11.1

$z = f(x,y)$ ex. $z = x^2 + y^2$

The domain of this is $D \subseteq \mathbb{R}^2$, $R \subseteq \mathbb{R}$ → Range

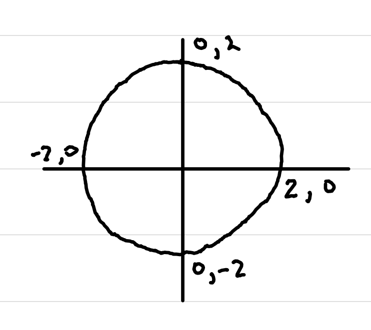
$T = g(x,y,z) \to T = x^2 + y^2 + z^2$ ★
$D \subseteq \mathbb{R}^3$ $R \subseteq \mathbb{R}$ → Range

$T = g(x,y,z)$ is a rule that for all (x,y,z) in the domain D associates a unique value in R

If domain is not given, assume it is the largest set for which the function is defined

ex. $z = x^2 + y^2$ (x,y) in $x^2 + y^2 \leq 4$

1. $x^2 + y^2 = 4$
2. $x^2 + y^2 \leq 4$ ★ → is everything <u>inside</u> the circle



Range?
1. find minimum (lower bound)

$x^2 + y^2 \geq 0$ so $z = 0$ ★

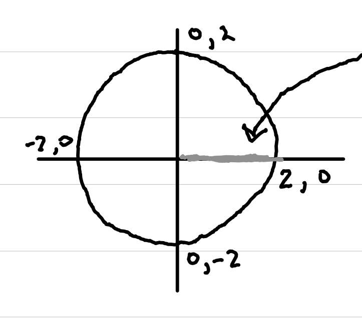
2. find maximum (upper bound)

every point on the circle goes to $z = 4$

so $\boxed{R = [0, 4]}$

$z(x,0) = x^2$ on $0 \leq x \leq 2$

we get this from the line segment here



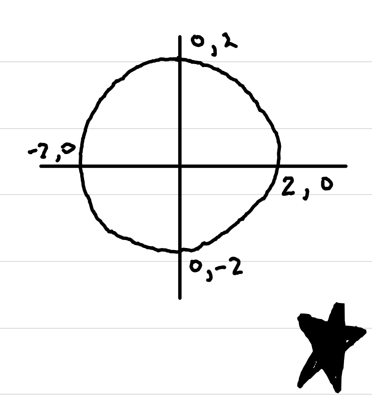
with just $z = x^2 + y^2$,
D: for all (x,y) in $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$
R: $[0, +\infty)$

$z = \frac{1}{\sqrt{4 - x^2 - y^2}}$ find the domain

to make this work, $4 - x^2 - y^2 > 0$
(not $\geq$ because $\frac{1}{0}$)

$\boxed{x^2 + y^2 < 4}$

Find range



★

$z(x,0) = \frac{1}{\sqrt{4 - x^2}}$

$x = 0 \to \frac{1}{2}$

$x = 2 \to \frac{1}{0}$

$\boxed{R: [\frac{1}{2}, +\infty)}$

<u>Implicit Functions</u>

$F(x,y,z) = C$ ex. $x^2 + y^2 + z^2 = 4$

no variables are shown as dependent or independent

$x^2 + y^2 + z^2 = 4 \to z = \pm\sqrt{4 - x^2 - y^2}$
$x^2 + y^2 - z = 4 \to z = x^2 + y^2 - 4 = f(x,y)$
$x^2 + y^2 + z^2 = -4$

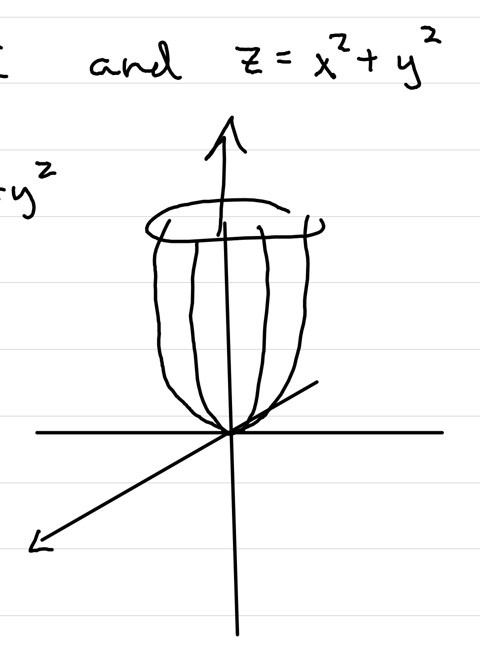
this is now explicit

not a function because two values on same z

$x^2 + y^2 + z^2 + \tan z^3 + e^z = 9$
↓
this gets shady

$F(x,y,z) = C$ and $z = x^2 + y^2$ in D

<u>graph</u> $z = x^2 + y^2$



★

a function like $F(x,y,z) = C$ still lives in 3D space alongside normal explicit functions

$T = x^2 + y^2 + z^2$ $\underbrace{D \subseteq \mathbb{R}^3 \quad R \subseteq \mathbb{R}}_{\text{this is in 4D}}$

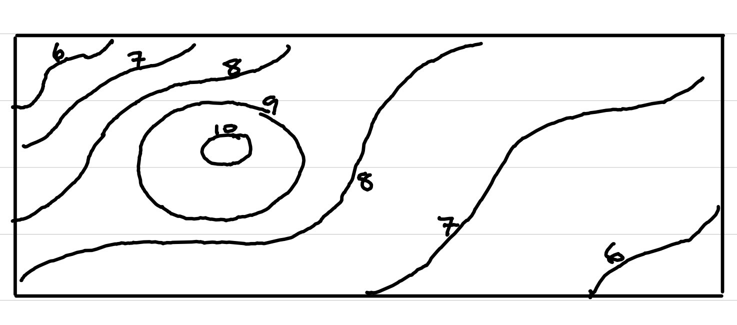
R: $[a, b] \to \left[\begin{matrix}\text{cold} \\ \text{blue}\end{matrix} \to \begin{matrix}\text{hot} \\ \text{red}\end{matrix}\right]$
↓
this is an example of using color maps to show values in another dimension

$T = (x^2 + y^2 + z^2)t^2 \to \mathbb{R}^5$
↓
we would add movement of time to visualize this

$F(x,y,z,t) = C$

(we will be working in 3D space)

<u>Pre-Computer Visualization (Level set curves)</u>
AKA
<u>Contour maps</u>



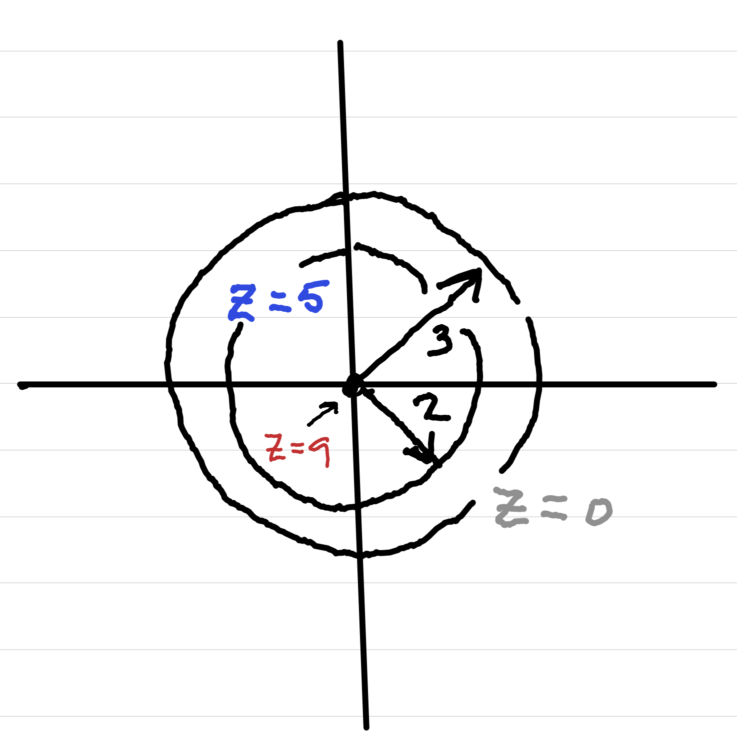
#'s by curves show the value taken on that curve, like topography

$z = 9 - x^2 - y^2$, D: $x^2 + y^2 \leq 9$
R: $[0, 9]$

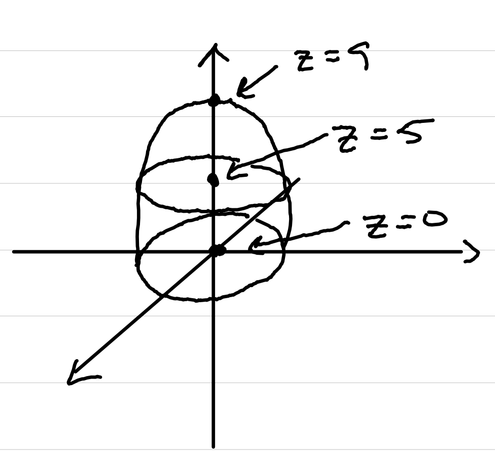
start at $z = 0$
★ $0 = 9 - x^2 - y^2 \to x^2 + y^2 = 9$

$z = 5$ ★
$5 = 9 - x^2 - y^2$ | $x^2 + y^2 = 4$

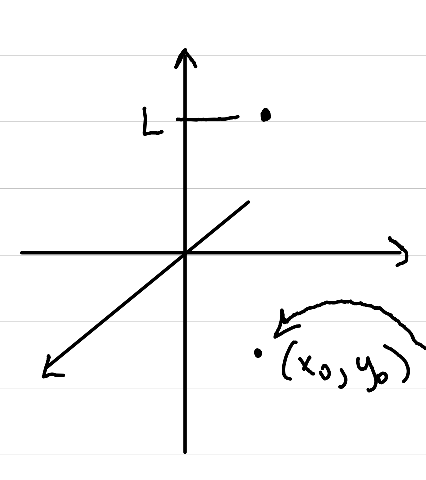
$z = 9$ ★
$9 = 9 - x^2 - y^2$ $x^2 + y^2 = 0$



this translates to a picture of a paraboloid:

