

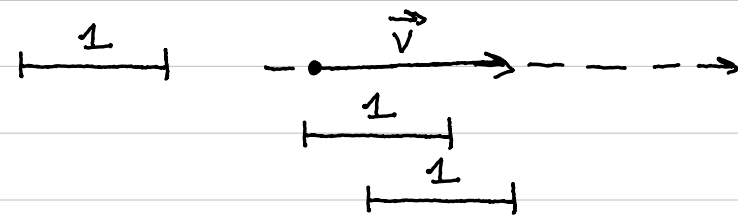
9.1 (Review)

Scalar vs. Vector

↓

↓
temperature
weight
speed
...

↓
magnitude == $\lambda = 0$
+
direction
↓
forces
displacement
velocity
...



magni. $\vec{v} = 1.5v$

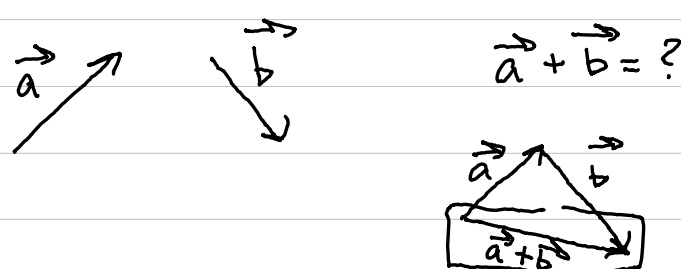
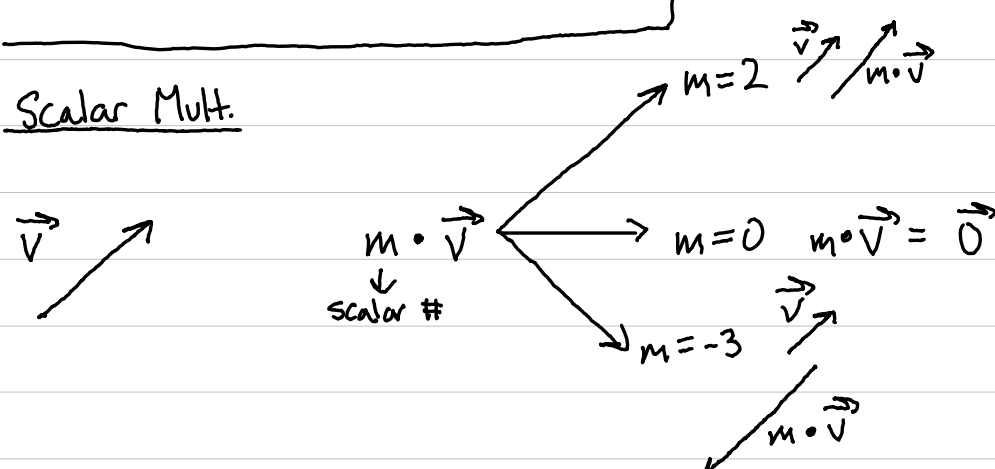
$\vec{0}$ (zero vector) has no direction $\|\vec{0}\| = 0$

$\|\vec{v}\|$ means magnitude of $\vec{v}$

$-\vec{v}$ is in the opposite direction of $\vec{v}$

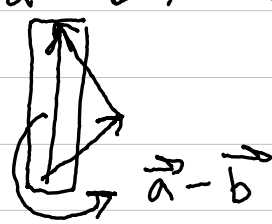
$\hat{u} \rightarrow$ unit vector ($\|\hat{u}\| = 1$)

Scalar Mult.



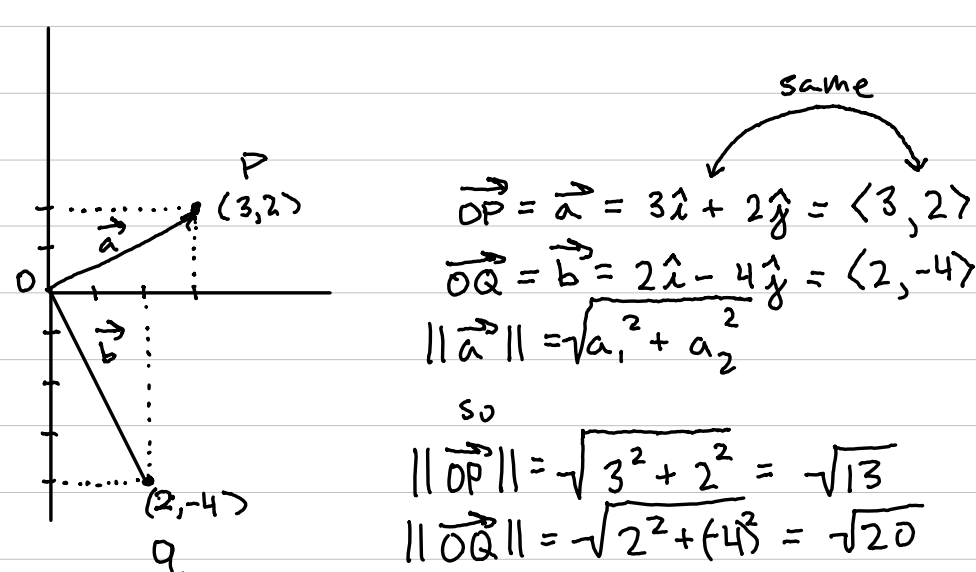
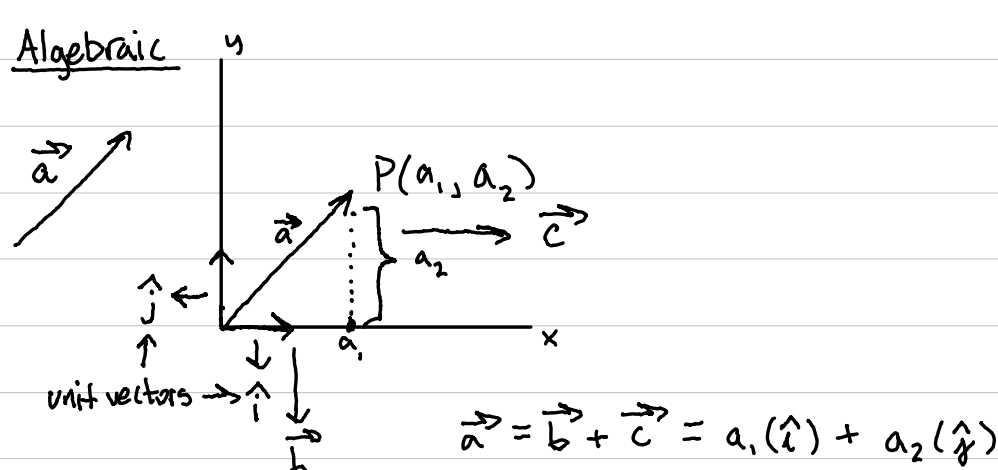
* Put 2nd foot where 1st head is

$\vec{a} - \vec{b} = ?$ $\vec{a} + (-\vec{b})$

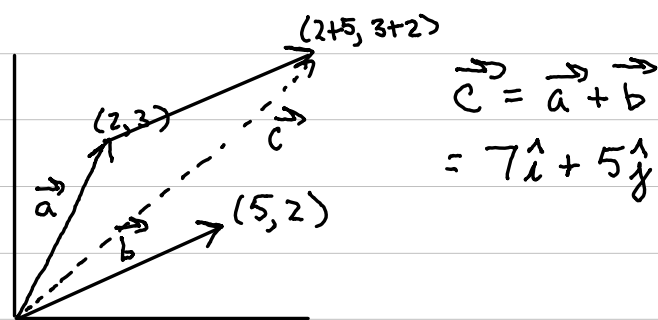


find $\hat{u}$ $\nearrow \nearrow \vec{v}$ $\hat{u} = \frac{\vec{v}}{\|\vec{v}\|}$
(Parallel to)

Algebraic



Find $\hat{u}$ $\nearrow \nearrow \vec{OP}$ $\hat{u} = \frac{\vec{OP}}{\|\vec{OP}\|} = \frac{\langle 3, 2 \rangle}{\sqrt{13}} = \left\langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right\rangle$



$\vec{a} = \langle a_1, a_2 \rangle$ $\vec{b} = \langle b_1, b_2 \rangle$ $\vec{c} = \langle a_1 + b_1, a_2 + b_2 \rangle$

Examples

1.) $\vec{a} = \langle 3, -2 \rangle$ $\vec{b} = \langle 5, 3 \rangle$

a) $\vec{a} + \vec{b}$

b) $2\vec{a} - 3\vec{b}$

$\langle 8, 1 \rangle$

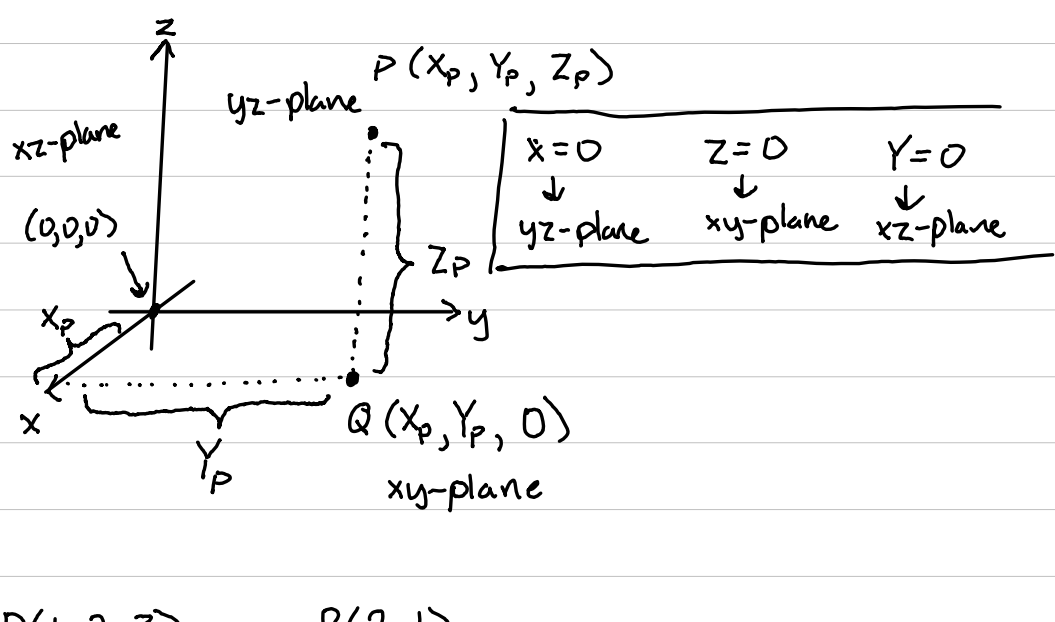
$\langle 6, -4 \rangle - \langle 15, 9 \rangle$

$\langle -9, -13 \rangle$

Genetic Rule

$m\vec{a} + n\vec{b} = \langle ma_1, ma_2 \rangle + \langle nb_1, nb_2 \rangle$
 $= \langle ma_1 + nb_1, ma_2 + nb_2 \rangle$

9.2



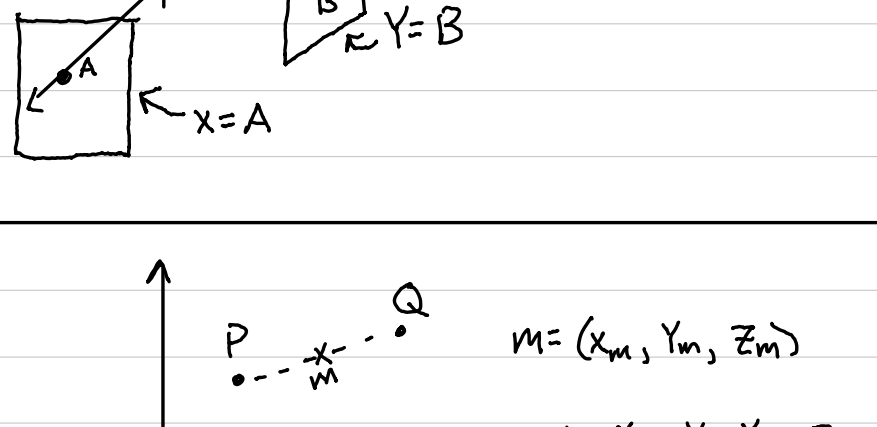
$$P(1, 2, 3) \quad P(2, 1)$$

$$\begin{cases} x=1 \\ y=2 \\ z=3 \end{cases} \quad \begin{cases} x=2 \\ y=1 \end{cases}$$

$$x\text{-axis} \rightarrow \begin{cases} z=0 \\ y=0 \end{cases}$$

$$y\text{-axis} \rightarrow \begin{cases} z=0 \\ x=0 \end{cases}$$

$$z\text{-axis} \rightarrow \begin{cases} x=0 \\ y=0 \end{cases}$$



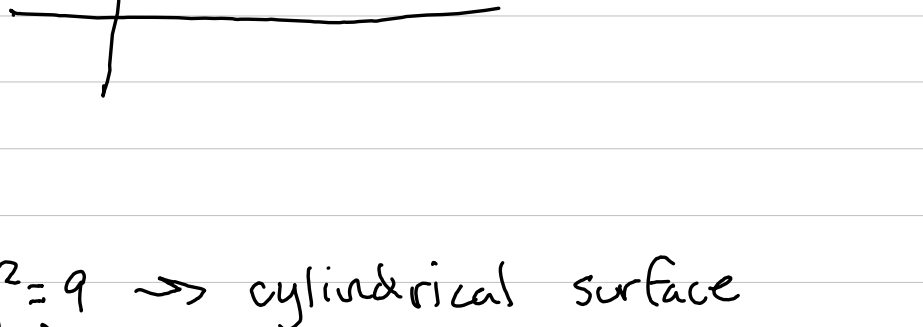
$$P \quad Q \quad m = (x_m, y_m, z_m) = \left(\frac{x_p + x_q}{2}, \frac{y_p + y_q}{2}, \frac{z_p + z_q}{2} \right)$$

$$\text{Distance: } d = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2 + (z_q - z_p)^2}$$

$$P(1, 2, 3) \quad Q(2, 4, -5)$$

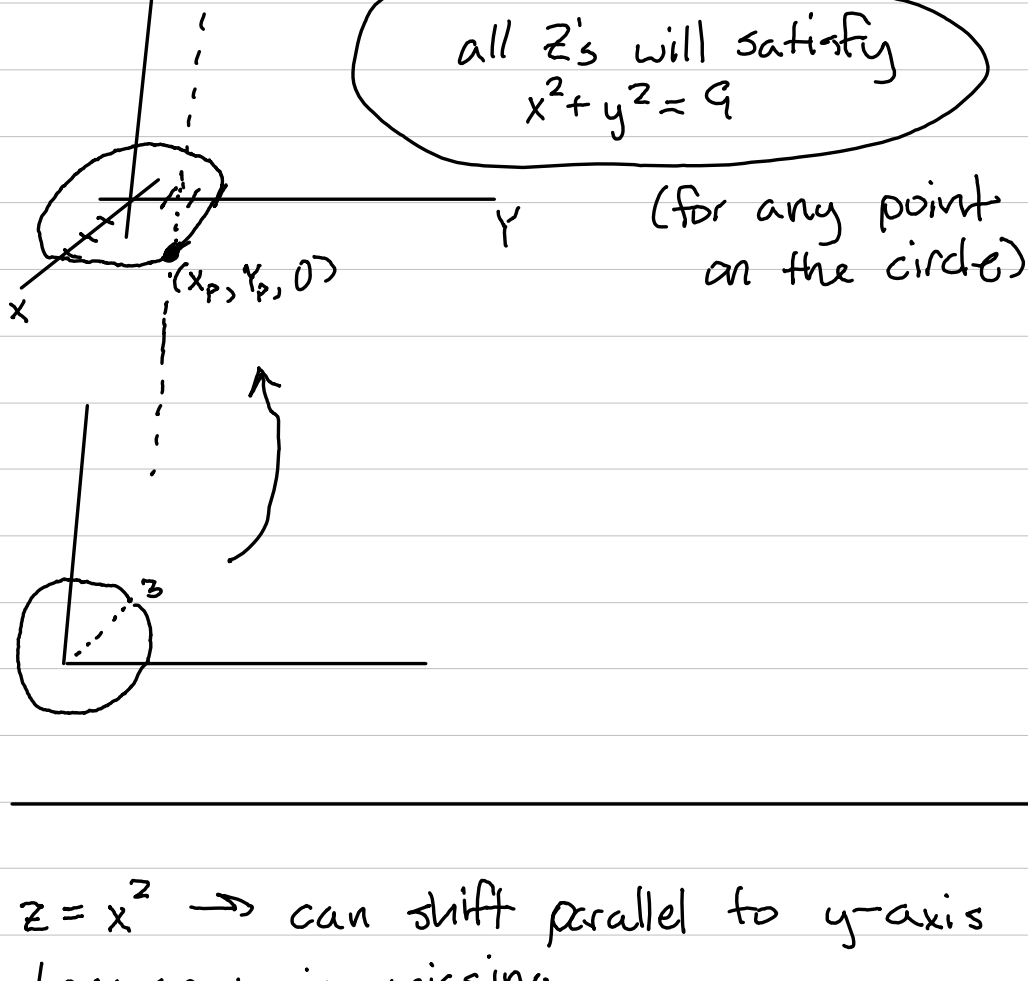
$$m = \left(\frac{1+2}{2}, \frac{2+4}{2}, \frac{3-5}{2} \right) = \left(\frac{3}{2}, 3, -1 \right)$$

