$

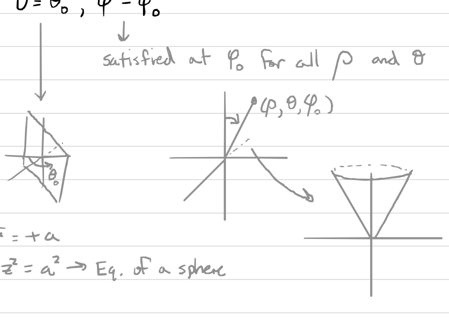
We sometimes make these restrictions:

$\begin{cases} \rho \ge 0 \to \cos\varphi = \frac{z}{\sqrt{x^2+y^2+z^2}} \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{cases}$

These mean each point is only taken once

$\rho = a,\ \theta = \theta_0,\ \varphi = \varphi_0$

satisfied at φ_0 for all ρ and θ



$\sqrt{x^2+y^2+z^2} = +a$
$x^2 + y^2 + z^2 = a^2$ → Eq. of a sphere

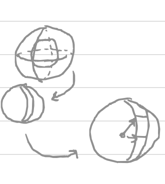
Spheres centered at origin are easiest in spherical coords

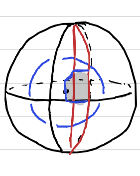
Interesting cases: ★

$\varphi = 0$ → positive z-axis
$\varphi = \frac{\pi}{2}$ → xy-plane
$\varphi = \pi$ → negative z-axis

Box:

$\begin{cases} a \le \rho \le b & \text{spherical shell} \\ \theta_0 \le \theta \le \theta_1 & \text{slice vertically} \\ \varphi_0 \le \varphi \le \varphi_1 & \text{horz. slice} \end{cases}$





$dV = \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$ ★

* Ask Prof. about this for more visual detail

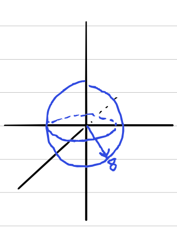
Examples:

#1] Find volume w/ spherical coords of the sphere of radius 8

$V = \frac{4}{3}\pi a^3$

$\text{Volume} = \iiint_V 1\,dV$

this is the simplest example



$\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \\ 0 \le \rho \le 8 \end{cases}$

$= \int_0^{2\pi} \int_0^{\pi} \int_0^8 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

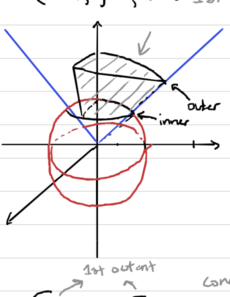
$= \int_0^{2\pi} \int_0^{\pi} \frac{\rho^3}{3}\Big|_0^8 \sin\phi\,d\phi\,d\theta$

$= \frac{a^3}{3} \int_0^{2\pi} -\cos\phi \Big|_0^{\pi} d\theta$

$= \frac{2}{3}a^3(2\pi) = \frac{4}{3}\pi a^3$ □

#2] $\iiint_V \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV$ where

$V: \begin{cases} x^2+y^2+z^2 \le 4 & \text{inside sphere } a=2 \\ x^2+y^2+z^2 \ge 1 & \text{outside sphere } a=1 \\ z \ge \sqrt{x^2+y^2} & \text{above cone} \\ x \ge 0,\ y \ge 0,\ z \ge 0 & \text{1st octant} \end{cases}$



1st octant — cone is at 45°

$\begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le \phi \le \frac{\pi}{4} \\ 1 \le \rho \le 2 \end{cases}$ ↳ radii

$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_1^2 \frac{1}{\sqrt{\rho^2}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$ — simplify

$f(\rho)\,g(\phi)\,h(\theta)$

$= \int_0^{\frac{\pi}{2}} 1\,d\theta \int_0^{\frac{\pi}{4}} \sin\phi\,d\phi \int_1^2 \rho\,d\rho$

since each is just $\int$ number, this is = the product of

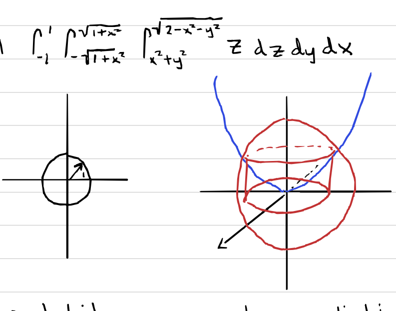
$= \frac{\pi}{2}\left[-\cos\phi \Big|_0^{\frac{\pi}{4}}\right]\left[\frac{\rho^2}{2}\Big|_1^2\right]$