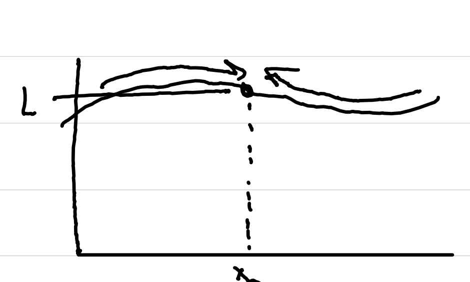


$\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L$ ★



we care how f(x,y) behaves here

no matter how we approach, it must go to L much like in Cal 2 we care how it approaches, not if it's defined



in Cal 3, there are infinitely many lines that hit (x_0, y_0) and it doesn't have to be a straight line ★

Trajectory <u>can only prove</u> the limit <u>does not exist</u>

11.2

$\lim_{(x,y)\to(2,2)} e^{xy}$

1. check if $(2,2)$ is in the domain of the function

   D: $\mathbb{R}^2$ because e^{xy} defined everywhere

2. get lim

   $= e^{(2)(2)} = \boxed{e^4}$

$\lim_{(x,y)\to(0,0)} \frac{x+1}{y}$

1. except for

   D: (x,y) in $\mathbb{R}^2$ / $y = 0$

$= \frac{1}{0}$ = undefined, $\boxed{\text{does not exist (DNE)}}$

$\lim_{(x,y)\to(0,0)} \frac{x^2+x-xy-y}{x-y}$ $x \neq y$

$= \frac{0}{0}$, more work needed

$= \frac{x(x+1)-y(x+1)}{x-y} = \lim_{(x,y)\to(0,0)} \frac{\cancel{(x-y)}(x+1)}{\cancel{(x-y)}}$

$\lim_{(x,y)\to(0,0)} x+1 = \boxed{1}$

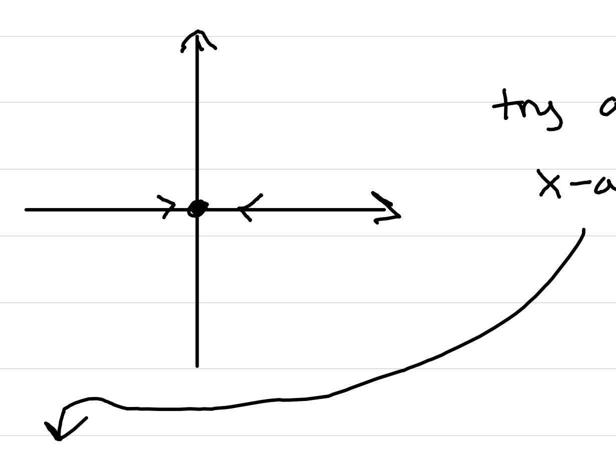
★★★

L'HOPITAL RULE DOES NOT WORK HERE!!!

Example

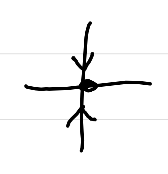
Show $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ DNE

$= \frac{0}{0}$, more work needed



try approaching w/ x-axis, $y=0$

$\lim_{\substack{x\to0\\y=0}} \frac{2x(0)}{x^2+0^2} = 0$

then try w/ y-axis 

$\lim_{\substack{y\to0\\x=0}} \frac{2y(0)}{0^2+y^2} = 0$

try $y = x$?

$\lim_{\substack{x\to0\\y=x}} \frac{2x(x)}{x^2+x^2} = \frac{2x^2}{2x^2} = 1$ ★

since we found 2 different limits, the limits $\boxed{\text{does not exist}}$

Trick:

$\lim_{\substack{x\to0\\y=mx}} \frac{2x(mx)}{x^2+mx^2} = \frac{2m}{1+m^2} = L(m)$

since this is many diff. things, the limit $\boxed{\text{DNE}}$

Example Disprove existence

$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \frac{0}{0}$, more work needed

$y = mx$

$\lim_{\substack{x\to0\\y=mx}} \frac{x^2(mx)}{\underbrace{x^4+m^2x^2}_{x^2(x^2+m^2)}} = \lim_{x\to0} \frac{mx}{x^2+m^2} = \frac{0}{m^2} = 0$

all $y=mx$ go to zero, but this doesn't mean L exists, because trajectory cannot prove

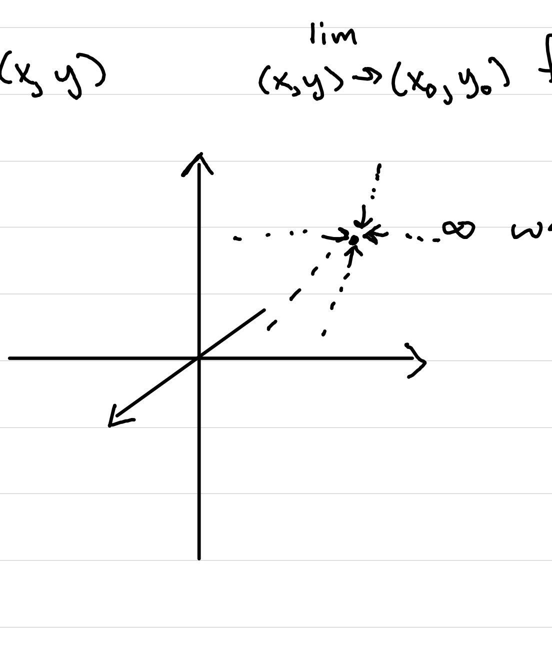
try $y=x^2$ (parabola)

$\lim_{\substack{x\to0\\y=x^2}} \frac{x^2(x^2)}{x^4+x^4} = \frac{x^4}{2x^4} = \frac{1}{2}$

2 limits so limit $\boxed{\text{DNE}}$

9-25-24 summary

$z = f(x,y)$ $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

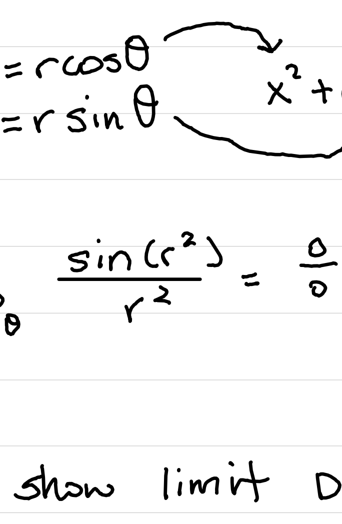


∞ ways to get this point

ex.

Assume the limit exists, use polar coordinates to find it

$\lim_{(x,y)\to(x_0,y_0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \frac{\sin(0)}{0} = \frac{0}{0}$ X



$r=0$ for all θ

1. put the limit to a single variable r

   $x = r\cos\theta$
   $y = r\sin\theta$
   $x^2+y^2 = r^2$

$\lim_{\substack{r\to0\\\text{for all }\theta}} \frac{\sin(r^2)}{r^2} = \frac{0}{0} \ldots = \boxed{1}$

★

Ex. show limit DNE w/ polar coordinates

$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

$\lim_{\substack{r\to0\\\text{for all }\theta}} \frac{\cancel{r}\cos\theta\,\cancel{r}\sin\theta}{\cancel{r^2}} = \frac{1}{2}\sin(2\theta) = L(\theta)$

Limit depends on θ, thus it changes.

since it changes, it $\boxed{\text{DNE}}$

---

3 conditions for Continuity ★★★

$F(x,y)$ is cont. at (x_0,y_0) IF:

i) (x_0,y_0) is in D $\to$ $F(x_0,y_0)$ is defined

ii) $\lim_{(x,y)\to(x_0,y_0)} F(x,y) = L$ (limit exists)

iii) $F(x_0,y_0) = L$

All 3 combined:

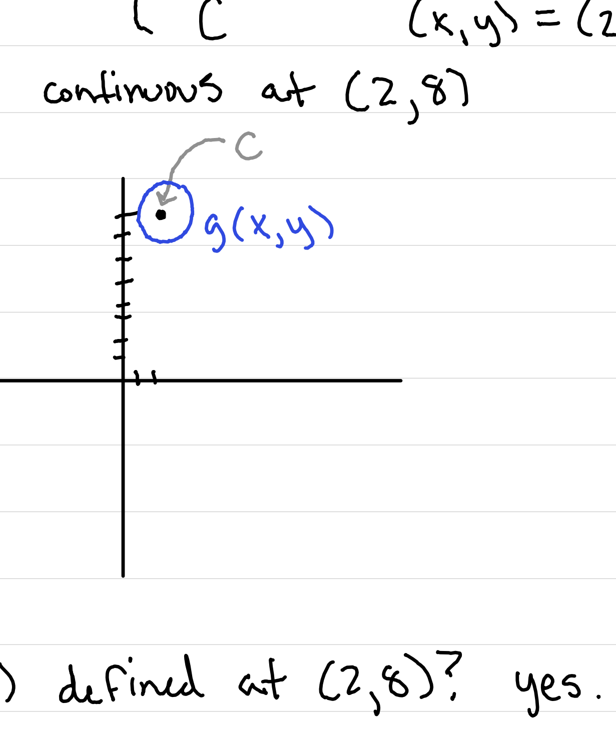
$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

Piecewise + Continuity

Ex. Find the constant C such that

$$F(x,y) = \begin{cases} \underbrace{\frac{4x-y}{\sqrt{4x-y+4}-2}}_{g(x,y)} & (x,y) \neq (2,8) \\ C & (x,y) = (2,8) \end{cases}$$

is continuous at $(2,8)$



i) defined at $(2,8)$? yes. it = C

ii) limit exists?