2D

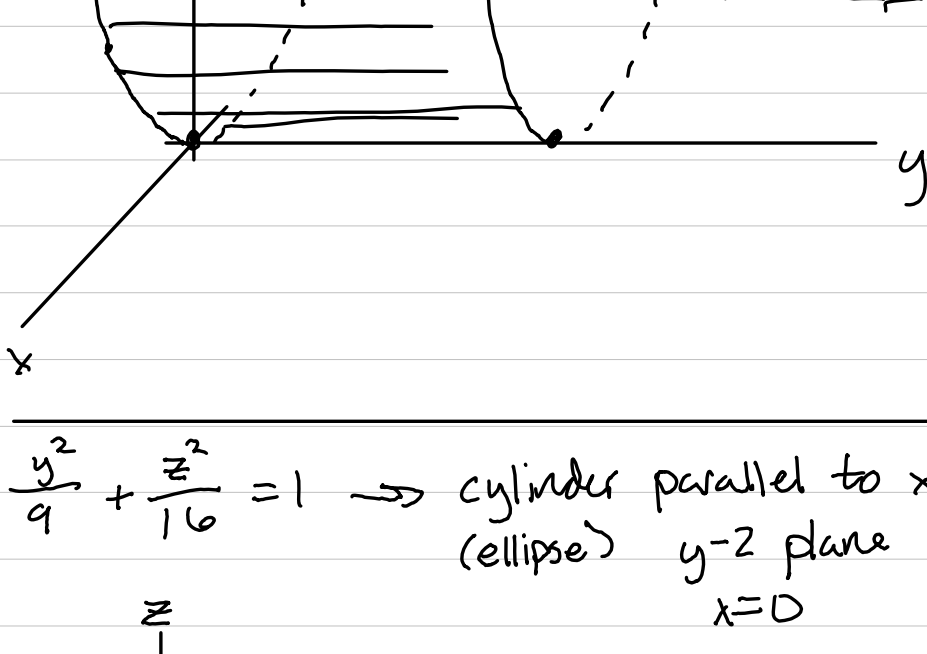


3D

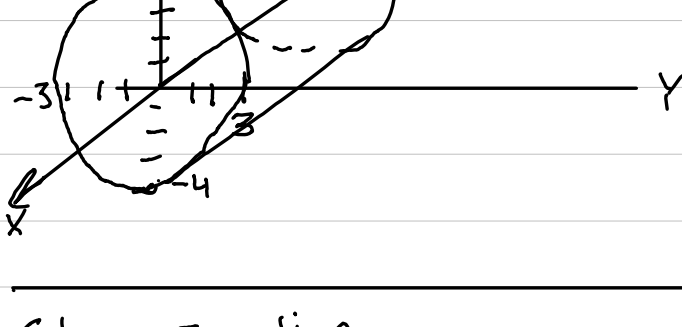
$$x^2 + y^2 = 9 \rightarrow \text{cylindrical surface (circle eq.)}$$



$$z = x^2 \rightarrow \text{can shift parallel to } y\text{-axis because } y \text{ is missing}$$



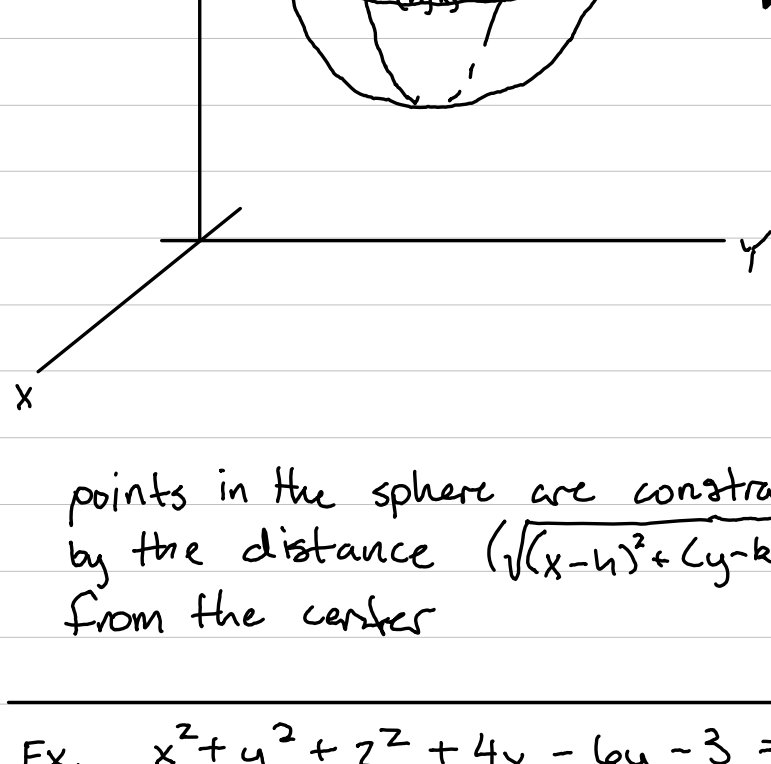
$$\frac{y^2}{9} + \frac{z^2}{16} = 1 \rightarrow \text{cylinder parallel to } x\text{-axis (ellipse) } y\text{-}z \text{ plane } x=0$$



Sphere Equation

$$2D \text{ circle} \rightarrow (x-h)^2 + (y-k)^2 = a^2$$

$$\text{Sphere} \rightarrow (x-h)^2 + (y-k)^2 + (z-l)^2 = a^2$$



points in the sphere are constrained by the distance $\sqrt{(x-h)^2 + (y-k)^2 + (z-l)^2}$ from the center

$$\text{Ex. } x^2 + y^2 + z^2 + 4x - 6y - 3 = 0$$

complete the square

1) put together like-terms

$$(x^2 + 4x) + (y^2 - 6y) + z^2 = 3$$

$$(x-h)^2 \rightarrow x^2 - 2xh + h^2$$

$$+4x = -2xh \quad (h = -2)$$

so

$$(x^2 + 4x + 4)$$

add to the right

$$(x^2 + 4x + 4) + \dots = 3 + 4$$

$$(x+2)^2$$

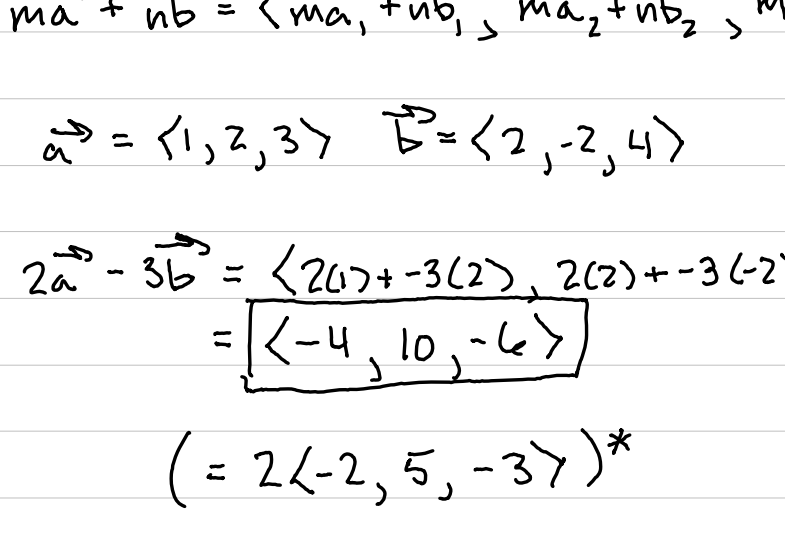
$$y^2 - 2yk + k^2$$

$$K = \frac{-6}{-2} = 3 \rightarrow y^2 - 6y + 9$$

$$(x+2)^2 + (y-3)^2 + z^2 = 16$$

$$(h, k, l) = (-2, 3, 0) \quad a = \sqrt{16} = 4$$

Vectors in 3D



$$\vec{OP} = x_p \hat{i} + y_p \hat{j} + z_p \hat{k} = \langle x_p, y_p, z_p \rangle$$

$$\|\vec{OP}\| = \sqrt{x_p^2 + y_p^2 + z_p^2}$$

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \quad \vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$m\vec{a} + n\vec{b} = \langle ma_1 + nb_1, ma_2 + nb_2, ma_3 + nb_3 \rangle$$

$$\vec{a} = \langle 1, 2, 3 \rangle \quad \vec{b} = \langle 2, -2, 4 \rangle$$

$$2\vec{a} - 3\vec{b} = \langle 2(1) - 3(2), 2(2) - 3(-2), 2(3) - 3(4) \rangle = \langle -4, 10, -6 \rangle$$

$$= \langle -2, 5, -3 \rangle^*$$

$$\|\vec{a}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

★ find unit vec $\parallel \vec{a}$

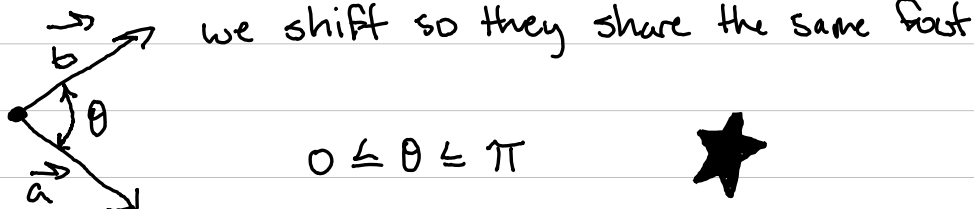
$$\hat{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{\langle 1, 2, 3 \rangle}{\sqrt{14}}$$

9.3

DOT Product (Scalar)

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \quad \vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\vec{a} \cdot \vec{b}$$



$$\vec{a} \cdot \vec{b} = \|\vec{a}\|(\|\vec{b}\|)(\cos \theta) \leftarrow \text{scalar}$$

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\text{if } \vec{a} \nearrow \vec{b}, \theta = 0, \text{ so } \vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos 0 = \|\vec{a}\| \|\vec{b}\| \quad \cos 0 = 1$$

$$\text{anti-parallel (opposite directions)} \quad \vec{a} \nwarrow \vec{b}, \theta = \pi, \text{ so } \vec{a} \cdot \vec{b} = -(\|\vec{a}\| \|\vec{b}\|) \quad \cos \pi = -1$$

$$\text{if } \vec{a} \perp \vec{b}, \theta = \frac{\pi}{2} \text{ so } \vec{a} \cdot \vec{b} = 0 \quad (\cos \frac{\pi}{2} = 0)$$

$$0 \leq \theta < \frac{\pi}{2}, \vec{a} \cdot \vec{b} > 0$$

$$\frac{\pi}{2} < \theta \leq \pi, \vec{a} \cdot \vec{b} < 0$$

$$\vec{a} \nearrow \vec{b}, \text{ or } \vec{a} \nwarrow \vec{b}, \quad \vec{a} = m\vec{b}$$

$$\vec{a} = \langle 3, 2, 7 \rangle \quad \vec{b} = \langle 6, 4, 14 \rangle \quad \downarrow \text{this needs to exist for them to be } \nearrow \text{ or } \nwarrow$$

$$\langle 6, 4, 14 \rangle = 2 \langle 3, 2, 7 \rangle = 2\vec{a}$$

$$\text{so } \vec{a} \nearrow \vec{b}$$

Find the angle θ between $\vec{a} = \langle 3, -2, 4 \rangle$ and $\vec{b} = \langle 2, -1, 7 \rangle$

$$\vec{a} \cdot \vec{b} = (\|\vec{a}\|)(\|\vec{b}\|)\cos \theta$$

$$\theta = \cos^{-1} \left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} \right)$$

$$\|\vec{a}\| = \sqrt{9+4+16} = \sqrt{29}$$

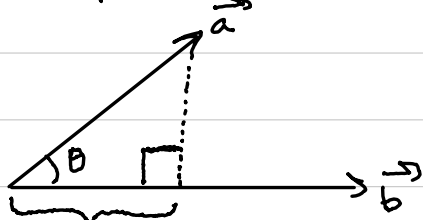
$$\|\vec{b}\| = \sqrt{4+1+49} = \sqrt{54}$$

$$\vec{a} \cdot \vec{b} = 3(2) + (-2)(-1) + 4(7) = 6 + 2 + 28 = 36$$

$$\theta = \cos^{-1} \left(\frac{36}{\sqrt{29}\sqrt{54}} \right) \quad \text{this is far enough}$$

Scalar projection of $\vec{a}$ into $\vec{b}$

Vector projection of $\vec{a}$ into $\vec{b}$



$$\text{scalar projection} = \|\vec{a}\|(\cos \theta) \left(\frac{\|\vec{b}\|}{\|\vec{b}\|} \right) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|}$$

($\vec{a}$ into $\vec{b}$)

Vec. Proj. = vector whose magnitude = scalar Proj.

$$\text{V.P.}_{\vec{a} \rightarrow \vec{b}} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|} \left(\frac{\vec{b}}{\|\vec{b}\|} \right) = \frac{\vec{a} \cdot \vec{b}}{(\|\vec{b}\|)^2} \left(\vec{b} \right)$$

$$\downarrow \nearrow \vec{b}$$

Find scalar & vector proj. of $\vec{a}$ into $\vec{b}$

$$\vec{a} = \langle 2, 3 \rangle \quad \vec{b} = \langle -4, 1 \rangle \quad \sqrt{16+1}$$

$$\text{scalar proj} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|} = \frac{(-8+3)}{\sqrt{17}} = \frac{-5}{\sqrt{17}}$$

$$\text{vect. proj.} = \frac{-5}{\sqrt{17}} \left(\frac{\langle -4, 1 \rangle}{\sqrt{17}} \right) \leftarrow \vec{b}$$

$\leftarrow \|\vec{b}\|$

$$= \left[\frac{-5}{17} \langle -4, 1 \rangle \right]$$

WW Ex. 16) $\vec{a} = \langle \sqrt{3}, -1 \rangle \quad \vec{b} = \langle 0, 5 \rangle$