$= \boxed{\frac{\pi}{2}\left[1 - \frac{\sqrt{2}}{2}\right]\left(\frac{3}{2}\right)}$

If you would get a function for a limit of int., do not use spherical coords

#3] Choose cylindrical or spherical:

Eval $\int_{-1}^{1} \int_{-\sqrt{1+x^2}}^{\sqrt{1+x^2}} \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} z\,dz\,dy\,dx$



the paraboloid means we choose cylindrical

$\int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} z\,dz\,dr\,d\theta$ ★

spherical would be tricky because you split into like 3 regions because of the cone

$z = x^2 + y^2 \to \rho\sin\phi = \rho^2\cos\phi^2$

$\rho = \frac{\sin\phi}{\cos\phi}$ $\quad$ nightmare.

12.8 Jacobian

$(x,y) \to (r,\theta)$ $\quad dA = dxdy \to r\,drd\theta$

$(x,y,z) \to (r,\theta,z)$ $\quad dr = dxdydz \nearrow r\,dzdrd\theta$

$\searrow (\rho,\theta,\varphi)$ $\quad \searrow \rho^2\sin\varphi\, d\rho d\varphi d\theta$

The circled parts are the Jacobians for these reference systems

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$(x,y) \to (u,v)$

$\begin{cases} x(u,v) \\ y(u,v) \end{cases}$ Imagine we are moving this to another system

Jacobian matrix $(J_m) = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$

$J = |\text{determinate}(J_m)|$

let's check w/ polar coords. $(x,y) \to (r,\theta)$

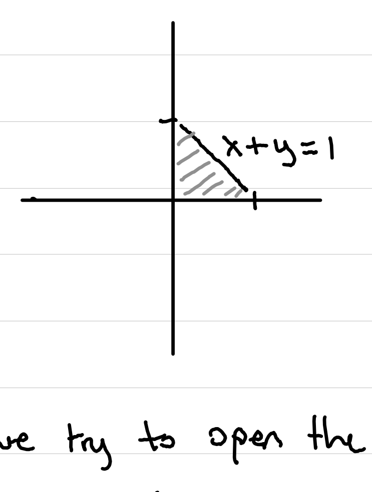
$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad J_m = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$

$J = |r\cos^2\theta - -r\sin^2\theta| = |r| \longrightarrow r\,drd\theta$

EXAMPLES

#1] Compute $\iint_D \left(\frac{x-y}{x+y}\right)^4 dydx$ where

D:



If we try to open the formula up, it no work

So we say:

$u = x-y \qquad v = x+y$

To transform like this, we need to:

1. get x + y in terms of u, v

$$\begin{aligned} u &= x-y \\ + \; v &= x+y \\ \hline u+v &= 2x \end{aligned} \qquad x = \frac{u+v}{2} = \frac{u}{2} + \frac{v}{2}$$

$$\begin{aligned} -\,u &= x-y \\ v &= x+y \\ \hline v-u &= 2y \end{aligned} \qquad y = \frac{v-u}{2} = \frac{v}{2} - \frac{u}{2}$$

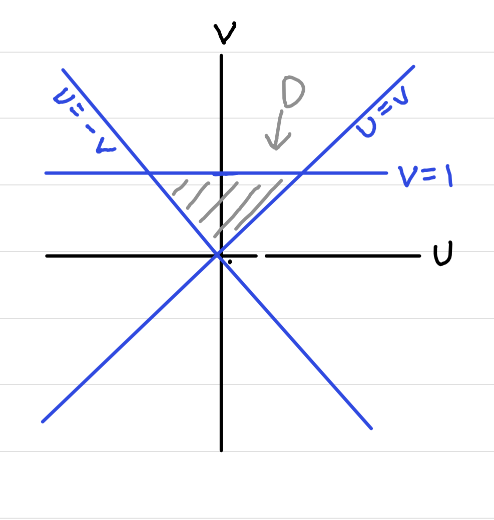
2. Do J_m

$J_m = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ $\quad J = |\det(J_m)| = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