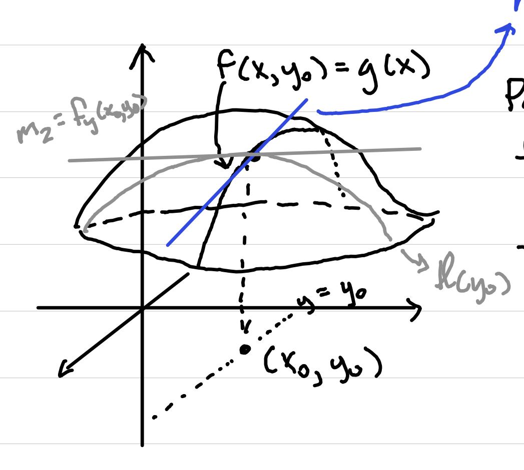
$\lim_{(x,y)\to(2,8)} \frac{4x-y}{\sqrt{4x-y+4}-2} = \frac{8-8}{\sqrt{8-8+4}-2} = \frac{0}{0}$ X

a − b so mult by a+b for a^2-b^2

$= \lim \frac{4x-y \quad (\sqrt{4x-y+4}+2)}{(\sqrt{4x-y+4}-2)(\sqrt{4x-y+4}+2)}$

$= \lim_{(x,y)\to(2,8)} \frac{\cancel{4x-y}(\sqrt{4x-y+4}+2)}{\cancel{4x-y}+\cancel{4}-\cancel{4}} = \boxed{4}$

11.3



$m_1 = f_x(x_0,y_0)$

Partial derivative of $f(x,y)$ in the direction x at (x_0, y_0)

we freeze y so that we see what it does on the x

Partial deriv.

$$\frac{\partial f}{\partial x}(x_0,y_0) = f_x(x_0,y_0) = \partial_x f(x_0,y_0)$$

$$= \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

$$= \lim_{x\to x_0} \frac{f(x,y_0) - f(x_0,y_0)}{x - x_0}$$

$$\lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h} = g'(x_0)$$ ★

$$\frac{\partial f}{\partial y}f(x_0,y_0) = f_y(x_0,y_0) = \partial_y f(x_0,y_0)$$

$$= \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$ this is freezing x instead

$$= \lim_{h\to 0} \frac{\ell(y_0+h) - \ell(y_0)}{h} = \ell'(y_0)$$ ★

Example

$z = f(x,y) = x^3y + 3xy^2 + y^4$

find $f_x(1,2)$ and $f_y(0,3)$

1. $f_x(x,y) \to$ treat y like a constant

$f_x(x,y) = 3x^2y + 3y^2 \cancel{+0}$

$f_x(1,2) = 3(1)^2(2) + 3(2)^2$

$6 + 12 = \boxed{18}$

2. $f_y(x,y) \longrightarrow$ treat x like constant

$f_y(x,y) = x^3 + 6xy + 4y^3$

$f_y(0,3) = 0^3 + 6(0)(3) + 4(3)^3$

$0 + 0 + 108 = \boxed{108}$

Linearity Rule ★

$f(x,y) \quad g(x,y) \quad a \quad b$

$\partial_{x \text{ or } y}(af(x,y) + bg(x,y))$

$= af_{x \text{ or } y}(x,y) + bg_{x \text{ or } y}(x,y)$

Product Rule ★

$\partial_{x \text{ or } y}(f(x,y)\,g(x,y))$

$= f_{x \text{ or } y}(x,y)\,g(x,y) + f(x,y)\,g_{x \text{ or } y}(x,y)$

Quotient Rule ★

$\partial_{x \text{ or } y}\left(\frac{f(x,y)}{g(x,y)}\right)$

$$= \frac{f_{x \text{ or } y}(x,y)\,g(x,y) - f(x,y)\,g_{x \text{ or } y}(x,y)}{(g(x,y))^2}$$

Partial derivatives of implicit functions

$F(x,y,z) = C \qquad x^2 + x^2yz + z^3 = 7$

Find z_x, z_y $\longrightarrow$ this means z is the dependent $(z(x,y))$

$\partial_x(F(x,y,z))$

$\partial_x(x^2 + x^2y\,z(x,y) + z^3(x,y) = 7)$

Product Rule — Regular chain rule

$= 2x + y\left[2xz + x^2z_x\right] + 3z^2(z_x) = 0$

z_x: short for $z(x,y)$

move over everything w/out z_x

$z_x(yx^2 + 3z^2) = -2x - 2xyz$

$$\boxed{z_x = \frac{-2x - 2xyz}{yx^2 + 3z^2}}$$

$\partial_y(x^2 + x^2yz + z^3 = 7)$

$= 0 + x^2[z + yz_y] + 3z^2z_y = 0$

$$\boxed{z_y = \frac{-x^2z}{x^2y + 3z^2}}$$

Multi-order derivatives

I order | II order | III order

$f \xrightarrow{\partial_x} f_x \xrightarrow{\partial_x} f_{xx} \longrightarrow = \frac{\partial^2 f}{\partial x^2} = \partial_{xx} f$

$f_x \xrightarrow{\partial_y} f_{xy} \longrightarrow = \frac{\partial^2 f}{\partial y \partial x} = \partial_{xy} f$

$f \xrightarrow{\partial_y} f_y \xrightarrow{\partial_x} f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \partial_{yx} f$

$f_y \xrightarrow{\partial_y} f_{yy} = \frac{\partial^2 f}{\partial y^2} = \partial_{yy} f$

There is a theorem that says ★

$f_{xy} = f_{yx} \qquad f_{xyx} = f_{yxx} = f_{xxy}$

order doesn't matter, the count does.

Example

$F(x,y) = \cos(x^2y + x^3)$

Find all 4 II order derivs.

$f_x(x,y) = -\sin(x^2y+x^3)[2xy + 3x^2]$

$f_y(x,y) = -\sin(x^2y+x^3)[x^2]$

$f_{xx} = -\cos(x^2y+x^3)[2xy+3x^2]^2 - \sin(x^2y+x^3)(2y+6x)$

$f_{xy} = -\cos(x^2y+x^3)(2xy+3x^2)x^2 - \sin(x^2y+x^3)2x$

$\Updownarrow$

$f_{yx} = -\cos(x^2y+x^3)(2xy+3x^2)x^2 - \sin(x^2y+x^3)2x$

$f_{yy} = -\cos(x^2y+x^3)[x^2]^2$

9-27-24

has 3 I-order derivatives

$T(x,y,z) = e^{x^2yz}$

$T_x = e^{x^2yz^3}(2xyz^3)$

$T_y = e^{x^2yz^3}(x^2z^3)$

$T_z = e^{x^2yz^3}(3x^2yz^2)$

II-ORDER

T_{xx}

T_{yy}

T_{zz}

$T_{xy} = T_{yx}$

$T_{xz} = T_{zx}$

$T_{yz} = T_{zy}$

III-ORDER

there would be 27 of these

Gradient Operator ∇

$\nabla = (\partial_x, \partial_y)$ so $\nabla f = (f_x, f_y)$

heshian

$$Hf = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$\nabla f = \langle f_x, f_y, f_z \rangle$

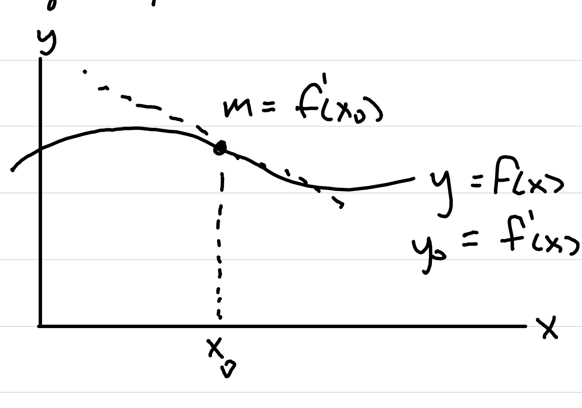
$$Hf = \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$$

A is symmetric if $A_{ij} = A_{ji}$

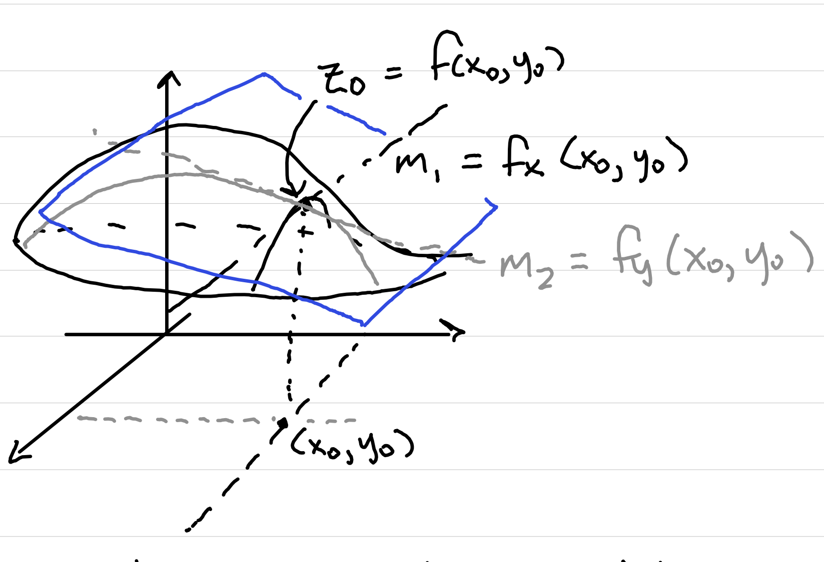
11.4

Tangent planes

CAL 1



$y - y_0 = m(x - x_0)$



Tangent plane is the plane containing m_1 and m_2

$$z - z_0 = \underset{\text{number}}{m_1}(x - x_0) + \underset{\text{number}}{m_2}(y - y_0)$$ ★

$$\begin{cases} y = y_0 \\ z - z_0 = m_1(x - x_0) \end{cases} \qquad \begin{cases} x = x_0 \\ z - z_0 = m_2(y - y_0) \end{cases}$$

Example

Find eq. of tangent plane $z = x^2 + 3y^3$ at $(1,2)$

$z(1,2) = 1^2 + 3(2)^3 = \boxed{25}$

$z_x(1,2) = 2(1) = \boxed{2} = m_1$
$z_y(1,2) = 9(2)^2 = \boxed{36} = m_2$

$z - 25 = 2(x-1) + 36(y-2)$

$2x + 36y - z = -25 + 2 + 72 = 49$

$= \boxed{2x + 36y - z - 49}$

Example 2

Find eq. of tangent plane $z = \cos(x+y)$ at $(\frac{\pi}{4}, \frac{\pi}{4})$

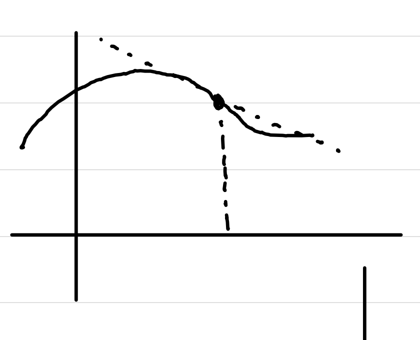
$z(\frac{\pi}{4}, \frac{\pi}{4}) = \cos(\frac{\pi}{2}) = 0$