$\theta = ?$

$$\vec{a} \cdot \vec{b} = \sqrt{3}(0) + (-1)(5) = -5$$

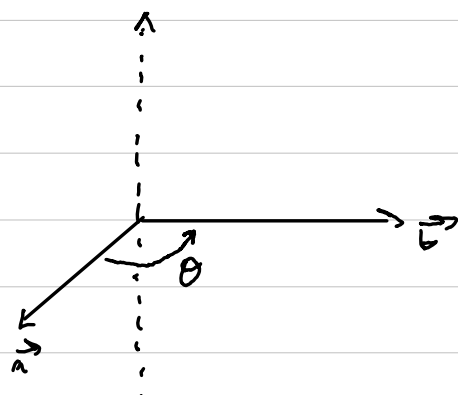
$$\|\vec{a}\| = \sqrt{3+1} = 2$$

$$\|\vec{b}\| = \sqrt{0+25} = 5$$

$$\theta = \cos^{-1} \left(\frac{-5}{2(5)} \right)$$

9.4

Cross Product (Vector Product)



$$\vec{a} \times \vec{b} = \vec{c}$$

not commutative

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta$$

$$0 \leq \theta \leq \pi \quad \sin \theta \geq 0$$



$$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$$

Algebraic Interpretation

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \quad \vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

* product of diagonal - product of diagonal

$$= \hat{i}(a_2 b_3 - a_3 b_2) - \hat{j}(a_1 b_3 - a_3 b_1) + \hat{k}(a_1 b_2 - a_2 b_1)$$

$$\hat{i} = \langle 1, 0, 0 \rangle \quad \hat{j} = \langle 0, 1, 0 \rangle \quad \hat{k} = \langle 0, 0, 1 \rangle$$

$$\hat{i} \times \hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \hat{i}(0 \cdot 0 - 0 \cdot 1) - \hat{j}(1 \cdot 0 - 0 \cdot 0) + \hat{k}(1 \cdot 1 - 0 \cdot 0) = \hat{k}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and in turn, } \hat{j} \times \hat{k} = \hat{i} \text{ and } \hat{k} \times \hat{i} = \hat{j}$$

This is for a right-reference system

Example: $\vec{a} = \langle 2, -1, 3 \rangle \quad \vec{b} = \langle 4, 1, 0 \rangle$

find $\vec{a} \times \vec{b}$ then $\sin \theta$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 4 & 1 & 0 \end{vmatrix} = \hat{i}(0 - 3) - \hat{j}(0 - 12) + \hat{k}(2 - 4) = -3\hat{i} + 12\hat{j} - 2\hat{k}$$

now find $\sin \theta$

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta$$

$$\|\vec{a}\| = \sqrt{4+1+9} = \sqrt{14}$$

$$\|\vec{b}\| = \sqrt{16+1+0} = \sqrt{17}$$

$$3\sqrt{1+16+4} = 3\sqrt{21} = (\sqrt{14})(\sqrt{17}) \sin \theta$$

$$\sin \theta = \frac{3\sqrt{21}}{\sqrt{14}\sqrt{17}} = \frac{\|\vec{a} \times \vec{b}\|}{\|\vec{a}\| \|\vec{b}\|}$$

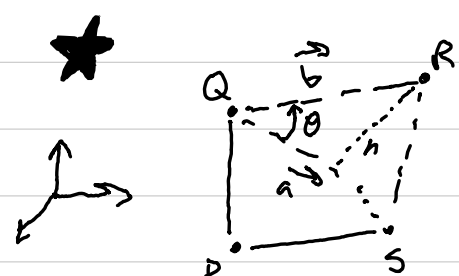
we know



$$\theta = \cos^{-1} \left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} \right)$$

$$\theta = \sin^{-1} \left(\frac{\|\vec{a} \times \vec{b}\|}{\|\vec{a}\| \|\vec{b}\|} \right)$$

BUT! This does not work unless $\theta \leq 90^\circ$
so use the dot product formula instead.



how do we find the area of the triangle?

$$\text{Area}_{\text{QRS}} = \frac{\|\vec{a}\| (h)}{2} = \frac{\|\vec{a}\| \|\vec{b}\| \sin \theta}{2} = \frac{\|\vec{a} \times \vec{b}\|}{2}$$

$$\text{AREA}_{\text{PQRS}} = \|\vec{a} \times \vec{b}\|$$

this is very easy

Ex. Find the area of triangle

$$P(1, 2, 3) \quad Q(2, -1, 4) \quad R(2, 1, 3)$$

1) Find an $\vec{a}$ & $\vec{b}$ (head-foot)

$$\vec{a} = \vec{PR} = \langle 2-1, 1-2, 3-3 \rangle = \langle 1, -1, 0 \rangle$$

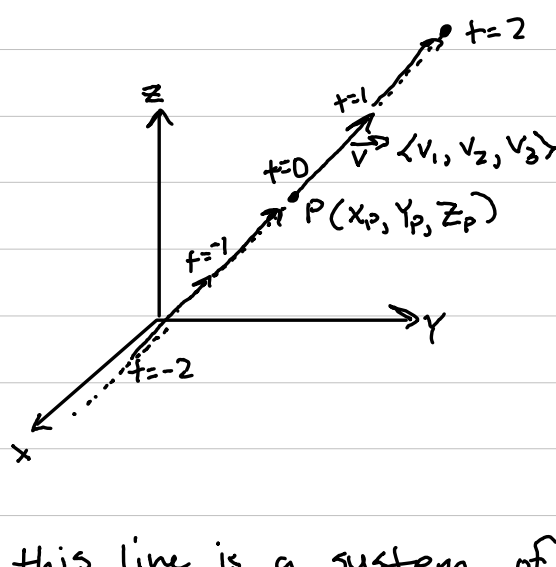
$$\vec{b} = \vec{PQ} = \langle 2-1, -1-2, 4-3 \rangle = \langle 1, -3, 1 \rangle$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & -3 & 1 \end{vmatrix} = \hat{i}(-1-0) - \hat{j}(1-0) + \hat{k}(-3+1) = \langle -1, -1, -2 \rangle$$

$$\|\vec{a} \times \vec{b}\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

$$\text{Area} = \frac{\sqrt{6}}{2}$$

9.5



this line is a system of 3 eq. $\hat{=}$

(Parametric Equation)

$$\begin{cases} x(t) = x_p + v_1 t \\ y(t) = y_p + v_2 t \\ z(t) = z_p + v_3 t \end{cases} \quad t \in (-\infty, +\infty)$$



when $t=0$, we are at the point P

Vector form of Par. Eq.

$$\langle x(t), y(t), z(t) \rangle = \langle x_p, y_p, z_p \rangle + t \langle v_1, v_2, v_3 \rangle$$

when we move up or down w/ t (ex. $t=1$), we move up or down one length of $\vec{v}$

Find par. eq. of line through $P(3, -1, 4)$

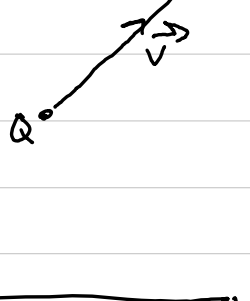
and $\nearrow \nearrow$ to $\vec{v} = \langle 1, 2, 1 \rangle$

$$\begin{cases} x(t) = 3 + 1t \\ y(t) = -1 + 2t \\ z(t) = 4 + 1t \end{cases} \quad t \in (-\infty, +\infty)$$

Find par. eq. of line through $P(2, 1, 4)$

and $\nearrow \nearrow$ to

$$\begin{cases} x(t) = 3 + 3t \\ y(t) = 2 - 2t \\ z(t) = 1 + 7t \end{cases}$$



$$\vec{v} = \langle 3, -2, 7 \rangle \text{ so}$$

$$\begin{cases} x(t) = 2 + 3t \\ y(t) = 1 - 2t \\ z(t) = 4 + 7t \end{cases} \quad t \in (-\infty, +\infty)$$

Symmetric Eq.

$$P(x_p, y_p, z_p) \quad \vec{v} = \langle v_1, v_2, v_3 \rangle$$

Par. Eq.

$$\begin{cases} x(t) = x_p + v_1 t \\ y(t) = y_p + v_2 t \\ z(t) = z_p + v_3 t \end{cases} \quad \begin{cases} t = \frac{x - x_p}{v_1} \\ t = \frac{y - y_p}{v_2} \\ t = \frac{z - z_p}{v_3} \end{cases} \quad \text{ONLY IF } v_1, v_2, v_3 \neq 0$$



$$\frac{x - x_p}{v_1} = \frac{y - y_p}{v_2} = \frac{z - z_p}{v_3}$$

THIS is the symmetric equation of the line for P $\nearrow \nearrow \vec{v}$

Ex. Find an intersection between

$$L_1: \frac{x-1}{2} = \frac{y+1}{1} = \frac{z-2}{4} \quad \text{if it exists}$$

$$L_2: \frac{x+2}{4} = \frac{y}{-3} = \frac{z-1/2}{-1}$$

Param = t

$$L_1: \begin{cases} x = 1 + 2t \\ y = -1 + t \\ z = 2 + 4t \end{cases}$$

param = s

$$L_2: \begin{cases} x = -2 + 4s \\ y = 0 - 3s \\ z = 1/2 - 1s \end{cases}$$

an intersect implies some $t =$ some s

so compare eq.s

$$\begin{cases} 1 + 2t = -2 + 4s \\ -1 + t = -3s \\ 2 + 4t = 1/2 - s \end{cases} \quad \begin{matrix} \text{this is a linear system} \\ \text{most of the time, 3 eqs. and} \\ \text{2 unknowns means no intersect.} \end{matrix}$$

to check for a solution, solve 2 eqs. for s and t

then check if they satisfy the 3rd eq.

$$\begin{cases} 2t - 4s = -3 \\ t + 3s = 1 \end{cases} \quad \begin{matrix} 0 - 10s = -5 \\ s = 1/2 \end{matrix} \quad \begin{matrix} t + 3(1/2) = 1 \\ t = -1/2 \end{matrix}$$

$$2 + 4(-1/2) = 1/2 - (1/2)$$

$$2 - 2 = 1/2 - 1/2$$

$$0 = 0 \quad \checkmark$$

the lines intersect

$$L_2(s = 1/2): \begin{cases} x = -2 + 4(1/2) & 0 \\ y = 0 - 3(1/2) & -3/2 \\ z = 1/2 - 1/2 & 0 \end{cases}$$

$$\text{Int. at } P = (0, -3/2, 0)$$

8-30-24

At the lowest level, a parametric eq. in 3D is

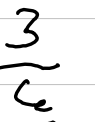
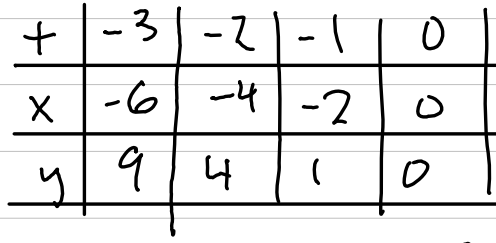
$$\begin{cases} x(t) = f(t) \\ y(t) = g(t) \\ z(t) = h(t) \end{cases} \quad t \in I$$

In 2D, it's

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \quad t \in I$$

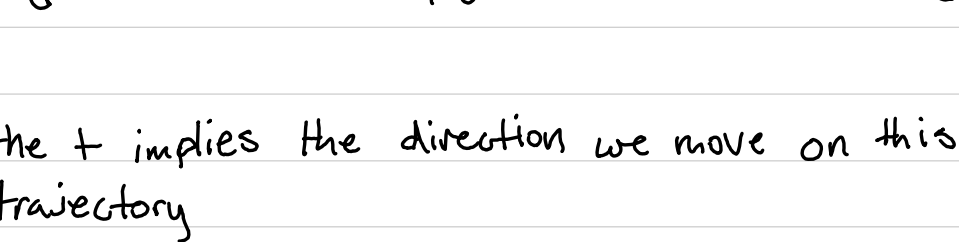
Think of t as the time parameter, determining how far along you are on a trajectory

A par. eq. is a rule that for all $t \in I$, there exists a unique point $(x(t), y(t), z(t))$ in the space (for 3D)



$$\begin{cases} x = 2t \\ y = t^2 \end{cases} \quad t \in [-3, 3]$$

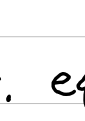
t	-3	-2	-1	0	1	2	3
x	-6	-4	-2	0	2	4	6
y	9	4	1	0	1	4	9



the t implies the direction we move on this trajectory

now we try to get the symmetric from this eq.

$$\begin{cases} x = 2t \\ y = t^2 \end{cases} \quad t \in I \quad \begin{matrix} t = \frac{x}{2} \\ y = \frac{x^2}{4} \end{matrix}$$



this is the system of eq. as one function, which we see on the graph

It is not the same as the par. eq.

We lose info from par. eq. $\rightarrow$ basic function

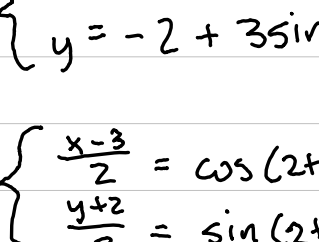
$$\begin{cases} x = 2 \cos \theta \\ y = 2 \sin \theta \end{cases} \quad \theta \in [0, 2\pi]$$

because $\sin^2 + \cos^2 = 1$, we will square both

$$\begin{cases} x^2 = 4 \cos^2 \theta \\ y^2 = 4 \sin^2 \theta \end{cases}$$

$$= x^2 + y^2 = 4(\cos^2 + \sin^2)$$

$$= x^2 + y^2 = 4 \rightarrow \text{function of a circle}$$



Example

* how can we manipulate this?

$$\begin{cases} x = 3 + 2 \cos(2t) \\ y = -2 + 3 \sin(2t) \end{cases} \quad t \in [0, 2\pi]$$

$$\begin{cases} \frac{x-3}{2} = \cos(2t) \\ \frac{y+2}{3} = \sin(2t) \end{cases} \quad \text{then square all, which gives } \cos^2 + \sin^2 = 1$$

$$\left(\frac{x-3}{2} \right)^2 + \left(\frac{y+2}{3} \right)^2 = 1$$

An ellipse

Standard ellipse form:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

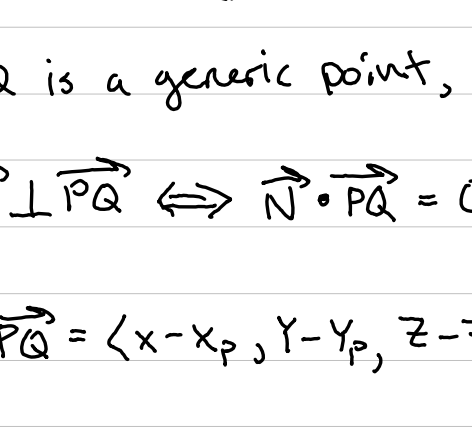


? ellipse? because $\cos$ and $\sin$ had $(2t)$, we move twice as fast

9.6

Plane in 3D

To simply define a plane:
a point $P = (x_p, y_p, z_p)$ that is in the plane
an $\vec{N} = \langle A, B, C \rangle$