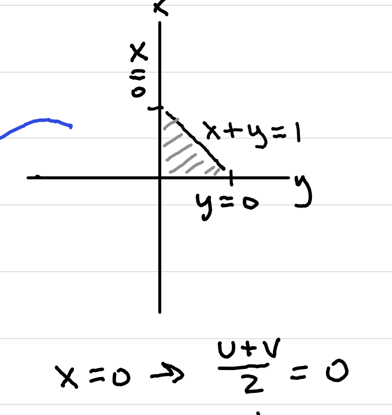
$\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$

$dA = \boxed{\frac{1}{2}du\,dv}$

3. Figure out the domain D in the (u, v) reference system



original:



$x = 0 \to \frac{u+v}{2} = 0 \to u = -v$

$y = 0 \to \frac{u-v}{2} = 0 \to u = v$

$x+y=1 \to v = 1$

4. Dew it

$\begin{cases} 0 \le v \le 1 \\ -v \le u \le v \end{cases}$ u-simple

← from original eq.

$I = \int_0^1 \int_{-v}^{v} \frac{u^4}{v^4} \frac{1}{2}\, du\,dv$

↑ Jacobian

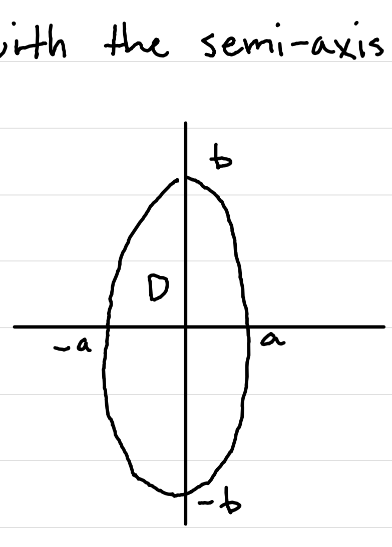
$= \frac{1}{2}\int_0^1 \frac{1}{v^4} \left.\frac{u^5}{5}\right|_{-v}^{v} dv$

simplify

$= \frac{1}{2}\int_0^1 \frac{1}{v^4} \frac{v^5 - (-v^5)}{5} dv \to \frac{v^5}{v^4}$ $\quad (2v^5)$

$= \frac{1}{5}\int_0^1 v\,dv = \frac{1}{5}\left.\frac{v^2}{2}\right|_0^1 = \boxed{\frac{1}{10}}$

#2] Find formula for area of an ellipse with the semi-axis a, b



Area $= \iint_D 1\,dA$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

we want to

↓ Reshape the ellipse to look like a unit circle

$u: \frac{x}{a} \qquad v: \frac{y}{b} \to u^2 + v^2 = 1$

This simplifies the region, not the formula

$x = au \qquad y = bv$

Area $= \iint_D 1\,dA$

$= \iint_{Ball(1)} J\,du\,dv$

(Ball(1): circle of r=1; J: some Jacobian)

$J_m = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \to J = |ab| = ab$

$= \iint_{Ball(1)} ab\,du\,dv = ab\pi(1)^2 = \boxed{\pi ab}$

We have no real way to know how the Jacobian matrix should be structured

WebWork notation:

$\frac{\partial xy}{\partial uv}$ ← this fixes the matrix to $\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$

$\frac{\partial xy}{\partial vu} \to \begin{bmatrix} x_v & x_u \\ y_v & y_u \end{bmatrix}$ ← opposite sign

★

Jacobian Inverse

we have:

$x(u,v)$
$y(u,v)$ $\quad J$

we want:

$u(x,y)$
$v(x,y)$ $\quad J^{-1}$

$$J^{-1} = \frac{1}{J}$$

we know:

$(x,y) \to (r,\theta) \quad J = r$

$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases} \to J^{-1} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \frac{1}{J} = \frac{1}{r}$

$= \frac{1}{\sqrt{x^2+y^2}}$

$\left|\frac{\partial uv}{\partial xy}\right| = J^{-1}$

Now in 3D:

$(x,y,z) \to (u,v,w)$

$J = \left|\frac{\partial xyz}{\partial uvw}\right| = \left|\begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}\right|$

Cylindrical:

$\begin{cases} x = r\cos\theta \\ y = r\cos\theta \\ z = z \end{cases}$

$J_m = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$J = |r\cos^2\theta - (-r\sin\theta)\sin\theta| = r$

Spherical:

$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\theta \end{cases}$ trust it.