$z_x = -\sin(x+y) = -\sin(\frac{\pi}{2}) = -1 = m_1$
$z_y = -\sin(x+y) = -\sin(\frac{\pi}{2}) = -1 = m_2$

$z - 0 = -1(x - \frac{\pi}{4}) + -1(y - \frac{\pi}{4})$

$\boxed{x + y + z = \frac{\pi}{2}}$

9-30-24



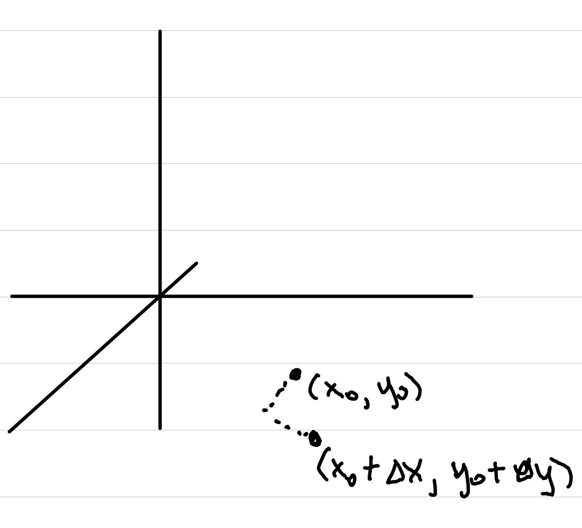
Cal 1
$y - f(m) = f'(x_0)(x - x_0)$

$y - y_0 = m(x - x_0)$

Increments

$\Delta y = \Delta f = f(x_0 + \Delta x) - f(x_0)$
$\Delta y \simeq m(x - x_0) = f'(x_0)\Delta x$

Cal 3



$\Delta z = z(x_0 + \Delta x, y_0 + \Delta y) - z(x_0, y_0) \simeq \Delta z_{TP}$

$\Delta z_{TP} = m_1 \Delta x + m_2 \Delta y$

$\Delta z \big|_{(\Delta x, \Delta y)} \simeq f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$ ★

Ex.

$z = e^{xy}$ eval $z(1.0003, 0.9992)$

$z = e^{1.0003(0.9992)}$ (with calculator)

1.
comes from the eq.?
$\Delta z = z(1 + \Delta x, 1 + \Delta y) - z(1,1) =$
$\Delta x = 0.0003$ $\Delta y = -0.0008$
from chosen point @ 1

2.
$\approx f_x(1,1)(0.0003) + f_y(1,1)(-0.0008)$

3.
$f_x = ye^{xy}$ $f_y = xe^{xy}$

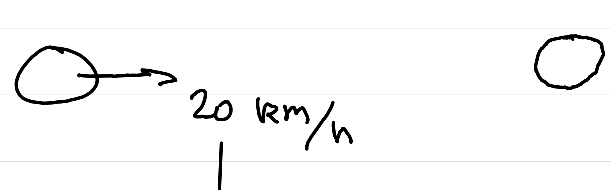
comes from $f_y(1,1) = 1e^{(1\cdot 1)}$

4.
$z - e = e(0.0003) - e(0.0008)$

$z \simeq e(0.9995)$

WW Ex.

Pressure
$P(x,t)$



$\Delta p = 20$ bars

$P(2) - P(0) = \Delta P$

travelled 40 km

$P_x = 10$ bar/km P_t ?

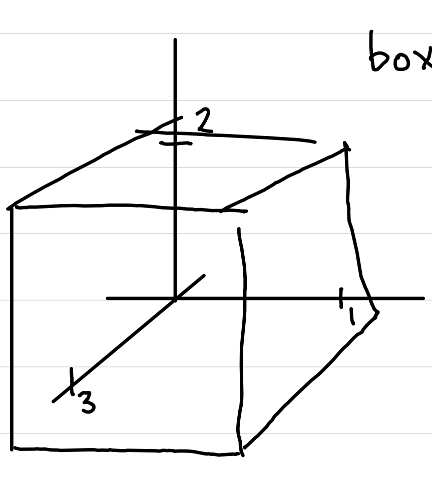
1.
$\Delta P = P_x \Delta x + P_t \Delta t$

2.
$20 = 10(20 \cdot 2) + P_t(2)$
2 hrs.

$P_t = \frac{20 - 400}{2} = -\frac{380}{2}$ bar/h

Ex.

box



$x = 3$ft $y = 1$ft $z = 2$ft

Base costs $3\$/ft^2$
Sides cost $2\$/ft^2$

estimate change in cost when x, y increase 3 in.
and z decreases 4 in.

ΔC

1.
$C = xy3 + (2xz + 2yz)2$

2.
$\Delta C \simeq C_x \Delta x + C_y \Delta y + C_z \Delta z$
3 in.
$\Delta x = 0.25$ ft. $\Delta y = 0.25$ ft.
$\Delta z = \frac{1}{3}$ ft.
4 in.

3.
$C_x = 3y + 4z$ $C_x(3,1,2) = 11$
$C_y = 3x + 4z$ $C_y(3,1,2) = 17$
$C_z = 4x + 4y$ $C_z(3,1,2) = 16$

4.
$\Delta C \simeq \frac{11}{4} + \frac{17}{4} - \frac{16}{3} = \boxed{\frac{5}{3}\$}$

Differential ★

dz of a function at $z = f(x,y)$ at x_0, y_0

$dz(x_0, y_0) = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$

this does not give us a number | for any x, y

$f = \cos(x+y) + y^2$ at $(0, \frac{\pi}{2})$

$f_x = -\sin(x+y)$ $f_y = -\sin(x+y) + 2y$

$f_x(0, \frac{\pi}{2}) = -1$ $f_y(0, \frac{\pi}{2}) = -1 + \pi$ ★

$\boxed{dz(0, \frac{\pi}{2}) = -1dx + (\pi - 1)dy}$

Differentiability

→ existence of tangent plane

$z = f(x,y)$ is differentiable at (x_0, y_0)
IF:
$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$
$= f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$
where
$(\varepsilon_1 + \varepsilon_2) \to (0,0)$ if $(\Delta x, \Delta y) \to (0,0)$

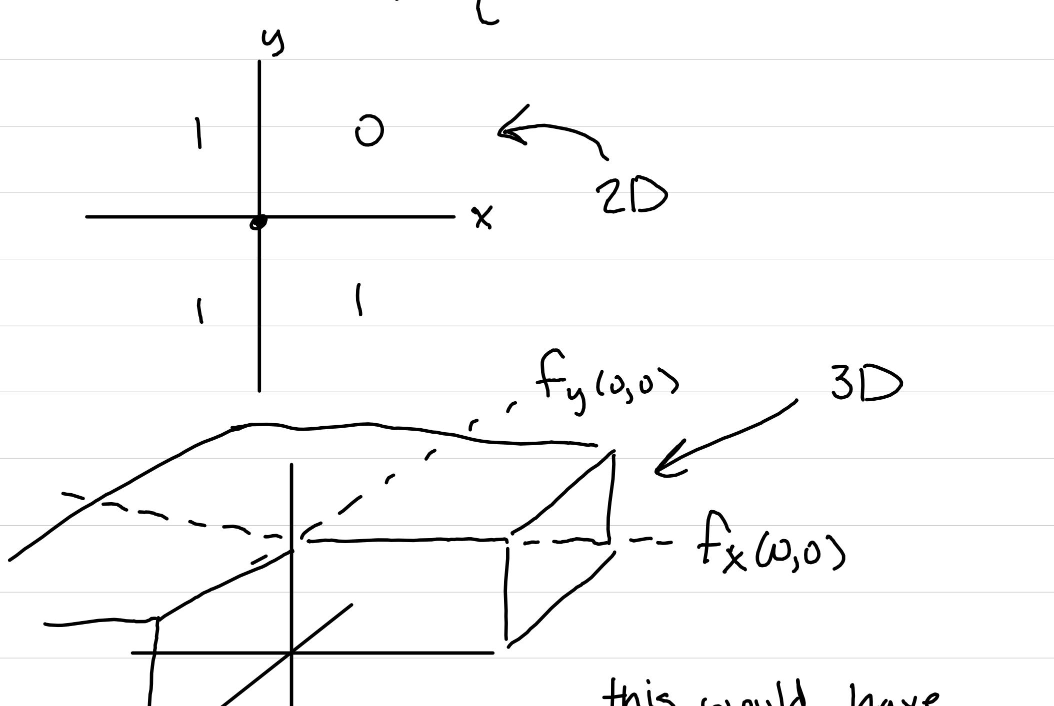
these are already going to zero linearly but need to go a little faster for this to be differentiable ★

Differentiable → smooth in each direction

Theorem: f is diff. at $(x_0, y_0) \iff$ ★
1) f is cont. at (x_0, y_0)
2) f_x is cont. at (x_0, y_0)
3) f_y is cont. at (x_0, y_0)

Example

$$f(x,y) = \begin{cases} 0 & \text{if } x > 0 \text{ and } y > 0 \\ 1 & \text{otherwise} \end{cases}$$



this would have f discontinuous but f_x and f_y still exist and are cont.