Q is a generic point, and

$$\vec{N} \perp \vec{PQ} \Leftrightarrow \vec{N} \cdot \vec{PQ} = 0$$

$$\vec{PQ} = \langle x - x_p, y - y_p, z - z_p \rangle$$

$$\langle A, B, C \rangle \cdot \langle x - x_p, y - y_p, z - z_p \rangle = 0$$

$$Ax + By + Cz = Ax_p + By_p + Cz_p$$

$$Ax + By + Cz + D = 0$$

This is the standard eq. for a plane in 3D

In 2D, would be a line, but here it's a plane

Example

Find eq. of plane through $P = (2, 1, 4)$ and $\perp$ to $\vec{N} = \langle 3, -2, 1 \rangle$

$$3x - 2y + z = 3(2) - 2(1) + 1(4)$$

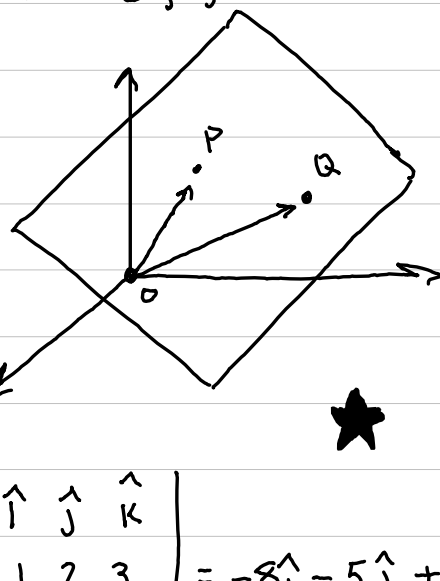
$$= 8$$

$$3x - 2y + z - 8 = 0$$

* we can also divide the whole thing to change / fix coefficients

Example

Find eq. of plane passing for $Q(1, 2, 3)$ and $\nearrow$ to $7x + 3y = 4$



$(x, y, z \text{ coord})$

$$\vec{N} = \langle 7, 3, 0 \rangle$$

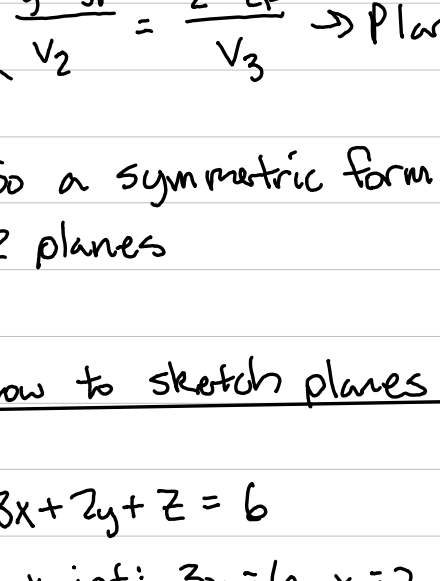
$$7x + 3y = 7(1) + 3(2) + 0(3)$$

$$= 7 + 6 = 13$$

$$7x + 3y - 13 = 0$$

Example

Find a plane through $P(1, 2, 3)$, $Q(1, 4, 2)$, and $O(0, 0, 0)$



The cross product between $\vec{OP}$ and $\vec{OQ}$ will give us the normal

$$\vec{N} = \vec{OP} \times \vec{OQ}$$

$$\vec{OP} = \langle 1, 2, 3 \rangle \quad \vec{OQ} = \langle 1, 4, 2 \rangle$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 4 & 2 \end{vmatrix} = -8\hat{i} - 5\hat{j} + 6\hat{k} = \langle -8, -5, 6 \rangle = \vec{N}$$

$$\text{so, } -8x - 5y + 6z = 0$$

* used $(0, 0, 0)$ as reference

$$\begin{cases} \frac{x - x_p}{v_1} = \frac{y - y_p}{v_2} \rightarrow \text{Plane 1} \\ \frac{y - y_p}{v_2} = \frac{z - z_p}{v_3} \rightarrow \text{Plane 2} \end{cases}$$

So a symmetric form is the intersection of 2 planes

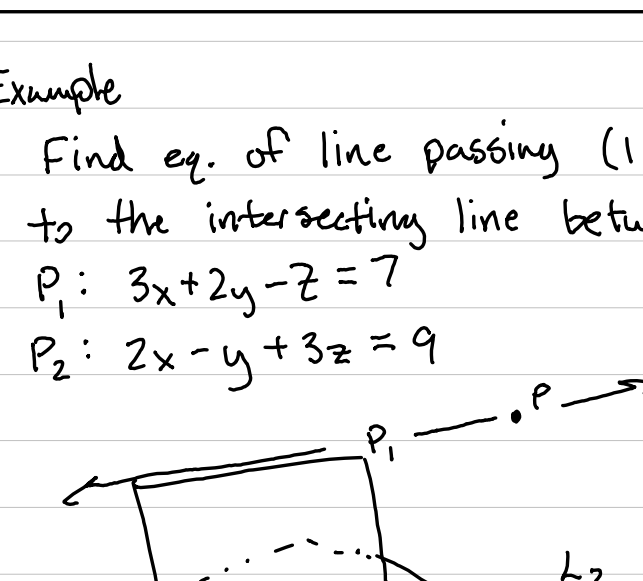
How to sketch planes

$$3x + 2y + z = 6$$

$$x\text{-int: } 3x = 6 \quad x = 2$$

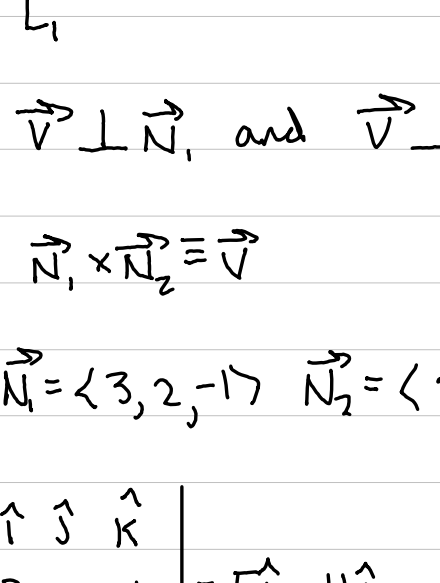
$$y\text{-int: } 2y = 6 \quad y = 3$$

$$z\text{-int: } z = 6$$



Example

Find eq. of line passing for $P(2, -2, 4)$ and $\perp$ to the plane $3x + 2y + 3z = 7$



$$\text{so } \vec{v} \nearrow \vec{N} \text{ and } \vec{v} \equiv \vec{N} = \langle 3, 2, 3 \rangle$$

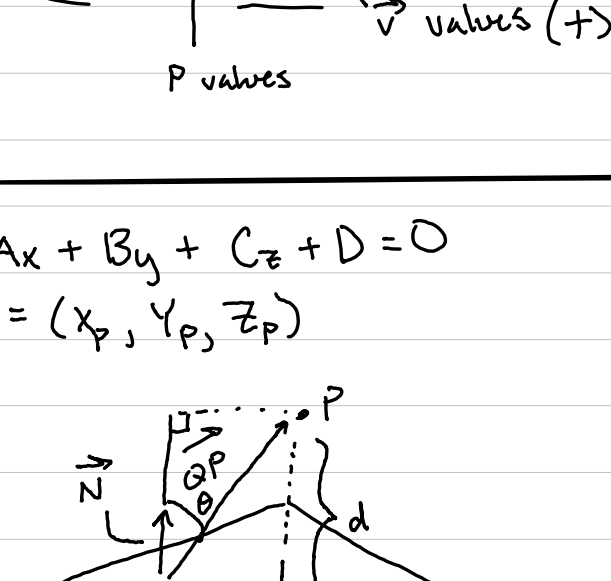
$$\text{so the line is } \frac{x-2}{3} = \frac{y+2}{2} = \frac{z-4}{3}$$

Example

Find eq. of line passing $(1, 2, 3)$ and $\nearrow$ to the intersecting line between

$$P_1: 3x + 2y - z = 7$$

$$P_2: 2x - y + 3z = 9$$



If we get a $\vec{v}$ for L_2 , we can use it for L_1

$$\vec{v} \perp \vec{N}_1 \text{ and } \vec{v} \perp \vec{N}_2, \text{ so we cross product } \vec{N}_1 \times \vec{N}_2$$

$$\vec{N}_1 \times \vec{N}_2 \equiv \vec{v}$$

$$\vec{N}_1 = \langle 3, 2, -1 \rangle \quad \vec{N}_2 = \langle 2, -1, 3 \rangle$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 5\hat{i} - 11\hat{j} - 7\hat{k} = \langle 5, -11, -7 \rangle = \vec{v}$$

so the line is

$$\begin{cases} x = 1 + 5t \\ y = 2 - 11t \\ z = 3 - 7t \end{cases} \quad t \in (-\infty, \infty)$$

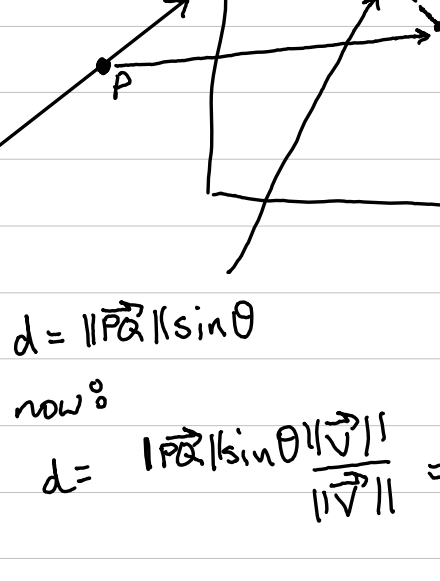
P values

t values (+)

$$Ax + By + Cz + D = 0$$

$$P = (x_p, y_p, z_p)$$

Find distance between point & the plane



Q is a generic point on the plane

$$d = |\vec{PQ}| \cos \theta \quad \text{where } Q \text{ is any point on the plane}$$

$$= |\vec{PQ}| \cos \theta \left(\frac{|\vec{N}|}{|\vec{N}|} \right) = \left| \frac{\vec{PQ} \cdot \vec{N}}{|\vec{N}|} \right|$$

$$= \frac{|Ax_p + By_p + Cz_p + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Ex. Find eq. of the sphere centered at $C(3, -1, 4)$ and tangent to the plane $3x + 4y - z + 7 = 0$

$$3x + 4y - z + 7 = 0$$

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

$$= d^2$$

plug in coords of center to eq. of plane

$$d = \frac{|3(3) + 4(-1) - 4 + 7|}{\sqrt{3^2 + 4^2 + (-1)^2}}$$

$$d = \frac{8}{\sqrt{26}}$$

$$\text{eq. of sphere: } (x-3)^2 + (y+1)^2 + (z-4)^2 = \frac{64}{26}$$

$$(x-3)^2 + (y+1)^2 + (z-4)^2 = \frac{32}{13}$$

$$d = |\vec{PQ}| \sin \theta$$

now

$$d = \frac{|\vec{PQ}| \sin \theta |\vec{N}|}{|\vec{N}|} = \frac{|\vec{v} \times \vec{PQ}|}{|\vec{v}|}$$

Find the dist between $L_1: \begin{cases} x = 1 + 3t \\ y = -2 + 4t \\ z = -t \end{cases}$ and $Q = (1, 0, 3)$

$$P = (1, -2, 0) \quad \vec{v} = \langle 3, 4, -1 \rangle$$

$$\vec{PQ} = \langle 1-1, 0-(-2), 3-0 \rangle = \langle 0, 2, 3 \rangle$$

$$\vec{v} \times \vec{PQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & -1 \\ 0 & 2 & 3 \end{vmatrix} = (12+2)\hat{i} - (9-0)\hat{j} + (6-4)\hat{k} = \langle 14, -9, 2 \rangle$$

$$d = \frac{\sqrt{14^2 + (-9)^2 + 2^2}}{\sqrt{3^2 + 4^2 + (-1)^2}}$$

9.7

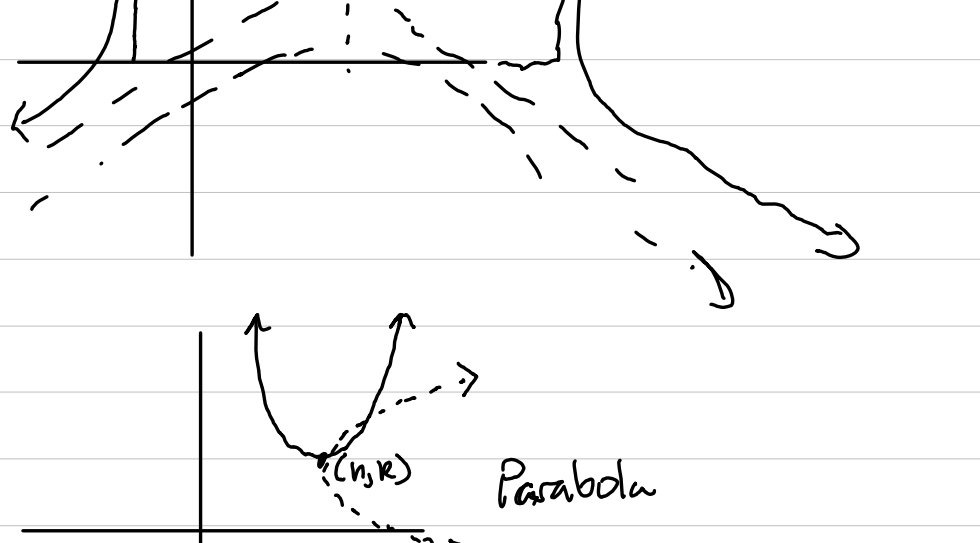
Conics

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \rightarrow \text{2D Ellipse}$$

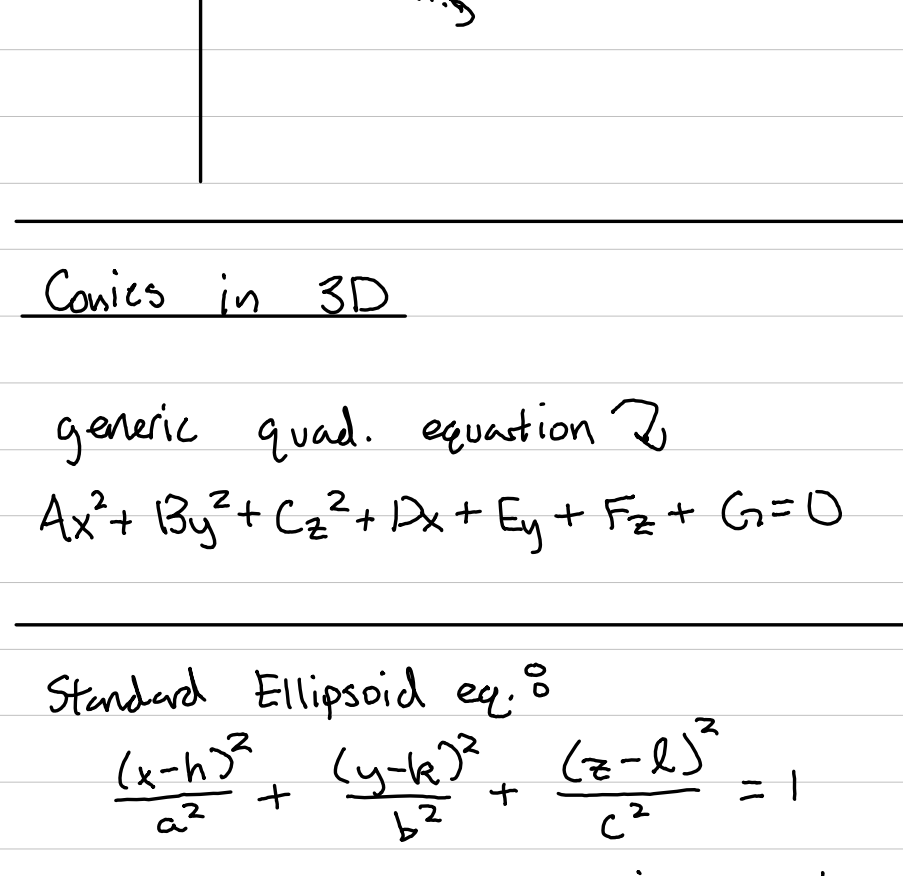
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \rightarrow \text{2D Hyperbola}$$

$$(y-k) = a(x-h)^2 \rightarrow \text{2D Parabola}$$

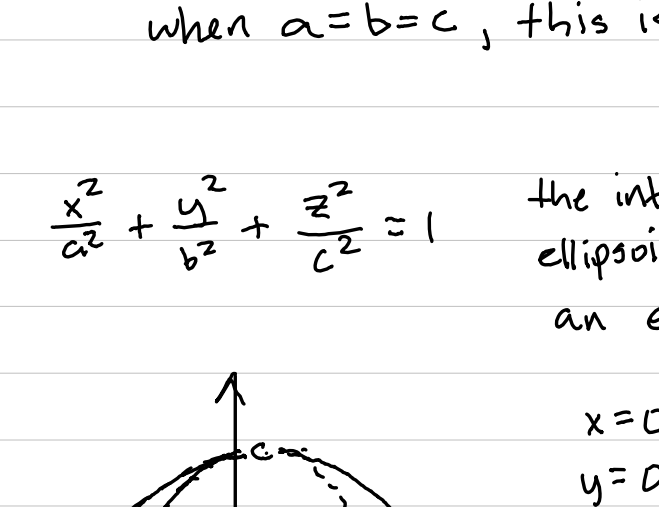
$$(x-h) = a(y-k)^2 \rightarrow \text{2D Parabola}$$



Ellipse (kinda)



Hyperbolas



Parabola

Conics in 3D

generic quad. equation ↴

$$Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz + G = 0$$

Standard Ellipsoid eq.º

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

when $a=b=c$, this is a sphere

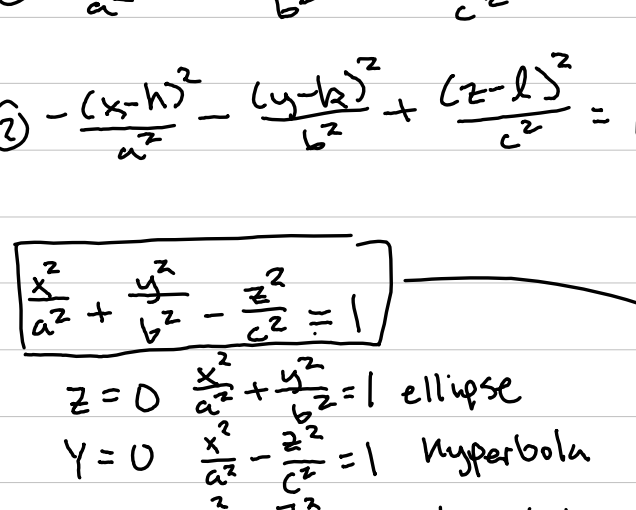
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

the intersection of an ellipsoid w/ a plane is an ellipse

$$x=0 \rightarrow \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$y=0 \rightarrow \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$$

$$z=0 \rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

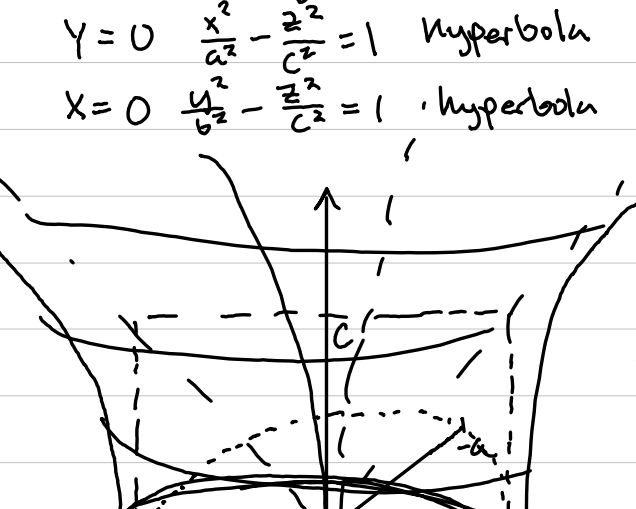


now slice it w/ another planeº

$$z = \frac{c}{2} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{\frac{c}{2}}{c^2} = 1 \rightarrow \frac{x^2}{\frac{3}{4}a^2} + \frac{y^2}{\frac{3}{4}b^2} = 1$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\boxed{\frac{x^2}{\frac{3}{4}a^2} + \frac{y^2}{\frac{3}{4}b^2} = 1}$$



It'll look something like this

Hyperboloids → 2 families

$$\textcircled{1} \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} - \frac{(z-l)^2}{c^2} = \pm 1 \quad \text{2 binomials w/ same sign as 1}$$

$$\textcircled{2} -\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1 \quad \text{1 binomial w/ same sign as 1}$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1}$$

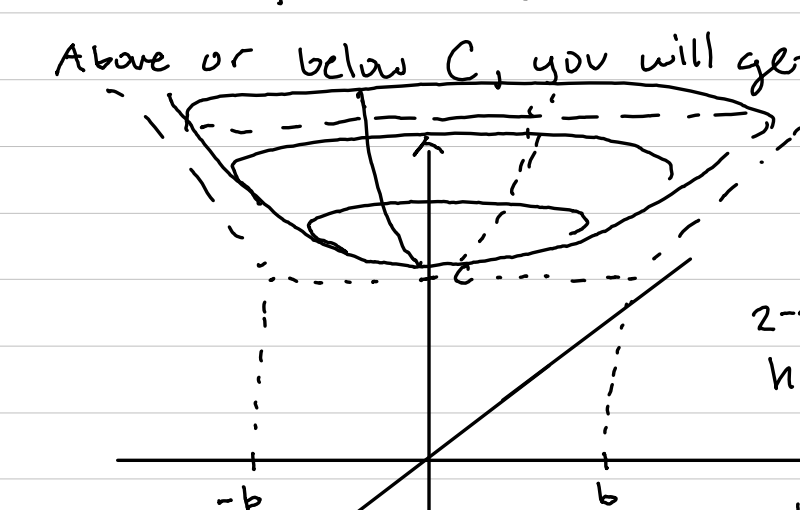
$$z=0 \rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ ellipse}$$

$$y=0 \rightarrow \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \text{ hyperbola}$$

$$x=0 \rightarrow \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \text{ hyperbola}$$

1 surface

this is a one-sheet elliptic hyperboloid



position of the minus sign determines axis alignment of the funnel

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

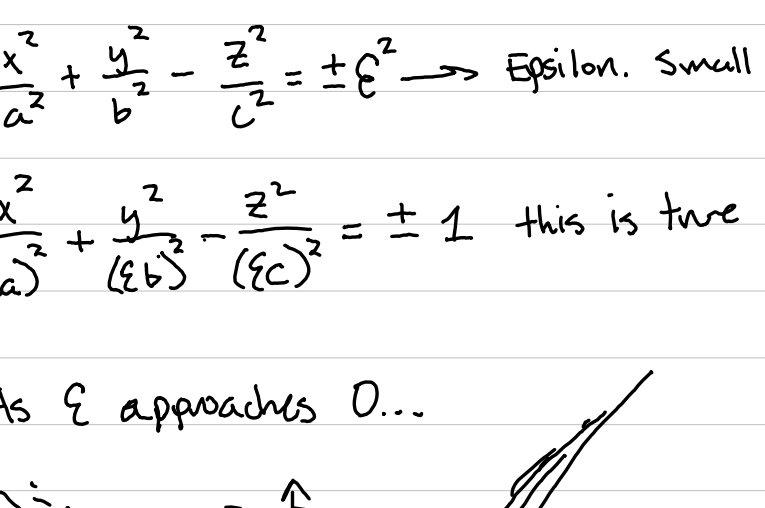
$$z=0 \rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = -1 \rightarrow \text{this has no solution}$$

hyperbola $y=0 \rightarrow \frac{x^2}{a^2} - \frac{z^2}{c^2} = -1$ so this object is not intercepting the xy plane

$$\text{hyperbola } x=0 \rightarrow \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

$$\text{if } z = \pm c, \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{c^2}{c^2} = -1 \rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 0 \quad (0, 0, \pm c)$$

Above or below C, you will get ellipses



2-sheet elliptic hyperboloid

$$=-1 \rightarrow \text{up + down hyperboloid}$$

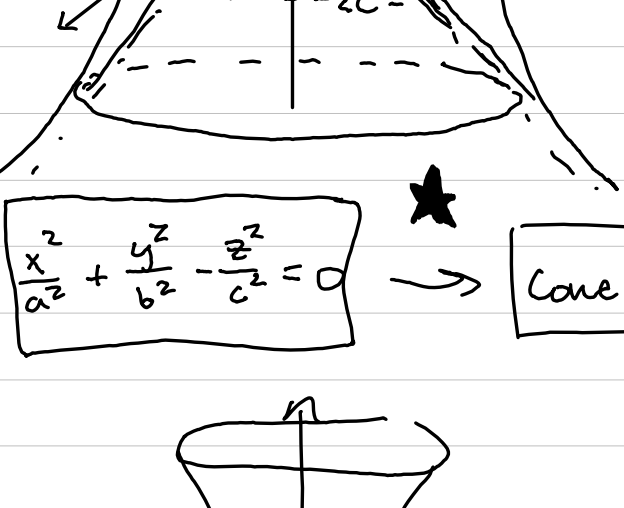
$$=1 \rightarrow \text{left + right hyperboloid}$$

position of minus sign determines axis alignment

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \pm \epsilon^2 \rightarrow \text{Epsilon. Small but not 0}$$

$$\frac{x^2}{(\epsilon a)^2} + \frac{y^2}{(\epsilon b)^2} - \frac{z^2}{(\epsilon c)^2} = \pm 1 \text{ this is true for } \epsilon > 0$$

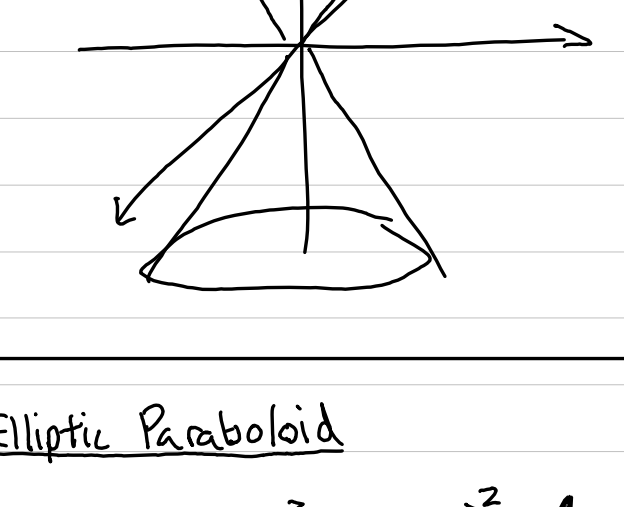
As ϵ approaches 0...



it forms cones

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0}$$

$$\rightarrow \text{Cone formula}$$



Elliptic Paraboloid

$$(z-l) = \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} \quad \star$$

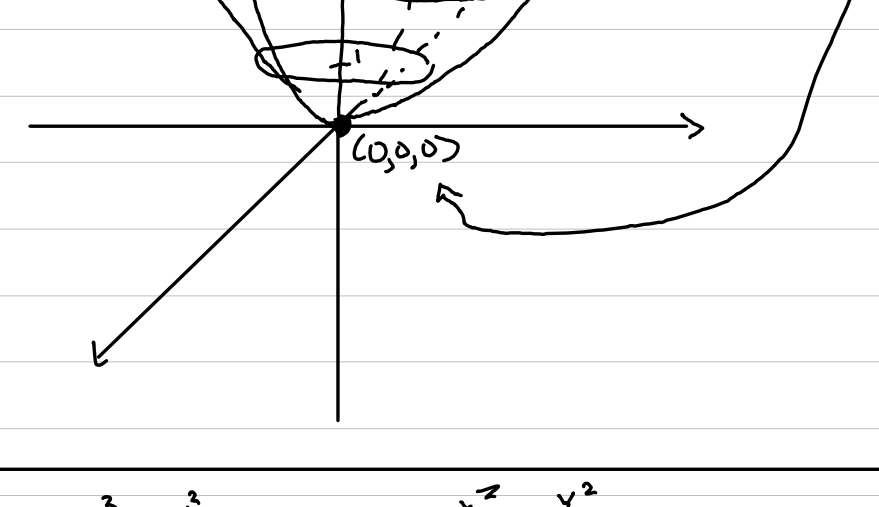
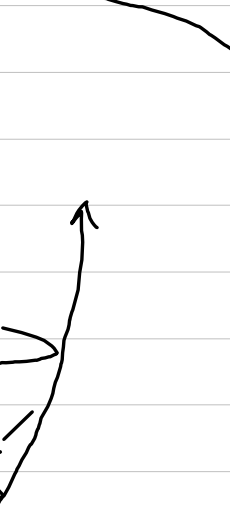
can be negative, making it face downwards

$$\uparrow \quad z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$x=0 \rightarrow z = \frac{y^2}{b^2}$$

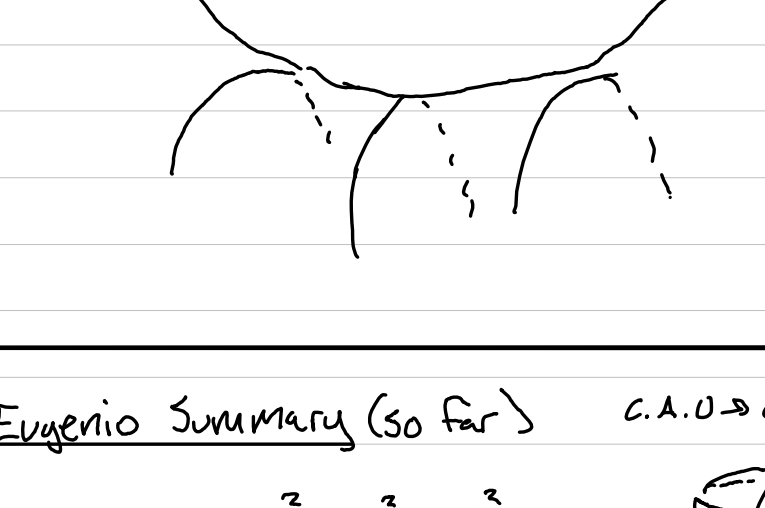
$$y=0 \rightarrow z = \frac{x^2}{a^2}$$

$$z=0 \rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$$



$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2} \text{ or } z = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

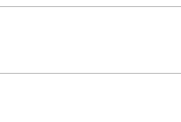
Hyperbolic Paraboloid



Eugenio Summary (so far)

C.A.O. → centered at origin

$$\text{ellipsoid } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



$$\text{1-sheet hyperboloid } \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \pm 1$$

$$\text{2-sheet hyperboloid } \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \pm 1$$

$$\text{cone } \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

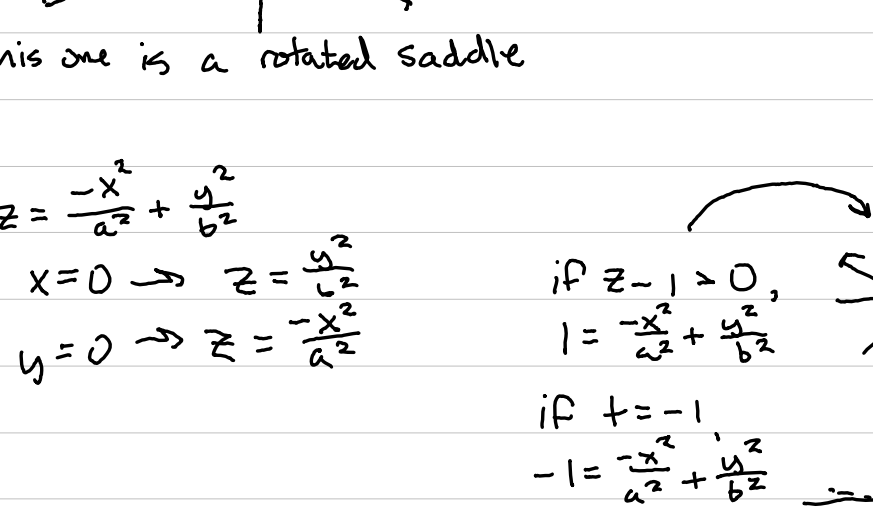
C.A.O.

$$\text{elliptic paraboloid } z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



$$\text{hyperbolic paraboloid C.A.O.}$$

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}, \quad z = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$



saddle-esque

this one is a rotated saddle

$$z = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$x=0 \rightarrow z = \frac{y^2}{b^2}$$

$$y=0 \rightarrow z = -\frac{x^2}{a^2}$$

$$\text{if } z=1 > 0, \quad 1 = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\text{if } z=-1, \quad -1 = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\text{if } z=1, \quad 1 = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\text{if } z=-1, \quad -1 = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\text{if } z=1, \quad 1 = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