Ex. $z = x^3y^2 + 3xy$

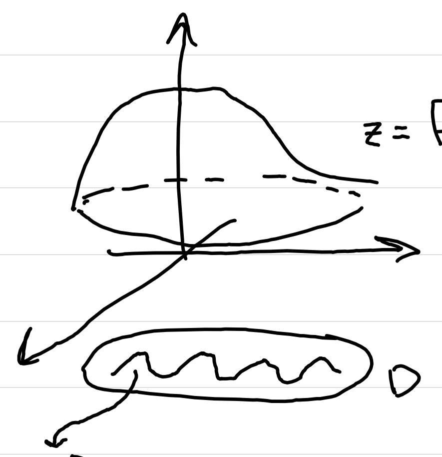
z is cont (polynomial)
$z_x = 3x^2y^2 + 3y$
$z_y = 2x^3y + 3x$ (Polynomial)

$\boxed{\text{Diff. everywhere in } \mathbb{R}^2}$

11.5

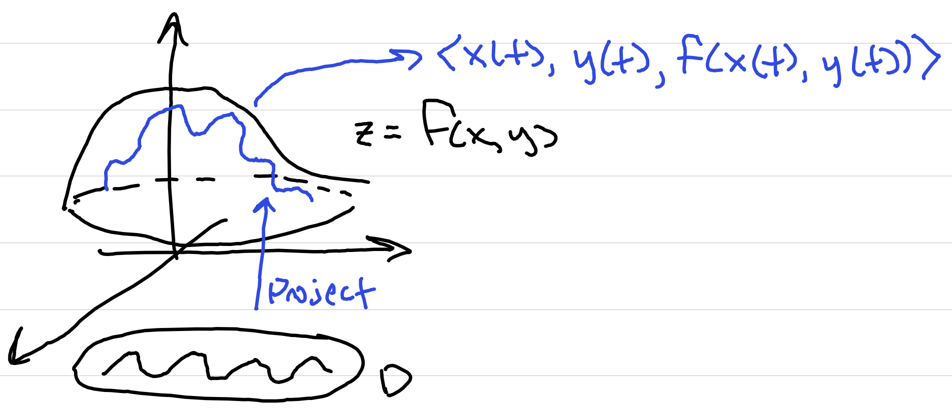
Chain rule for multivariate functions

$z = f(x,y)$ is diff in D



$\vec{R}(t) = \langle x(t), y(t) \rangle \quad t \in I$

we're looking for $\frac{d f(x(t), y(t))}{dt}$ ★



$= f_x(x(t), y(t))(x'(t)) + f_y(x(t), y(t))(y'(t))$

or

$= f_x(x') + f_y(y')$

$\frac{\Delta f}{\Delta t} = f_x \frac{\Delta x}{\delta t} + f_y \frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$

$\lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t}$

$\frac{df}{dt} = f_x x' + f_y y' + \varepsilon_1 x' + \varepsilon_2 y'$

$\Delta t \to 0$ implies $(\Delta x, \Delta y) \to (0,0)$

implies $(\varepsilon_1, \varepsilon_2) \to (0,0)$

Example

$f(x,y) = x^3 y^2 \qquad x(t) = \cos(t), \; y(t) = \sin(t)$

$\frac{df}{dt}$ by substitution and chain rule

A) By substitution

1.

$f(\cos(t), \sin(t)) = \cos^3(t)\sin^2(t) = g(t)$

$\frac{df}{dt} = g'(t) = 3\cos^2(t)(-\sin(t))(\sin^2(t)) + \cos^3(t)\, 2\sin(t)\cos(t)$

$= 3\cos^2(t)\sin^3(t) + 2\cos^4(t)\sin(t)$

B) Chain Rule

$f_x x' + f_y y'$

$f_x = 3x^2y^2 \qquad f_x(\cos, \sin) = 3\cos^2(t)\sin^2(t)$

$f_y = 2x^3 y \qquad f_y(\cos, \sin) = 2\cos^3(t)\sin(t)$

$= (3\cos^2(t)\sin^2(t))\underbrace{(-\sin(t))}_{x'} + (2\cos^3(t)\sin(t))\underbrace{(\cos t)}_{y'}$

Both give the same answer

Chain Rule in 3D

$\frac{\partial f}{\partial t} \vec{R}(t) = f_x(x') + f_y(y') + f_z(z')$

$\langle x(t), y(t), z(t) \rangle$

11.6

CHECK MICAELA'S NOTES!

directional

$D_{\hat{u}} f(x_p, y_p) = f_x(x_p, y_p)u_1 + f_y(x_p, y_p)u_2$

$= \nabla f(x_p, y_p) \cdot \hat{u}$ ★

gradient

The largest rate of increase occurs for $\hat{u} \nearrow\nearrow \nabla f(x_p, y_p)$ and equals $\underline{\underline{\|\nabla f(x_p, y_p)\|}}$

$\nabla f(x_p, y_p) \cdot \hat{u} = \|\nabla f(x_p, y_p)\| \, \|\hat{u}\| \cos\theta$

$= \|\nabla f(x_p, y_p)\| \cos\theta$ ★ Evaluating gradient at point P

$\nabla f = \langle \partial x, \partial y \rangle f = \langle f_x, f_y \rangle$

def. in 2D

3D

$\nabla f(x,y,z) = \langle \partial x, \partial y, \partial z \rangle f = \langle f_x, f_y, f_z \rangle$

Differential Operator

function of x, y, z

Linearity Rules (for this)

$a, b \in \mathbb{R}$ $\quad f(x,y,z),\ g(x,y,z)$ on D (common domain)

$\nabla(af + bg)(x,y,z) = a\nabla f + b\nabla g$

Product Rule (for this) ★

$\nabla(fg) = g(\nabla f) + f(\nabla g)$

Quotient Rule (for this)

$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

the usual rules

Product Rule Proof:

$\nabla(gf) = \langle \partial x, \partial y, \partial z \rangle (gf)$

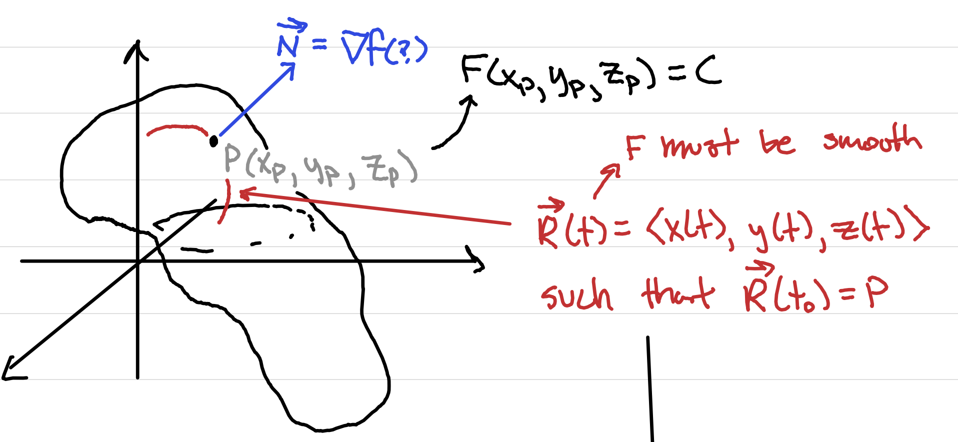
$= \langle \partial x(gf), \partial y(gf), \partial z(gf) \rangle$

$= \langle \underline{gf_x} + fg_x, \underline{gf_y} + fg_y, \underline{gf_z} + fg_z \rangle$

$= g\langle f_x, f_y, f_z \rangle + f\langle g_x, g_y, g_z \rangle$

$= \underline{g\nabla f + f\nabla g}$ ☐ proved

Take implicit function $F(x,y,z) = C$



$\nabla f(P) = \vec{N}$

normal to the surface at P

$\frac{d}{dt} C = 0$

$\frac{dF}{dt} = F_x(R(t))x' + F_y(R(t))y' + F_z(R(t))z' = 0$

$\nabla F(\vec{R}(t)) \cdot \langle x', y', z' \rangle = 0$

$\nabla F(\vec{R}(t)) \cdot \vec{R}'(t) = 0$

$\underline{\nabla F(P) \cdot \vec{R}'(t_0)} = 0$ ★

All possible R' will define the tangent plane

App. Ex. Find tan. plane for $x^2 + y^2 + z^3 = 3$ at $(1,1,1)$

we need:

1. the point
2. the $\vec{N}$ from $\nabla F(P)$

$\nabla F(P) = \langle 2x, 2y, 3z^2 \rangle (1,1,1)$

$= \langle 2, 2, 3 \rangle = \vec{N}$

Tangent Plane: $\boxed{2x + 2y + 3z^2 = 7}$

$z = f(x,y)$ $\quad (x_0, y_0)$

(from before)

$z - f(x_0,y_0) = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

$F = z - f(x,y) = 0$ $\quad (x_0, y_0, f(x_0,y_0))$ the point we need

$\vec{N} = \nabla F$

$= \langle -f_x, -f_y, 1 \rangle$

Tan. plane

TP = ★

$-f_x(x_0,y_0)(x - x_0) + -f_y(x_0,y_0)(y - y_0) + (z - f(x_0,y_0)) = 0$ from $A(x - x_p) + B(y - y_p) + C(z - z_p) = 0$

Gradient of implicit function $= \vec{N}$

App. Ex. $z = x^2 + y^3$ find eq. of TP at $(1,2)$

explicit

$F = z - x^2 - y^3 = 0$ ← implicit

$\nabla F(P) = \vec{N} = \langle -2x, -3y^2, 1 \rangle (1,2,9)$

$z_0 = (1^2 + 2^3) = 9$

$= \langle -2, -12, 1 \rangle$

$= -2(x-1) + (-12)(y-2) + 1(z-9) = 0$

or $z - 9 = 2(x-1) + 12(y-2)$

which is what you'd get from the 11.4 strat.

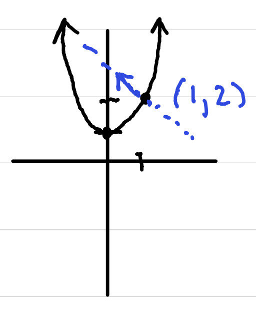
App. Ex. Find line ⊥ to surface at P

$P(1,1,1)$ $\vec{v} = \vec{N} = \langle 2,2,3 \rangle$

$$\begin{cases} x = 1 + 2t \\ y = 1 + 2t \\ z = 1 + 3t \end{cases} \quad t \in \mathbb{R} \equiv \langle 1+2t, 1+2t, 1+3t \rangle$$

$$\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-1}{3}$$

App. Ex. Find eq. of normal line to $y = x^2 + 1$ at $x = 1$ (2D)



$F(x,y) = y - x^2 - 1 = 0$

$\nabla F(1,2) = \langle -2x, 1 \rangle (1,2) = \underline{\langle -2, 1 \rangle = \vec{N}}$

$\frac{(x-1)}{-2} = \frac{(y-2)}{1} \implies x - 1 = -2y + 4$

$\boxed{x + 2y - 5 = 0}$

11.7

Max, min of $z = f(x,y)$

Absolute vs. Relative

→ (x_0, y_0) in D is an abs. max of f if $f(x_0,y_0) \geq f(x,y)$ for all (x,y) in D

(x_0,y_0) in D is an abs. min of f if $f(x_0,y_0) \leq f(x,y)$ for all (x,y) in D

→ (x,y) in D such that $(x-x_0)^2 + (y-y_0)^2 \leq \delta^2$

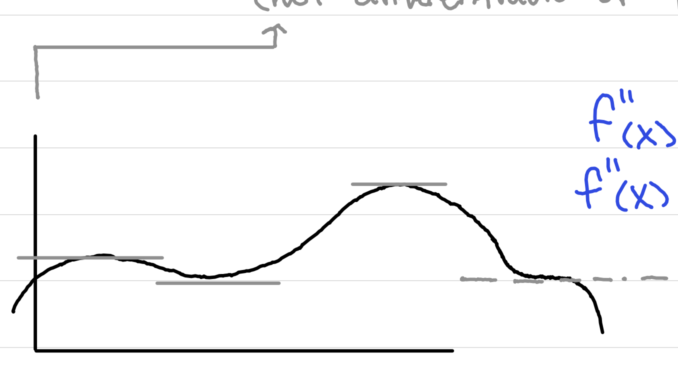
this is how we define a neighborhood for x and y from which we get rel. max/min

→ (x_0, y_0) in D is rel. min/max of f if $f(x_0,y_0) \leq , \geq f(x,y)$ for all (x,y) in D such that $(x-x_0)^2 + (y-y_0)^2 \leq \delta^2$

In Cal 1:

$y = f(x)$, find all critical points in D

(not differentiable or $f'(x) = 0$)



$f''(x) > 0$ rel. min.

$f''(x) < 0$ rel. max.

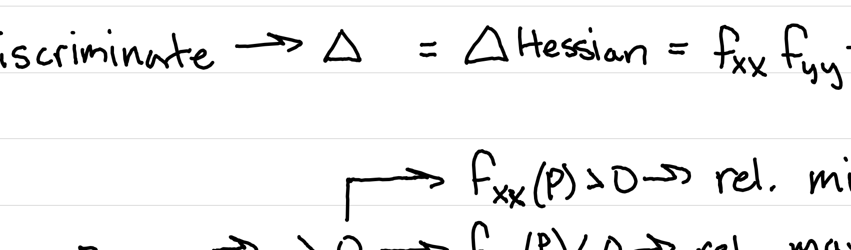
$f''(x) = 0$ inconclusive

In Cal 3:

$z = f(x,y)$, find all critical points in D

not differentiable or $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$

(horz. tan. plane)



critical points

Hessian matrix:

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

discriminate → Δ = ΔHessian $= f_{xx} f_{yy} - (f_{xy})^2$

$$\Delta(x_0, y_0) \; (P) = \begin{cases} > 0 \to & f_{xx}(P) > 0 \to \text{rel. min.} \\ & f_{xx}(P) < 0 \to \text{rel. max.} \\ 0 \to & \text{test inconclusive} \\ < 0 \to & \text{saddle point} \end{cases}$$

use this to identify these 4 things

1. Solve $\begin{cases} F_x = 0 \\ F_y = 0 \end{cases}$ → 2. Do this

---

App. Ex. Find relative max and min for

$F(x,y) = 2x^2 + 2xy + y^2 - 2x - 2y + 5$

1.

$$\begin{cases} F_x = 4x + 2y - 2 = 0 \\ F_y = 2x + 2y - 2 = 0 \end{cases}$$

sub the 2nd

$2x + 0y + 0 = 0$ $x = 0,\ y = 1$ so crit. point at $P(0,1)$

2.

$$H = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} = \underbrace{8-4}_{\text{Discrim.}} = \underbrace{4}_{f_{xx}(0,1)} > 0$$

both positive, so

$\boxed{P(0,1) \text{ is a rel. min.}}$

10-07-24

$z = f(x,y)$

have to find this

$$\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases} \Rightarrow P_1, P_2, \ldots, P_n$$

$$D, \Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2 \quad (x,y)$$

$$\Delta(P_I) \begin{cases} > 0 & \nearrow f_{xx}\ (P_I > 0) \text{ rel min.} \\ & \searrow f_{xx}\ (P_i < 0) \text{ rel max.} \\ = 0 & \text{Inconclusive} \\ < 0 & P_I = \text{Saddle point} \end{cases}$$

Ex. $z = 8x^3 - 24xy + y^3$

Find and classify all crit points

1.

$$\begin{cases} z_x = 24x^2 - 24y = 0 \\ z_y = -24x + 3y^2 = 0 \end{cases}$$

Rewrite

$$\begin{cases} y = x^2 \\ x(-8 + x^3) = 0 \end{cases}$$

$\nearrow x = 0$

$\searrow x^3 = 8 \to x = 2$

2.

$\begin{cases} x = 0 \\ y = 0 \end{cases}$ $P_1(0,0)$ $\begin{cases} x = 2 \\ y = 4 \end{cases}$ $P_2(2,4)$

3. Classify (Hessian Δ)

1. Find z_{xx}, z_{xy}, z_{yy}

$z_{xx} = 48x$

$z_{xy} = -24$

$z_{yy} = 6y$

$$= \begin{vmatrix} 48x & -24 \\ -24 & 6y \end{vmatrix} = \underline{\qquad}$$

$\to 48(6)xy - (-24)^2$ $f_{xx} f_{yy} - (f_{xy})^2$

$\Delta(0,0) = 24 \overset{(12)}{}(0-2) < 0$ $\boxed{P_1 = \text{Saddle Point}}$

$\Delta(2,4) = 24(12)(2(4)-2) > 0$ $\boxed{P_2 \text{ is rel. min.}}$

$f_{xx}(2,4) = 48(2) > 0$

Ex. $z = x^2 y^4$ Find & classify crit. points

1.

$$\begin{cases} z_x = 2x(y^4) - x^2(0) = 2xy^4 = 0 \\ z_y = 4x^2y^3 = 0 \end{cases}$$

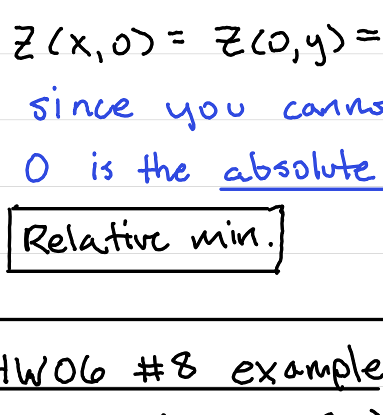
2.

$\begin{cases} x = 0 \\ 0 = 0 \end{cases}$ $\begin{cases} y = 0 \\ 0 = 0 \end{cases}$

we have infinitely many points

3.

$$\Delta = \begin{vmatrix} 2y^4 & 8xy^3 \\ 8xy^3 & 12x^2y^2 \end{vmatrix} = 24x^2y^6 - 64x^2y^6 = -40x^2y^4$$



$\Delta(x,0) = 0$

$\Delta(0,y) = 0$

more work needed

4.

$z(x,0) = z(0,y) = 0$, so...

since you cannot go < 0 with $z = x^2y^4$, 0 is the <u>absolute min</u>, therefore they are all

$\boxed{\text{Relative min.}}$

HW06 #8 example

$z = xy(1 - 6x - 9y)$ find and classify crit. point

$$\begin{cases} z_x = y(1-6x-9y) + xy(-6) \\ z_y = x(1-6x-9y) + xy(-9) \end{cases} = 0$$

collect x and y to do this simplification

$$= \begin{cases} z_x = y(1-12x-9y) = 0 \\ z_y = x(1-6x-18y) = 0 \end{cases}$$

P_1 $\begin{cases} y = 0 \\ x = 0 \end{cases} = (0,0)$ P_2 $\begin{cases} y = 0 \\ 1 - 6x - 18y = 0 \end{cases} = (1/6, 0)$

P_3 $\begin{cases} 1 - 12x - 9y = 0 \\ x = 0 \end{cases} = (0, 1/9)$ P_4 $\begin{cases} 1 - 12x - 9y = 0 \\ 1 - 6x - 18y = 0 \end{cases} = \left(\frac{1}{18}, \frac{1}{27}\right)$

(this is every permutation) solve

$z_{xx} = y(-12)$

$z_{xy} = (1-12x-9y) + y(-9) = 1 - 12x - 18y$

$z_{yy} = x(-18)$

$$\Delta = \begin{vmatrix} -12y & 1-12x-18y \\ 1-12x-18y & -18x \end{vmatrix}$$

$= -12(-18)xy - (1-12x-18y)^2$

$\Delta(0,0) = 0 - (1-0-0)^2 = -(1)^2$ <u>Saddle point</u>

$\Delta(1/6, 0) = < 0$ <u>saddle point</u>

$\Delta(0, 1/9) = < 0$ <u>saddle point</u>

$\Delta\left(\frac{1}{18}, \frac{1}{27}\right) = \frac{12}{27} - \left(1 - \frac{12}{18} - \frac{18}{27}\right)^2$

$= \frac{4}{9} - \left(1 - \frac{2}{3} - \frac{2}{3}\right)^2 = \ldots = \frac{4}{9} - \frac{1}{9} > 0$

$z_{xx}\left(\frac{1}{18}, \frac{1}{27}\right) < 0 \longrightarrow$ <u>rel. max.</u>