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$$\text{if } z=1, \quad 1 = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

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$$\text{if } z=1, \quad 1 = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

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$$\text{if } z=1, \quad 1 = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

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$$\text{if } z=-1, \quad -1 = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\text{if } z=1, \quad 1 = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\text{if } z=-1, \quad -1 = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\text{if } z=1, \quad 1 = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\text{if } z=-1, \quad -1 = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\text{if } z=1, \quad 1 = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

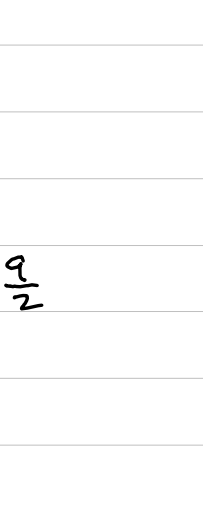
sometimes you only get partial eqs.

$$x^2 + y^2 - z^2 = 0 \text{ (cone)}$$

and you can take things out

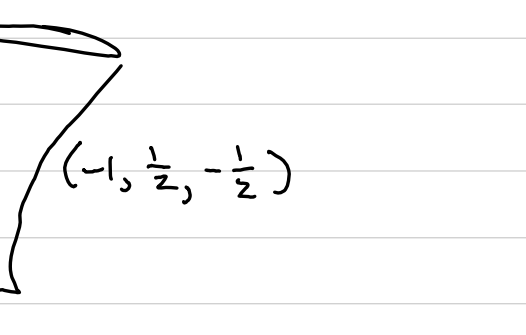
$$z^2 = x^2 + y^2 \quad \boxed{z = \sqrt{x^2 + y^2}}$$

this is the positive cone



$$x^2 + y^2 + z^2 = 4 \rightarrow z = -\sqrt{4 - x^2 - y^2}$$

(sphere)



bottom half of sphere

Book exampleº



$$z^2 = x^2 - y^2 + 2x + y - z + 5$$

① put in standard form by completing the square

$$(x^2 + 2x) + (-y^2 + y) + (-z^2 - z) = -5$$

② move signs

$$(x^2 + 2x + 1) + (-1)(y^2 - y + \frac{1}{4}) + (-1)(z^2 + z + \frac{1}{4}) = -5$$

③ h, k, l ↓

$$\text{find } h, k, l \downarrow$$

$$h = \frac{2}{2} = -1$$

$$\text{then add } h^2$$

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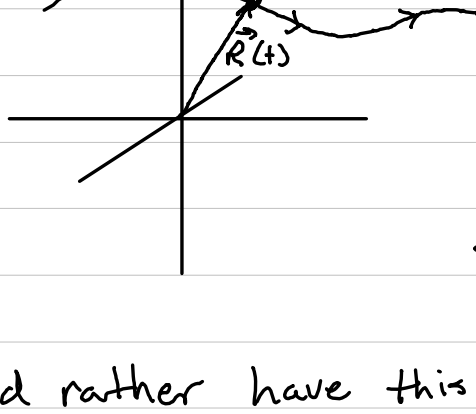
Ch. 9 - TO KNOW:

- Surfaces 
- Planes 
- Recognize & Sketch these

10.1

Value Vector Functions

$$\begin{cases} x(t) = f(t) \\ y(t) = g(t) \\ z(t) = h(t) \end{cases} \quad t \in I$$



We'd rather have this in a vector format:

$$\vec{R}(t) = \langle x(t), y(t), z(t) \rangle = \langle f(t), g(t), h(t) \rangle$$

this is a vector whose components are changing in time and points to our location on a line

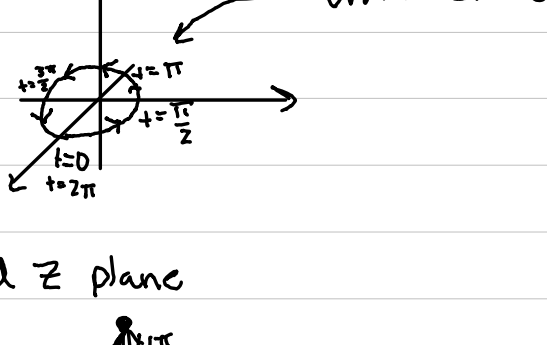
this allows us to use vector algebra on value vector functions

Examples

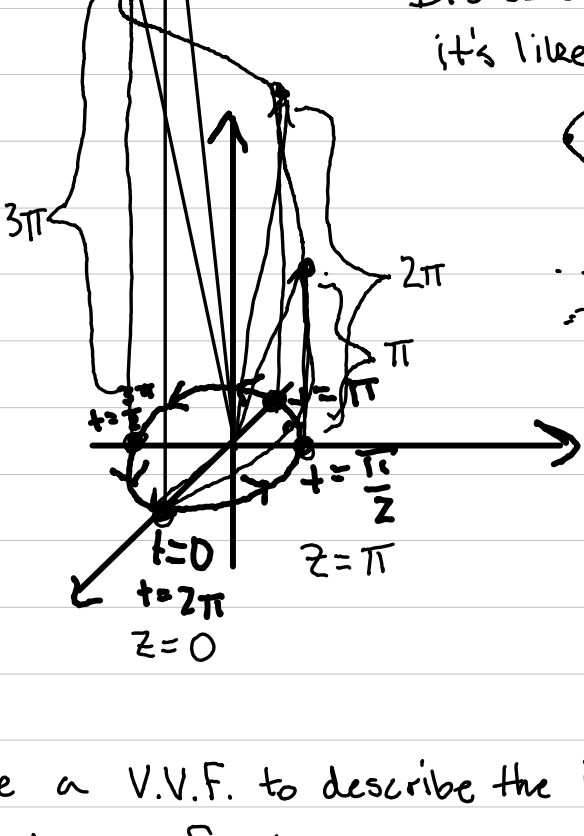
sketch $\vec{R}(t) = \langle \cos(t), \sin(t), 2t \rangle \quad t \in [0, 2\pi]$

$$\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \\ z(t) = 2t \end{cases} \quad t \in [0, 2\pi]$$

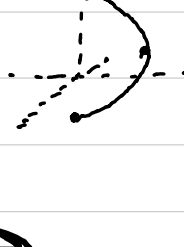
① Look at what happens in xy-plane



② Add z plane



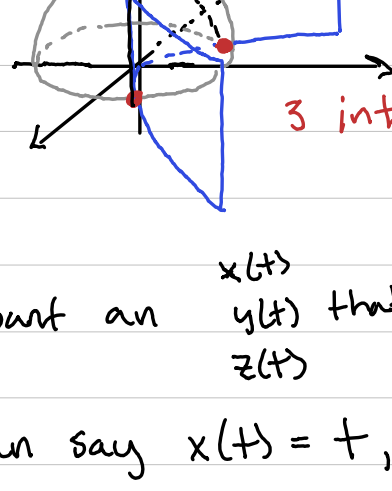
It's a bad drawing but it's like a helix



Use a V.V.F. to describe the intersection between 2 given surfaces

$$\begin{cases} z = \sqrt{4 - x^2 - y^2} \rightarrow \text{half-sphere} \\ y = x^2 \rightarrow \text{cylinder} \end{cases}$$

$$z^2 = 4 - x^2 - y^2 \Rightarrow x^2 + y^2 + z^2 = 4 \rightarrow \text{sphere}$$



3 intersection points

we want an $x(t), y(t), z(t)$ that satisfies both

we can say $x(t) = t$, so that

$$y(t) = t^2 \quad \text{and} \quad z(t) = \sqrt{4 - t^2 - t^4}$$

$$y = t^2$$

now we must find the interval I ($t \in I$)

where they are together

NOTE - t being an extra variable, there are infinite ways to describe I

(i.e. setting $x(t) = -t$ goes the other way on the curve)

$$t \in I = \{t : 4 - t^2 - t^4 \geq 0\}$$

"such that"

the element of $z(t)$'s



conventionally, we take the biggest set where $x(t), y(t)$ and $z(t)$ are defined (in this case, $z(t)$)

$$\vec{R}(t) \quad t \in I_1$$

$$\vec{Q}(t) \quad t \in I_2$$

$$a, b \in \mathbb{R}$$

$$f(t) \quad t \in I_3$$

what is $(\vec{R} + \vec{Q})(t)$?

$$= \vec{R}(t) + \vec{Q}(t)$$

the domain of this function is the intersection of each part's domain

$$t \in I_1 \cap I_2$$



$$\vec{R}(t) = \langle r_1(t), r_2(t), r_3(t) \rangle$$

$$\vec{Q}(t) = \langle q_1(t), q_2(t), q_3(t) \rangle$$

$$= \langle r_1(t) + q_1(t), r_2(t) + q_2(t), r_3(t) + q_3(t) \rangle$$

$$(a\vec{R} + b\vec{Q})(t) = a\vec{R}(t) + b\vec{Q}(t) \quad t \in I_1 \cap I_2$$

$$(f\vec{R})(t) = f(t)(\vec{R}(t)) \quad t \in I_1 \cap I_3$$

this is scalar (vector) multiplication

$$\left(\frac{\vec{R}}{f}\right)(t) = \frac{\vec{R}(t)}{f(t)} \quad t \in I_1 \cap I_3 \setminus \{t : f(t) = 0\}$$

removing the set where $f(t) = 0$

$$(\vec{R} \cdot \vec{Q})(t) = \vec{R}(t) \cdot \vec{Q}(t) \quad t \in I_1 \cap I_2$$



$$(\vec{R} \times \vec{Q})(t) = \vec{R}(t) \times \vec{Q}(t) \quad t \in I_1 \cap I_2$$

Be careful about the domains.

Examples

$$t > 0$$

$$\mathbb{R}$$

$$\mathbb{R}$$

this is the domain

$$\vec{R}(t) = \langle \ln t, e^{t^2}, t^3 \rangle$$

$$t > 0$$

$$\vec{Q}(t) = \langle \frac{1}{t+1}, t^2, 0 \rangle$$

$$t \neq -1$$

$$t > 0 \setminus \{1\}$$

$$\vec{R} \cdot \vec{Q} = \left[\frac{\ln t}{t+1} + t^2 e^{t^2} + t^3(0) \right]$$

$$\vec{R} \times \vec{Q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \ln t & e^{t^2} & t^3 \\ \frac{1}{t+1} & t^2 & 0 \end{vmatrix}$$

$$= \hat{i}(0 - t^5) - \hat{j}(0 - \frac{t^3}{t+1}) + \hat{k}(\ln t(t^2) - \frac{e^{t^2}}{t+1})$$

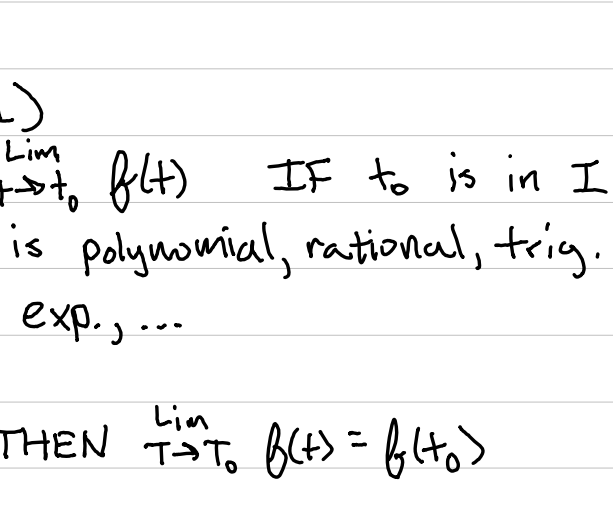
$$= \vec{H}(t) = \left\langle -t^5, \frac{t^3}{t+1}, \ln t - \frac{e^{t^2}}{t+1} \right\rangle$$

10.2

Limits, continuity, derivative, integral

$$f(t) \text{ as } \lim_{t \rightarrow t_0} f(t) = L$$

if for all $\epsilon > 0$ there exists $\delta > 0$ such that for t : $|t - t_0| < \delta$ then $|f(t) - L| < \epsilon$



1)
 $\lim_{t \rightarrow t_0} f(t)$ IF t_0 is in I AND the func. is polynomial, rational, trig., inv. trig., exp., ...

THEN $\lim_{t \rightarrow t_0} f(t) = f(t_0)$

2)

IF $t_0 \notin I$ you can get

$$\frac{1}{0}, +\infty \rightarrow \text{undefined / DNE}$$

$$\frac{0}{0}, \frac{\infty}{\infty}, +\infty - \infty, \dots \rightarrow \text{more work needed...}$$

Cal 3 Application

$$\vec{R}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\lim_{t \rightarrow t_0} \vec{R}(t) = \vec{L} = \langle L_1, L_2, L_3 \rangle$$

$\Leftrightarrow$

$$\begin{cases} \lim_{t \rightarrow t_0} x(t) = L_1 \\ \lim_{t \rightarrow t_0} y(t) = L_2 \\ \lim_{t \rightarrow t_0} z(t) = L_3 \end{cases}$$

$$\lim_{t \rightarrow 0} \langle e^{t^2-1}, \frac{\sin t}{t}, \frac{3t^3+2t^2}{2t^2+t} \rangle$$

$$\downarrow$$

$$\lim_{t \rightarrow 0} e^{t^2-1} = e^{-1} = \boxed{\frac{1}{e}}$$

$$\lim_{t \rightarrow 0} \frac{\sin(t)}{t} = \frac{0}{0} \text{ undetermined form}$$

$$i) \text{ use L'Hopital } \rightarrow \lim_{t \rightarrow 0} \frac{\cos(t)}{1} = \frac{1}{1} = \boxed{1}$$

(top / bottom)

$$\lim_{t \rightarrow 0} \frac{3t^3+2t^2}{2t^2+t} = \frac{0}{0} \text{ U.F.}$$

$$i) \text{ collect } t\text{'s} \rightarrow \lim_{t \rightarrow 0} \frac{t^2(3t+2)}{t(2t+1)}$$

ii) simplify

$$\lim_{t \rightarrow 0} \frac{t(3t+2)}{2t+1} = \frac{0(2)}{1} = \boxed{0}$$

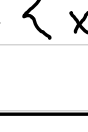
OR L'Hopital

$$i) \frac{4t^2+4t}{4t+1} = \frac{0}{1} = \boxed{0}$$

If a limit of one component DNE, the whole limit doesn't exist

Continuity

$$f(t) \text{ is cont. at } t_0 \Leftrightarrow \lim_{t \rightarrow t_0} f(t) = f(t_0)$$



conditions

$$\begin{cases} t_0 \in I \\ \lim_{t \rightarrow t_0} f(t) = L \\ L = f(t_0) \end{cases}$$

$$\vec{R}(t) \text{ is cont. at } t_0 \Leftrightarrow \lim_{t \rightarrow t_0} \vec{R}(t) = \vec{R}(t_0)$$

conditions

$$\begin{cases} t_0 \in I \\ \lim_{t \rightarrow t_0} \vec{R}(t) = \vec{L} \\ \vec{L} = \vec{R}(t_0) \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} x(t) \text{ is cont. at } t_0 \\ y(t) \text{ is cont. at } t_0 \\ z(t) \text{ is cont. at } t_0 \end{cases}$$

Example

$$\vec{R}(t) = \langle e^{t^2-1}, \frac{\sin t}{t}, \frac{3t^3+2t^2}{2t^2+t} \rangle$$

Is it continuous at 0? No

i) $0 \notin I$ because 0 denominator EVEN IF LIMIT EXISTS

Is it continuous at 1? Yes

i) $1 \in I$

ii) all the functions are standard

Derivatives

assuming all comps. are diff. at t_0 :

$$\vec{R}'(t_0) = \langle x'(t_0), y'(t_0), z'(t_0) \rangle$$

$$\vec{R}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

Example

$$\vec{R}(t) = \langle t^3 + \cos t, \ln(t), \sin^2(t) \rangle$$

$$\downarrow$$

$$\mathbb{R}$$

$$\downarrow$$

$$t > 0$$

$$\downarrow$$

$$\mathbb{R}$$

so $t > 0 = I$

find $\vec{R}'(t)$

$$\vec{R}'(t) = \langle 3t^2 - \sin t, \frac{1}{t}, 2\sin(t)\cos(t) \rangle \quad t > 0$$

$$(af(t) + bg(t))' = af'(t) + bg'(t)$$

Linearity rule



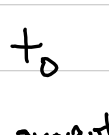
$$(a\vec{R}(t) + b\vec{Q}(t))' = a\vec{R}'(t) + b\vec{Q}'(t)$$