∇(explicit) = 2D vector 10-7-24

$D_{\hat{u}} f(P) = \nabla f(P) \cdot \hat{u}$

$= \|\nabla f(P)\| \, \cancel{\|\hat{u}\|}^{\,1} \cos\theta$

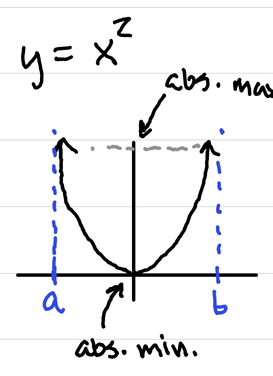
max rate of change

largest when $\theta = 0$

11.7B

Absolute Extrema

$y = x^2$ (Cal 1 Theorem)



$D = [a, b]$
for continuous functions

In Cal 3

D is closed. $z = f(x,y)$ is cont. on D

$\Rightarrow \exists$ absolute extrema (max + min)
$\Rightarrow$ all the crit. points in D or on the boundary of D

Ex. Find abs. extrema for $z = xy$ on the box

$D: \begin{cases} -1 \le x \le 1 \\ -1 \le y \le 1 \end{cases}$ (This is a closed domain)

1. Look for all crit. points in D

NOT on boundary

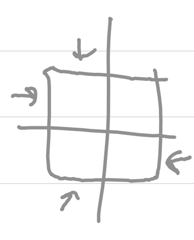
$\begin{cases} -1 < x < 1 \\ -1 < y < 1 \end{cases} \longrightarrow \begin{cases} f_x = 0 = y \\ f_y = 0 = x \end{cases}$ $P_1 = (0,0)$

★ $\partial D \Rightarrow$ "on the boundary of D"

2. Find crit points ∂D

1) $x = 1, \ -1 \le y \le 1$
2) $x = -1, \ -1 \le y \le 1$
3) $-1 \le x \le 1, \ y = 1$
4) $-1 \le x \le 1, \ y = -1$

$\rightarrow$ the 4 sides



1)
$z(1,y) = y = g(y)$
$g'(y) = 0$
$1 = 0$ X no solution

2)
$z(-1,y) = -y = g(y)$
$g'(y) = 0 \rightarrow -1 = 0$ X

3)
$z(x,1) = x = g(y)$
... X

4)
$z(x,-1) = -x = g(y)$
... X

2.
$z(P_1) = 0$

$z(P_2) = 1 = z(P_4)$ Abs. max
$z(P_3) = -1$ Abs. min.

Ex. $z = e^{x^2 - y^2}$ for $D: x^2 + y^2 \le 1$

1. crit. p. inside $x^2 + y^2 \le 1$ ★

$\begin{cases} z_x = 2x e^{x^2-y^2} \\ z_y = -2y e^{x^2-y^2} \end{cases} = 0$

$\begin{cases} x = 0 \\ y = 0 \end{cases}$

2. crit. p. on $x^2 + y^2 = 1$

$-1 \le x \le 1 \quad y = \pm\sqrt{1 - x^2}$

OR

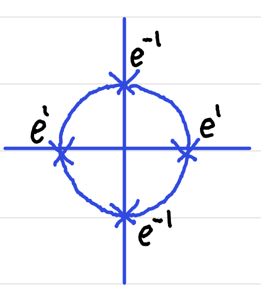
$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad 0 \le t \le 2\pi$

$z = e^{\cos^2 t - \sin^2 t} = e^{\cos(2t)} = g(t) \quad 0 \le t \le 2\pi$

$g'(t) = e^{\cos(2t)}[-2\sin(2t)] = 0$

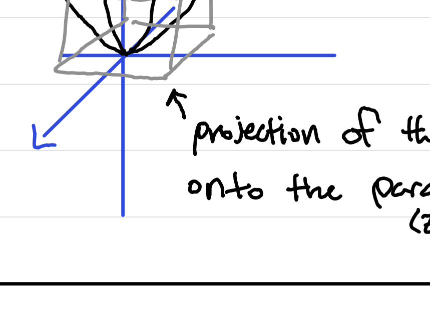
$\sin(2t) = 0 \rightarrow 2t = \pi \rightarrow t = \frac{\pi}{2}$
$2t = 2\pi \rightarrow t = \pi$
$2t = 3\pi \rightarrow t = \frac{3\pi}{2}$
$2t = 4\pi \rightarrow t = 2\pi$

abs. min.
abs. max



Ex. $z = x^2 + y^2$

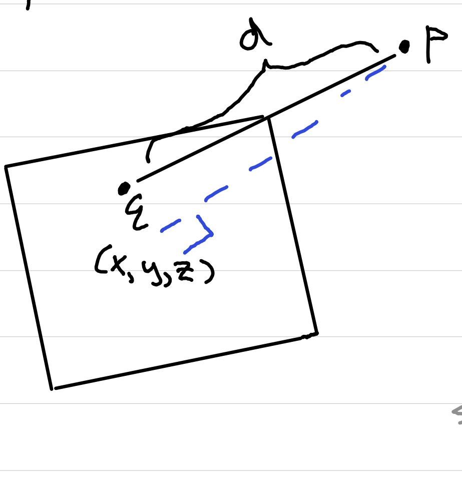
$D: \begin{cases} -1 \le x \le 1 \\ -1 \le y \le 1 \end{cases}$



projection of the box (D) onto the paraboloid ($z = x^2 + y^2$)

10-9-24

Ex. Find the point on $x + 2y + z = 5$ closest to $(0,3,4)$



solved for z

generic point Q: $(x, y, 5 - x - 2y)$

$d = \sqrt{(x - \underset{x_p}{0})^2 + (y - \underset{y_p}{3})^2 + (5 - x - 2y - \underset{z_p}{4})^2}$

$f(x,y) = d^2(x,y) = (x-0)^2 + (y-3)^2 + (5 - x - 2y - 4)^2$

1. Find $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$

$x^2 + (y-3)^2 + (1 - x - 2y)^2$

$\begin{cases} 2x - (1 - x - 2y) = 0 & f_x \\ y - 3 - 2(1 - x - 2y) = 0 & f_y \end{cases}$

$\downarrow$

$\begin{cases} 3x + 2y - 1 = 0 \\ 2x + 5y - 5 = 0 \end{cases}$

2. Find solution to $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$

cancel x's

$\begin{array}{r} 2 \\ -3 \end{array} \begin{cases} 3x + 2y - 1 = 0 \\ 2x + 5y - 5 = 0 \end{cases}$

$4y - 15y - 2 + 15 \longrightarrow -11y = -13 \longrightarrow \underline{y = \frac{13}{11}}$

cancel y's

$\begin{array}{r} 5 \\ -2 \end{array} \begin{cases} 3x + 2y - 1 = 0 \\ 2x + 5y - 5 = 0 \end{cases}$

$11x + 5 = 0 \longrightarrow \underline{x = \frac{-5}{11}}$

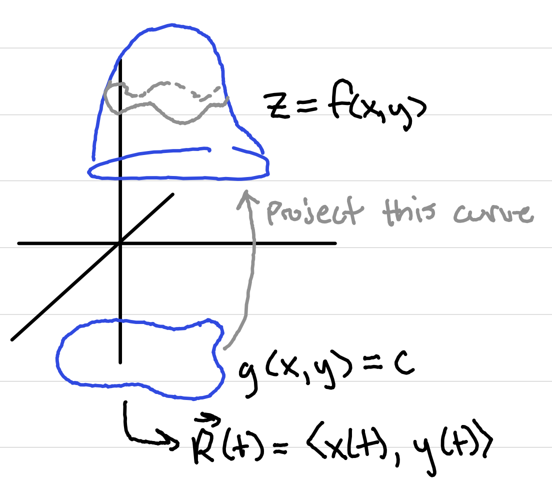
3. get z

from original eq. $x + 2y + z = 5$

$z = 5 - x - 2y$

$\boxed{z = 5 - \frac{5}{11} - 2\left(\frac{13}{11}\right)} = \frac{52}{11}$

11.8



we're already done this

$z=f(x(t), g(t)) = g(t)$, find $g'(t)=0$

LMT (when $f + g$ are smooth) ★

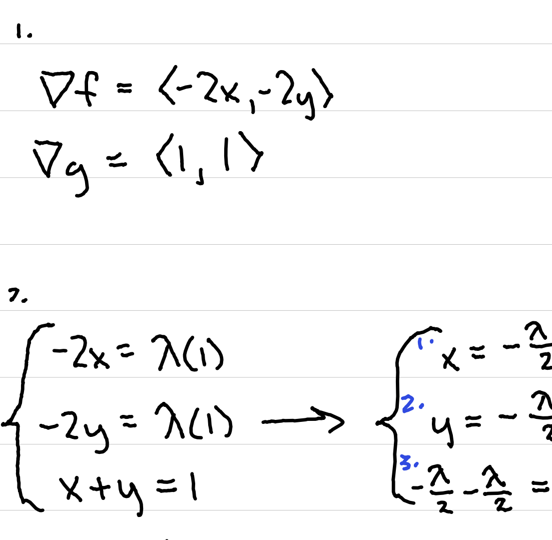
ALL CRIT. POINTS OF $f(x,y)$ constrained on $g(x,y)=c$ occurs when $\nabla f = \lambda \nabla g$
$\nabla f \uparrow\uparrow \nabla g$

$z(t) = f(x(t), y(t))$ $\quad \frac{dz}{dt} = 0 = f_x x'(t) + f_y y'(t)$ —

$\longrightarrow \nabla f \cdot \vec{R}(t), \quad \nabla f \perp \vec{T}$

g is implicitly defined, so ∇g is also $\perp$ to the curve

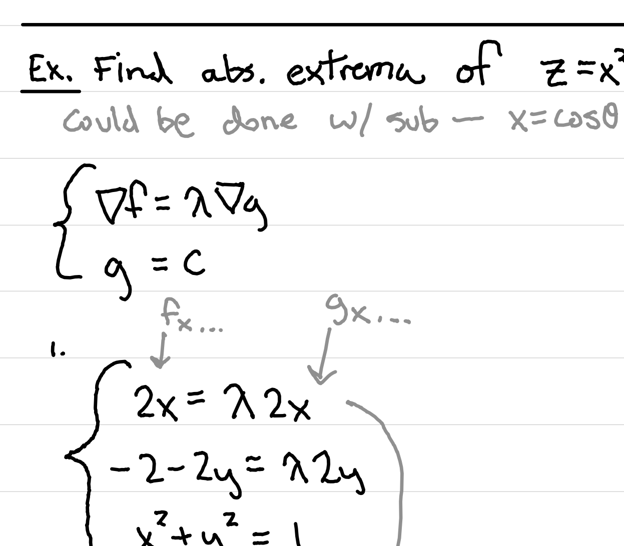
Ex. Find the largest and smallest values for $f(x,y) = 1-x^2-y^2$ constrained on $x+y=1$ for $x, y \geq 0$



$\nabla f(x,y) \uparrow\uparrow \nabla g$

1.
$\nabla f = \langle -2x, -2y\rangle$
$\nabla g = \langle 1, 1\rangle$

2.
$$\begin{cases} -2x = \lambda(1) \\ -2y = \lambda(1) \\ x+y=1 \end{cases} \longrightarrow \begin{cases} 1.\ x = -\frac{\lambda}{2} = \frac{1}{2} \\ 2.\ y = -\frac{\lambda}{2} = \frac{1}{2} \\ 3.\ -\frac{\lambda}{2}-\frac{\lambda}{2} = 1 \quad -\lambda = 1 \quad 4.\ \lambda=-1 \end{cases}$$
5.