Product Rule

$$(f(t)g(t))' = f'g + fg'$$

scalar mult:

$$(f(t)\vec{R}(t))' = f'\vec{R} + f\vec{R}'$$



dot. prod.:

$$(\vec{R}(t) \cdot \vec{Q}(t))' = \vec{R}' \cdot \vec{Q} + \vec{R} \cdot \vec{Q}'$$

cross prod.:

$$(\vec{R}(t) \times \vec{Q}(t))' = \vec{R}' \times \vec{Q} + \vec{R} \times \vec{Q}'$$

Quotient Rule

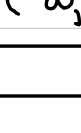
$$\left(\frac{f(t)}{g(t)}\right)' = \frac{f'g - fg'}{g^2}$$



$$\left(\frac{\vec{R}(t)}{g(t)}\right)' = \frac{\vec{R}'g - \vec{R}g'}{g^2}$$

Chain Rule

$$(f(g(t)))' = \frac{df}{dg} g'(t)$$



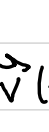
$$\vec{R}'(t_0) = \langle x'(t_0), y'(t_0), z'(t_0) \rangle$$

$$= \langle \lim_{t \rightarrow t_0} \frac{x(t) - x(t_0)}{t - t_0}, \lim_{t \rightarrow t_0} \frac{y(t) - y(t_0)}{t - t_0}, \lim_{t \rightarrow t_0} \frac{z(t) - z(t_0)}{t - t_0} \rangle$$

$$= \lim_{t \rightarrow t_0} \frac{\langle x(t) - x(t_0), y(t) - y(t_0), z(t) - z(t_0) \rangle}{t - t_0}$$

$$= \lim_{t \rightarrow t_0} \frac{\langle x(t), y(t), z(t) \rangle - \langle x(t_0), y(t_0), z(t_0) \rangle}{t - t_0}$$

$$\boxed{\vec{R}'(t_0) = \langle x'(t_0), y'(t_0), z'(t_0) \rangle = \lim_{t \rightarrow t_0} \frac{\vec{R}(t) - \vec{R}(t_0)}{t - t_0}}$$



the tangential vector given by derivative

ONLY IF $\vec{R}'(t_0)$ exists and $\neq \vec{0}$

THEN $\vec{R}'(t_0)$ has direction tangential to the curve at t_0 and points in the direction of motion

ALSO THEN $\vec{R}$ is smooth at t_0

smooth

not smooth

Example

Find interval of smoothness for

$$\vec{R}(t) = \langle t^2+1, \cos t, e^t + e^{-t} \rangle$$

$$\downarrow$$

$$\mathbb{R}$$

$$\downarrow$$

$$\mathbb{R}$$

$$\downarrow$$

$$\mathbb{R}$$

so $I = \mathbb{R} \equiv (-\infty, \infty)$

$$\vec{R}'(t) = \langle 2t, -\sin t, e^t - e^{-t} \rangle \text{ in } I$$

i) remove all t in I such that $\vec{R}'(t) = \vec{0}$

$$\begin{cases} 2t = 0 \\ -\sin t = 0 \\ e^t - e^{-t} = 0 \end{cases}$$

since $2t = 0 \Rightarrow t = 0$

$$-\sin(0) = 0$$

$$e^0 - e^{-0} = 1 - 1 = 0$$

ii) since 0 is the only solution for the $x'(t)$ component, it's the only problem overall

since $x'(t)$ is otherwise not = 0

$$\boxed{\vec{R}'(t) \text{ is smooth for all } t \in (-\infty, \infty) / \{0\}}$$

Let $\vec{R}(t)$ be smooth in I and $\|\vec{R}(t)\| = \text{const. in } I$

$$\Rightarrow \vec{R}(t) \perp \vec{R}'(t) \text{ for all } t \in I$$

= a

$$\vec{R} \cdot \vec{R} = x^2 + y^2 + z^2$$

$$x^2(t) + y^2(t) + z^2(t) = \text{const.} = a^2$$

$$\vec{R}(t) \perp \vec{R}'(t) \Leftrightarrow \vec{R}(t) \cdot \vec{R}'(t) = 0$$

$$a^2 = \vec{R} \cdot \vec{R}' = 0$$

$$0 = 2(\vec{R} \cdot \vec{R}')$$

"trajectory"

$$\vec{R}(t) = \langle x(t), y(t), z(t) \rangle \quad t \text{ is time}$$

$$\vec{R}'(t) = \langle x'(t), y'(t), z'(t) \rangle = \vec{v}(t) \text{ velocity}$$

$$\|\vec{v}(t)\| = \text{speed}$$

Physics Notation

$$\vec{v}'(t) = \vec{R}''(t) = \vec{a}(t) \text{ acceleration}$$



$$\int_a^b \vec{R}(t) dt = \left\langle \int_a^b x(t) dt, \int_a^b y(t) dt, \int_a^b z(t) dt \right\rangle = \vec{I}$$

Definite

$$\int \vec{R}(t) dt = \left\langle \int x(t) dt, \int y(t) dt, \int z(t) dt \right\rangle + \vec{C}$$

Indefinite

vector of
integration constants

Example 0

$$\begin{aligned} & \int_1^2 \langle t^3, e^{2t}, \sin(t) \rangle dt \\ &= \left\langle \int_1^2 t^3 dt, \int_1^2 e^{2t} dt, \int_1^2 \sin(t) dt \right\rangle \\ &= \left\langle \left. \frac{t^4}{4} \right|_1^2, \left. \frac{e^{2t}}{2} \right|_1^2, -\cos(t) \Big|_1^2 \right\rangle \\ &= \left\langle \frac{16}{4} - \frac{1}{4}, \frac{e^4 - e^2}{2}, -\cos(2) + \cos(1) \right\rangle \end{aligned}$$

Physics Example 0

$$\vec{v} = \langle t^2, \sin(2t), e^t \rangle$$

Find $\vec{a}$, ($\vec{R}$ such that $\vec{R}(0) = \langle 1, 2, 3 \rangle$)

$$\vec{a}(t) = \langle 2t, 2\cos(2t), e^t \rangle$$

$$\vec{R}'(t) = \vec{v}(t) \longrightarrow \int \vec{R}'(t) dt = \int \vec{v}(t) dt$$

$$\vec{R}(t) = \int \vec{v}(t) dt + \vec{C}$$

$$= \left\langle \int t^2 dt, \int \sin(2t) dt, \int e^t dt \right\rangle + \vec{C}$$

$$= \left\langle \frac{t^3}{3}, -\frac{\cos(2t)}{2}, e^t \right\rangle + \vec{C}$$

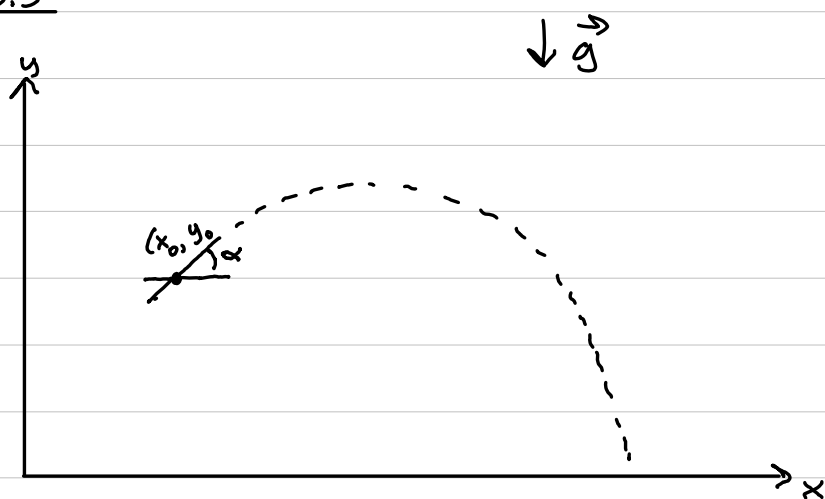
Plug in

$$\vec{R}(0) = \left\langle 0, -\frac{1}{2}, 1 \right\rangle + \vec{C} = \langle 1, 2, 3 \rangle$$

$$\begin{aligned} \vec{C} &= \langle 1, 2, 3 \rangle - \left\langle 0, -\frac{1}{2}, 1 \right\rangle \\ &= \left\langle 1, \frac{5}{2}, 2 \right\rangle \end{aligned}$$

$$\vec{R}(t) = \left\langle \frac{t^3}{3}, -\frac{\cos(2t)}{2}, e^t \right\rangle + \left\langle 1, \frac{5}{2}, 2 \right\rangle$$

10.3



$$\vec{v}(0) = \langle v_0 \cos \alpha, v_0 \sin \alpha \rangle$$

$$\vec{r}(0) = \langle x_0, y_0 \rangle$$

$$m\vec{a} = m\langle 0, -g \rangle \rightarrow \vec{a} = \langle 0, -g \rangle$$

$$\vec{v} = \int \vec{a} dt + \vec{c}$$

$$= \int \langle 0, -g \rangle dt + \vec{c} = \langle 0, -gt \rangle + \vec{c}$$

$$\vec{v}(0) = \vec{c}$$

$$\vec{r}(t) = \int \vec{v} dt + \vec{D} = \langle 0, -\frac{gt^2}{2} \rangle + \langle v_0 \cos \alpha t, v_0 \sin \alpha t \rangle + \vec{D}$$

$$\vec{r}(0) = \vec{D}$$

$$= \langle x_0 + v_0 \cos \alpha t, y_0 + v_0 \sin \alpha t - \frac{gt^2}{2} \rangle$$

$$\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$$

$$\vec{r}(t) = \langle v_0 \cos \alpha t, v_0 \sin \alpha t - \frac{1}{2}gt^2 \rangle$$

$$\vec{v}(t) = \langle v_0 \cos \alpha, v_0 \sin \alpha - gt \rangle$$

$T_F \rightarrow$ Time of flight

$$y(T_F) = 0 = T_F(v_0 \sin \alpha - \frac{1}{2}g T_F)$$

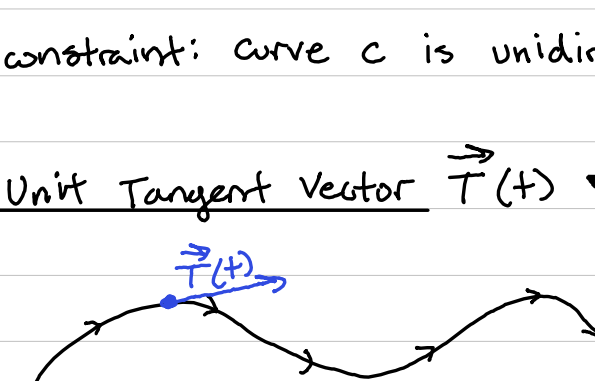
$$T_F = \frac{2v_0 \sin \alpha}{g}$$

$$\text{Range } R = x(T_F) = \frac{v_0 \cos \alpha (2v_0 \sin \alpha)}{g}$$

$$= \frac{v_0^2 \sin(2\alpha)}{g}$$

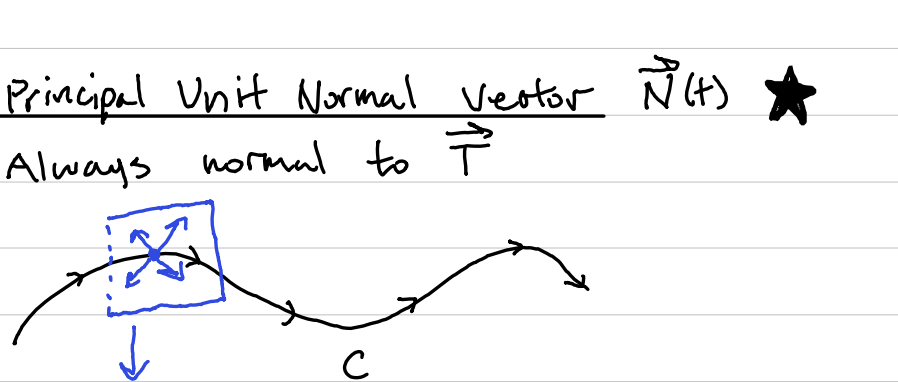
10.4

$$\vec{R}(t) = \langle x(t), y(t), z(t) \rangle$$



constraint: curve C is unidirectional

Unit Tangent Vector $\vec{T}(t)$ ★

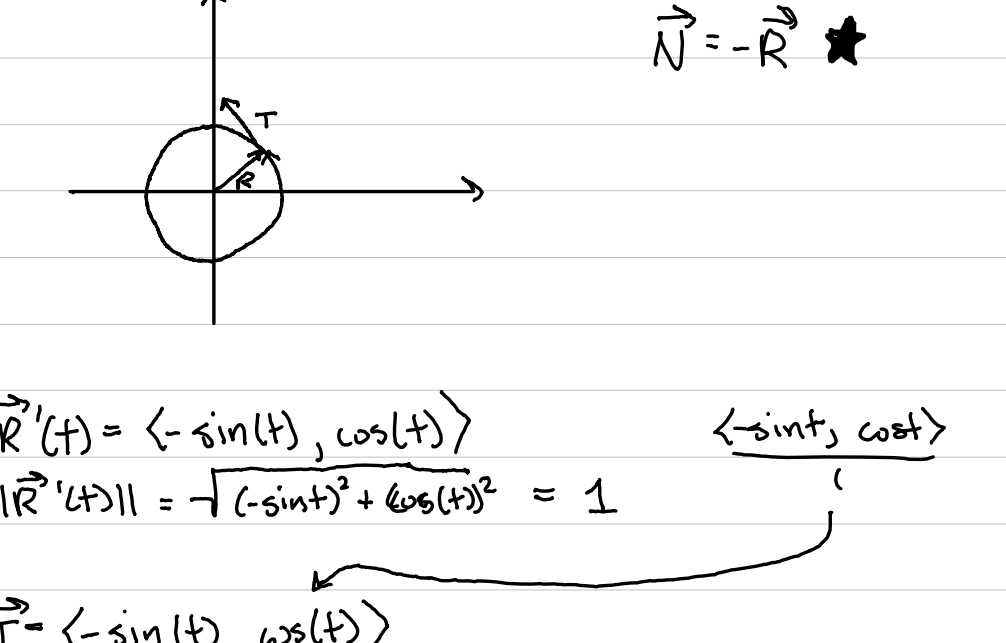


Always tangent to the curve

$$\vec{T}(t) = \frac{\vec{R}'(t)}{\|\vec{R}'(t)\|} \quad \text{* note: bottom} = \|\vec{R}'\|$$

Principal Unit Normal Vector $\vec{N}(t)$ ★

Always normal to $\vec{T}$

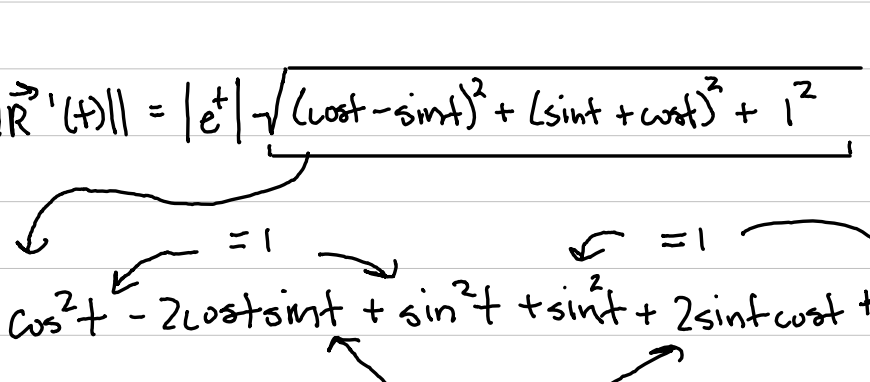


$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} \quad \vec{N} \perp \vec{T}$$

2D Example

Find $\vec{N}(t)$, $\vec{T}(t)$

$$\vec{R}(t) = \langle \cos(t), \sin(t) \rangle$$



$$\vec{N} = -\vec{R} \quad \star$$

$$\vec{R}'(t) = \langle -\sin(t), \cos(t) \rangle$$

$$\|\vec{R}'(t)\| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1$$

$$\vec{T} = \langle -\sin(t), \cos(t) \rangle$$

$$\vec{T}' = \langle -\cos(t), -\sin(t) \rangle$$

$$\|\vec{T}'\| = \sqrt{(-\cos(t))^2 + (-\sin(t))^2} = 1$$

$$\vec{N} = \langle -\cos(t), -\sin(t) \rangle = -\vec{R}(t)$$

★★★ Example

$$\vec{R}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle = \langle \cos t, \sin t, 1 \rangle e^t$$

Find $\vec{T}(t)$, $\vec{N}(t)$

$$\vec{R}'(t) = e^t \langle \cos t, \sin t, 1 \rangle + e^t \langle -\sin t, \cos t, 0 \rangle$$

$$= e^t \langle \cos t - \sin t, \sin t + \cos t, 1 \rangle$$

$$\|\vec{R}'(t)\| = |e^t| \sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1^2}$$

$$\cos^2 t - 2\cos t \sin t + \sin^2 t + \sin^2 t + 2\sin t \cos t + \cos^2 t + 1$$

$$= |e^t| \sqrt{1+1+1} = e^t \sqrt{3}$$

$$\vec{T} = \frac{\vec{R}'(t)}{\|\vec{R}'(t)\|} = \frac{1}{\sqrt{3}} \langle \cos t - \sin t, \sin t + \cos t, 1 \rangle$$

$$\vec{T}'(t) = \frac{1}{\sqrt{3}} \langle -\sin t - \cos t, \cos t - \sin t, 0 \rangle$$

$$\|\vec{T}'(t)\| = \frac{1}{\sqrt{3}} \sqrt{(-\sin t - \cos t)^2 + (\cos t - \sin t)^2 + 0^2}$$

$$\sin^2 t + 2\sin t \cos t + \cos^2 t + \cos^2 t - 2\sin t \cos t + \sin^2 t + 0$$

...

$$= \frac{\sqrt{2}}{\sqrt{3}} \quad \text{so} \quad \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} =$$

$$\vec{N}(t) = \frac{1}{\sqrt{2}} \langle \sin t - \cos t, \cos t - \sin t, 0 \rangle$$

WebWork Example

$$\vec{R}(t) = \langle t, 3t^2, 2 \rangle \quad \text{Find } \vec{T}(t), \vec{N}(t)$$

$$\vec{R}'(t) = \langle 1, 6t, 0 \rangle = \langle 1, t, 0 \rangle$$

$$\|\vec{R}'(t)\| = |6| \sqrt{1^2 + t^2 + 0^2} = 6\sqrt{1+t^2}$$

$$\vec{T}(t) = \frac{\langle 1, t \rangle}{\sqrt{1+t^2}} \quad \text{use quotient rule}$$

$$\vec{T}'(t) = \frac{\langle 0, 1 \rangle \sqrt{1+t^2} - \langle 1, t \rangle \frac{2t}{2\sqrt{1+t^2}}}{1+t^2}$$

$$= \frac{\langle 0, 1 \rangle \sqrt{1+t^2} - \langle 1, t \rangle \frac{t}{\sqrt{1+t^2}}}{1+t^2}$$

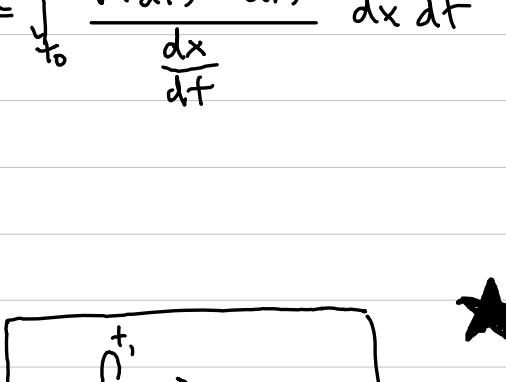
$$= \frac{\langle 0, 1 \rangle (1+t^2) - \langle t, t^2 \rangle}{(1+t^2)^{3/2}}$$

$$= \frac{\langle -t, 1 \rangle}{(1+t^2)^{3/2}}$$

$$\|\vec{T}'\| = \frac{\sqrt{t^2+1}}{(1+t^2)^{3/2}}$$

$$\vec{N}(t) = \frac{\langle -t, 1 \rangle}{\sqrt{t^2+1}}$$

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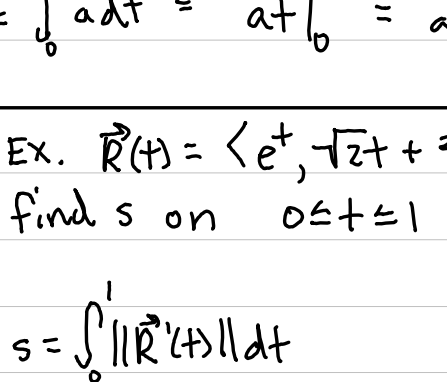


$$\vec{R}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\vec{T} = \frac{\vec{R}'}{\|\vec{R}'\|} \quad \text{unit tangent vector}$$

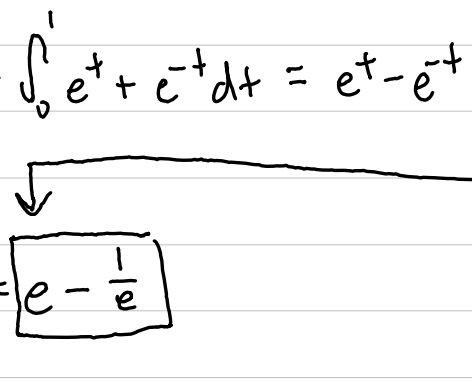
$$\vec{N} = \frac{\vec{T}'}{\|\vec{T}'\|} \quad \text{principal unit normal vector}$$

Arc Length ★



s = Arc Length

In Cal 2:



$$ds = \sqrt{1 + (y')^2} dx$$

$$s = \int_a^b ds = \int_a^b \sqrt{1 + (y')^2} dx$$

In Cal 3 (2D):

$$1 + (y')^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2 \approx 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

$$1 + \left(\frac{\Delta y}{\Delta x}\right)^2 = 1 + \left(\frac{\left(\frac{\Delta y}{\Delta t}\right)}{\left(\frac{\Delta x}{\Delta t}\right)}\right)^2 = \frac{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}{\left(\frac{\Delta x}{\Delta t}\right)^2}$$

Plug in to function:

$$s = \int_{t_0}^{t_1} \lim_{\Delta t \rightarrow 0} \sqrt{\frac{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}{\left(\frac{\Delta x}{\Delta t}\right)^2}} dx$$

$$= \int_{t_0}^{t_1} \sqrt{\frac{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}} dx dt = \int_{t_0}^{t_1} \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{\frac{dx}{dt}} dx dt$$

$$s = \int_{t_0}^{t_1} \|\vec{R}'(t)\| dt \quad \star$$

Ex. Find the perimeter of a circle of radius a

$$\vec{R}(t) = \langle a \cos t, a \sin t \rangle \quad 0 \leq t \leq 2\pi$$

$$\vec{R}'(t) = \langle -a \sin t, a \cos t \rangle$$

$$\|\vec{R}'(t)\| = a \sqrt{(-\sin t)^2 + (\cos t)^2} = a$$

$$s = \int_0^{2\pi} \|\vec{R}'(t)\| dt = \int_0^{2\pi} a dt = at \Big|_0^{2\pi} = a(2\pi - 0) = 2\pi a$$

Ex. $\vec{R}(t) = \langle e^t, \sqrt{2}t + 3, e^t \rangle$

Find s on $0 \leq t \leq 1$

$$s = \int_0^1 \|\vec{R}'(t)\| dt$$

$$\vec{R}'(t) = \langle e^t, \sqrt{2}, -e^t \rangle$$

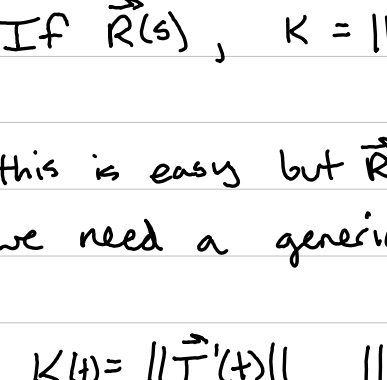
$$\|\vec{R}'(t)\| = \sqrt{(e^t)^2 + (\sqrt{2})^2 + (-e^t)^2}$$

$$= \sqrt{e^{2t} + 2 + e^{2t}} \quad (a+b)^2 = a^2 + 2ab + b^2$$

$$= \sqrt{(e^t + e^t)^2} = e^t + e^t \quad (e^t + e^t)^2 = e^{2t} + 2 + e^{2t}$$

$$= \int_0^1 e^t + e^t dt = e^t - e^{-t} \Big|_0^1 = (e^1 - e^{-1}) - (e^0 - e^0)$$

$$= e - \frac{1}{e}$$



$$s(t) = \int_{t_0}^t \|\vec{R}'(u)\| du \quad \star \quad \frac{ds(t)}{dt} = \|\vec{R}'(t)\|$$

what if we want $\vec{R}$ based on s?

$$\vec{R}(s) \quad \frac{ds}{ds} = \|\vec{R}'(s)\|$$

Ex. Given $\vec{R}(t)$ find $\vec{R}(s)$

$$\vec{R}(t) = \langle \sin t, \cos t, 2t \rangle \quad t = 0$$

$$1. \quad s(t) = \dots = f(t) \quad 2. \quad t = f^{-1}(s)$$

$$3. \quad \vec{R}(s)$$

$$1. \quad s(t) = \int_0^t \|\vec{R}'(u)\| du \quad \vec{R}'(u) = \langle \cos u, -\sin u, 2 \rangle$$

$$\|\vec{R}'(u)\| = \sqrt{(\cos u)^2 + (-\sin u)^2 + 2^2} = \sqrt{5}$$

$$s(t) = \int_0^t \sqrt{5} du = \sqrt{5} u \Big|_0^t = \sqrt{5} t$$

$$2. \quad t = \frac{s}{\sqrt{5}}$$

$$3. \quad \vec{R}(s) = \left\langle \sin\left(\frac{s}{\sqrt{5}}\right), \cos\left(\frac{s}{\sqrt{5}}\right), \frac{2s}{\sqrt{5}} \right\rangle$$

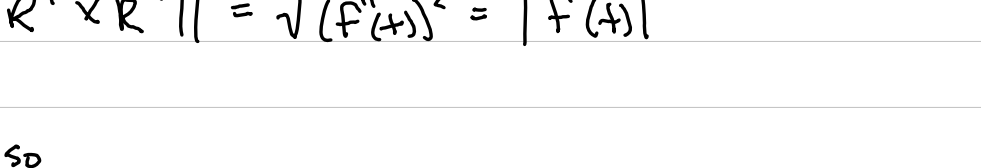
*** $\|\vec{R}'(s)\| = 1$ ALWAYS ★

Curvature

denoted by Kappa (K)

K is the reciprocal of the radius of the osculating circle

circle that best approx. the curve



where P_1, P_2 are radii

$$\text{Line:} \quad K = 0 \quad (\text{infinitely large radius}) \rightarrow \frac{1}{\infty}$$

$$\text{Circle:} \quad K = \frac{1}{a}$$

If $\vec{R}(s)$, $K = \|\vec{T}'(s)\|$ ★

this is easy but $\vec{R}(s)$ is hard to get so we need a generic form

$$K(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{R}'(t)\|} = \frac{\|\vec{R}'(t) \times \vec{R}''(t)\|}{\|\vec{R}'(t)\|^3} \quad \star \quad \text{this is for when } \vec{T} \text{ is very complicated}$$

given $\vec{R}(t)$ find $\vec{T}$, $\vec{N}$, and K

(use $K(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{R}'(t)\|}$ for this type of problem)

Ex. $\vec{R}(t) = \langle a \cos t, a \sin t, bt \rangle \leftarrow$ (helix)

find K

$$\vec{R}'(t) = \langle -a \sin t, a \cos t, b \rangle$$

$$\|\vec{R}'(t)\| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$\vec{T} = \frac{1}{\sqrt{a^2 + b^2}} \langle -a \sin t, a \cos t, b \rangle$$

$$\vec{T}' = \frac{1}{\sqrt{a^2 + b^2}} \langle -a \cos t, -a \sin t, 0 \rangle$$

$$\|\vec{T}'(t)\| = \frac{a}{\sqrt{a^2 + b^2}}$$

$$K(t) = \frac{\frac{a}{\sqrt{a^2 + b^2}}}{\sqrt{a^2 + b^2}} = \frac{a}{a^2 + b^2}$$

* For circles ($b=0$), $K(t) = \frac{a}{a^2} = \frac{1}{a}$ ★

Ex. $\vec{R}(t) = \langle 2 \cos t, 3 \sin t, 4t \rangle$

$$\vec{R}'(t) = \langle -2 \sin t, 3 \cos t, 4 \rangle$$

$$\|\vec{R}'(t)\| = \sqrt{4 \sin^2 t + 9 \cos^2 t + 16} = \sqrt{20 + 5 \cos^2 t}$$

$$\vec{T} = \frac{\langle -2 \sin t, 3 \cos t, 4 \rangle}{\sqrt{20 + 5 \cos^2 t}} \quad \vec{T}' \text{ would be gross so use other formula}$$

$$\vec{R}''(t) = \langle -2 \cos t, -3 \sin t, 0 \rangle$$

$$\vec{R}' \times \vec{R}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 \sin t & 3 \cos t & 4 \\ -2 \cos t & -3 \sin t & 0 \end{vmatrix} = \hat{i}(0 + 12 \sin t) - \hat{j}(0 + 8 \cos t) + \hat{k}(6 \sin^2 t + 6 \cos^2 t)$$

$$\|\vec{R}' \times \vec{R}''\| = \sqrt{144 \sin^2 t + 64 \cos^2 t + 36}$$

$$= \sqrt{100 + 80 \sin^2 t}$$

$$K = \frac{\sqrt{100 + 80 \sin^2 t}}{(20 + 5 \cos^2 t)^{3/2}} = \frac{\|\vec{R}' \times \vec{R}''\|}{\|\vec{R}'\|^3}$$

K(x) for $y = f(x)$

$$\vec{R}(t) = \langle t, f(t), 0 \rangle$$

$$\vec{R}'(t) = \langle 1, f'(t), 0 \rangle$$

$$\vec{R}''(t) = \langle 0, f''(t), 0 \rangle$$

$$\vec{R}' \times \vec{R}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}(f'(t)f''(t))$$

$$\|\vec{R}'(t)\| = \sqrt{1 + (f')^2}$$

$$\|\vec{R}' \times \vec{R}''\| = \sqrt{(f'(t)f''(t))^2} = |f''(t)|$$

so

$$K(t) = \frac{|f''(t)|}{(1 + (f')^2)^{3/2}}$$

so

$$K(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$$

Ex. Find curvature of $f(x) = 3e^{2x}$ at $x=3$

$$f'(x) = 6e^{2x} \quad f''(x) = 12e^{2x}$$

$$K(x) = \frac{|12e^{2x}|}{(1 + 36e^{4x})^{3/2}}$$

$$K(3) = \frac{12e^6}{(1 + 36e^{12})^{3/2}}$$