Ex. Find abs. extrema of $z=x^2-2y-y^2$ on $x^2+y^2=1$
could be done w/ sub — $x=\cos\theta \quad y=\sin\theta$
$x^2+y^2-1=0$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = c \end{cases}$$

$f_x \ldots \quad g_x \ldots$

1.
$$\begin{cases} 2x = \lambda 2x \\ -2-2y = \lambda 2y \\ x^2+y^2=1 \end{cases}$$

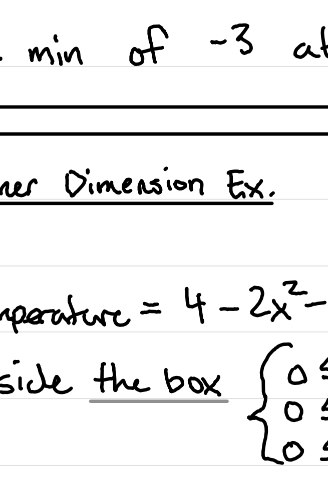
simplify

2.
$$\begin{cases} (\lambda-1)x = 0 \\ -2-2y = \lambda 2y \\ x^2+y^2=1 \end{cases}$$
$\longrightarrow$ either $x=0$ or $\lambda=1$

3.
#1
$$\begin{cases} x=0 \\ y^2=1 \quad y=\pm 1 \\ \lambda = \ldots \end{cases}$$

#2
$$\begin{cases} \lambda = 1 \\ -2-2y = 2y \quad -2=4y \\ \end{cases}$$
$y=-1/2$

$x^2=\frac{3}{4} \quad x=\pm\frac{\sqrt{3}}{2}$

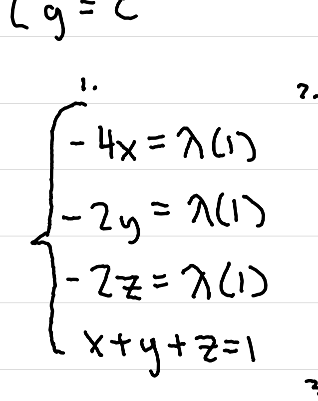


$z\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{3}{4} - 2\left(-\frac{1}{2}\right) - \frac{1}{4}$
$= 1 + \frac{2}{4} = \frac{3}{2}$

abs. max of $\frac{3}{2}$ at $\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
abs. min of -3 at $(0, 1)$

Higher Dimension Ex.

Temperature $= 4-2x^2-y^2-z^2$ on the plane $x+y+z=1$ inside the box $\begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \\ 0 \leq z \leq 1 \end{cases}$



1. Look for crit. points on $x+y+z=1$ (LMT on higher dim.)

$$\begin{cases} \nabla T = \lambda \nabla g \\ g = c \end{cases} \qquad \nabla T = \langle -4x, -2y, -2z\rangle \quad \nabla g = \langle 1, 1, 1\rangle$$

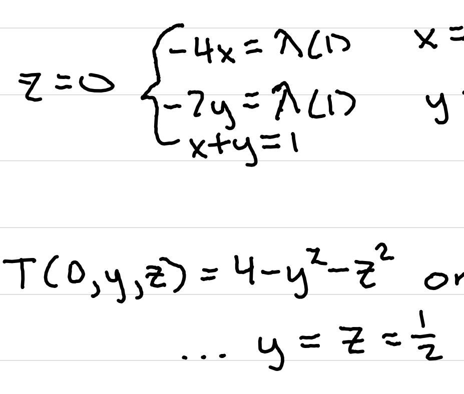
1.
$$\begin{cases} -4x = \lambda(1) \\ -2y = \lambda(1) \\ -2z = \lambda(1) \\ x+y+z=1 \end{cases}$$

2.
$x = \frac{-\lambda}{4}$
$y = \frac{-\lambda}{2}$
$z = \frac{-\lambda}{2}$
$-\frac{\lambda}{4} + -\frac{\lambda}{2} + -\frac{\lambda}{2} = 1$
3.

4. $-\frac{5}{4}\lambda = 1 \quad \lambda = -\frac{4}{5}$

5.
$x = -\frac{1}{4}\left(-\frac{4}{5}\right) = \frac{1}{5}$
$y = \frac{2}{5}$
$z = \frac{2}{5}$

$\boxed{\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)}$



etc.

2. check crit. points on edges

$T(x,y,0) = 4-2x^2-y^2$ on $x+y=1$ — this is a sub. of z to be zero

$z=0 \begin{cases} -4x = \lambda(1) \\ -2y = \lambda(1) \\ x+y=1 \end{cases}$ $\quad x = -\frac{\lambda}{4} \quad y = -\frac{\lambda}{2}$ $\quad \boxed{\left(\frac{1}{3}, \frac{2}{3}\right)}$

$-\frac{\lambda}{4} - \frac{\lambda}{2} = 1 = -\frac{3}{4}\lambda$
$\lambda = -\frac{4}{3}$
$x = \frac{1}{3} \quad y = \frac{2}{3}$

$T(0,y,z) = 4-y^2-z^2$ on $y+z=1$
$\ldots y = z = \frac{1}{2}$ $\quad \boxed{\left(0, \frac{1}{2}, \frac{1}{2}\right)}$

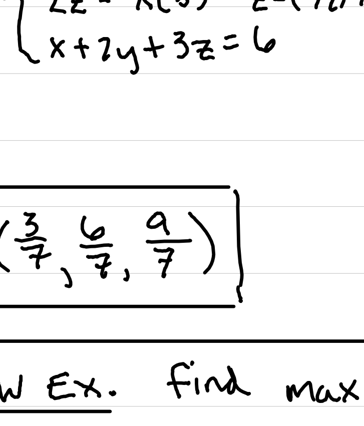
$T(x,0,z) = 4-2x^2-z^2$ on $x+z=1$
$\ldots x = \frac{1}{3}, \ z = \frac{2}{3}$
$\boxed{\left(\frac{1}{3}, 0, \frac{2}{3}\right)}$

3. Compare function at all 4 crit. points and 3 vertices

$T\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right) = 3\frac{3}{5}$ MAX
$T\left(\frac{1}{3}, \frac{2}{3}, 0\right) = 3\frac{1}{3} = T\left(\frac{1}{3}, 0, \frac{2}{3}\right)$
$T\left(0, \frac{1}{2}, \frac{1}{2}\right) = 3\frac{1}{2}$
] crit. points

$T(1,0,0) = 2$ MIN
$T(0,1,0) = 3 = T(0,0,1)$
] vertices

Ex. Find the point on plane $x+2y+3z=6$ closest to the origin



$f = d^2 = x^2+y^2+z^2$ on $x+2y+3z=6$
distance^2 — from origin

1.
$$\begin{cases} 2x = \lambda(1) \\ 2y = \lambda(2) \\ 2z = \lambda(3) \\ x+2y+3z=6 \end{cases} \quad \begin{aligned} x &= \tfrac{\lambda}{2} \\ y &= \lambda \\ z &= (3/2)\lambda \end{aligned}$$

$\frac{\lambda}{2} + 2(\lambda) + 3\left(\frac{3}{2}\right)\lambda = 6$
$\lambda = \frac{6}{7} \rightarrow x = \frac{3}{7}$
$y = \frac{6}{7}$
$z = \frac{9}{7}$

$= \boxed{\left(\frac{3}{7}, \frac{6}{7}, \frac{9}{7}\right)}$

WW Ex. find max/min of $x^2+y^2+z^2$ on $4x^2+2y^2+z^2=4 = g(x,y,z)$

$T(x,y,z) \quad D = \mathbb{R}^3$

$\nabla T = \lambda \nabla g$

1.
$$\nabla T = \begin{cases} 2x = \lambda 8x \\ 2y = \lambda 4y \\ 2z = \lambda 2z \\ 4x^2+2y^2+z^2=4 \end{cases}$$

2.
$$= \begin{cases} 2x(1-4\lambda) = 0 \\ \ldots \\ \ldots \\ \ldots \end{cases}$$
$\rightarrow$ make this true
either $x=0$ or $\lambda = \frac{1}{4}$

$$= \begin{cases} \ldots \\ 2y(1-2\lambda) = 0 \\ \end{cases}$$
$y=0$ or $\lambda = \frac{1}{2}$

$=$

Solutions possible:

1a:
$x=0$
$y=0$
$z=\pm 2$

1b:
$x=0$
$\lambda = \frac{1}{2}$
$2z = z$
$z=0$
$y = \pm\sqrt{2}$
$\hookrightarrow$ from plugging in to $4x^2+2y^2+z^2=4$

2:
$2\begin{cases} \lambda = \frac{1}{4} \\ \end{cases}$

...

$(0,0,\pm 2) \quad (0, \pm\sqrt{2}, 0) \quad (\pm 1, 0, 0)$
$\downarrow \quad \downarrow \quad \downarrow$
$T(\ \ ) = 4 \quad T(\ \ ) = 2 \quad T(\ \ ) = 1$