CHAPTER 13

Vector Analysis

Contents

In this chapter, we will combine all the information on differentiation, integration and vectors to study the calculus of vector functions defined on a set of points in $\mathbb{R}^2$ or $\mathbb{R}^3$.

13.1 Properties of a vector field: divergence and curl

Definition 13.1. A **Vector field** in $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns a vector to each point in its domain. A vector field with domain D in $\mathbb{R}^3$ has the form

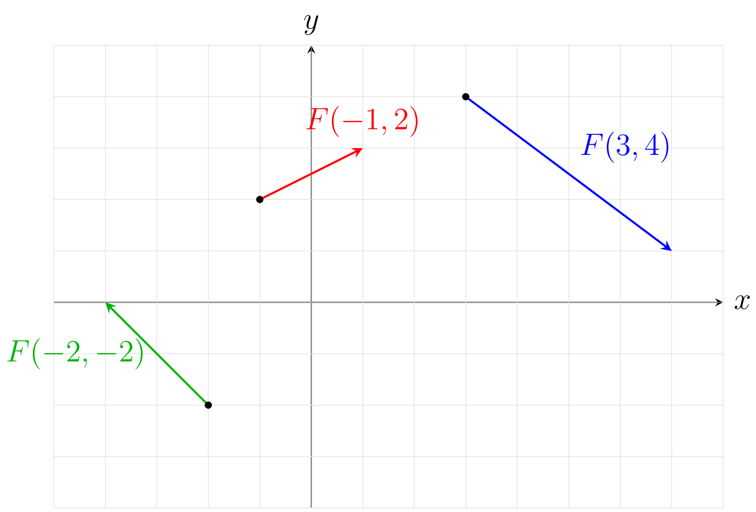
$$\mathbf{F}(x,y,z) = M(x,y,z)\,\hat{\imath} + N(x,y,z)\,\hat{\jmath} + P(x,y,z)\,\hat{k}\,,$$

where the scalar functions M, N, P are called the **components** of $\mathbf{F}$. A **continuous vector field F** has continuous components M, N, P. A **differentiable vector field** has all the components M, N, P that have all the partial derivatives.

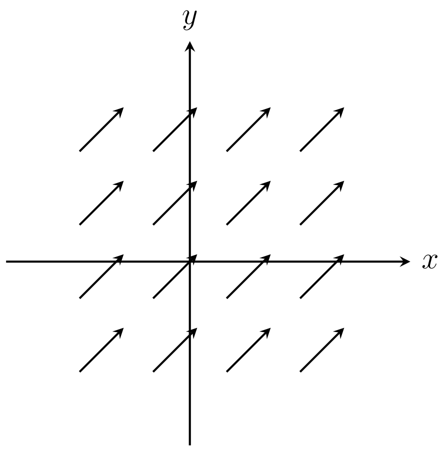
Example 13.1. Sketch the graph of the vector field $\mathbf{F}(x,y) = y\,\hat{\imath} - x\,\hat{\jmath}\,.$

Solution: The vector field has component y along $\hat{\imath}$ and $-x$ along $\hat{\jmath}$. It is not easy to draw the vector field: we consider $\mathbf{F}$ at various points.

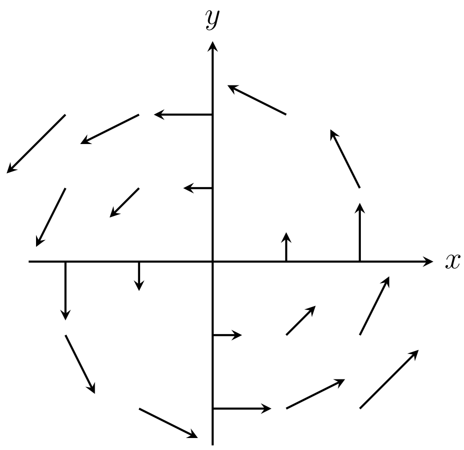
$$\mathbf{F}(3,4) = 4\,\hat{\imath} - 3\,\hat{\jmath}, \qquad \mathbf{F}(-1,2) = 2\,\hat{\imath} + \hat{\jmath} \qquad \mathbf{F}(-2,-2) = -2\,\hat{\imath} + 2\,\hat{\jmath}\,.$$



In general, the vector fields are numerically represented. One of the most important fields of application of vector fields is fluid dynamics. The vector field can be either irrotational or rotational.



IRROTATIONAL



ROTATIONAL

Divergence We define the divergence of a differentiable vector field $\mathbf{V}(x, y, z) = u(x, y, z)\,\hat{\imath} + v(x, y, z)\,\hat{\jmath} + w(x, y, z)\,\hat{k}$ as

$$\operatorname{div}\mathbf{V} = \frac{\partial u}{\partial x}(x, y, z) + \frac{\partial v}{\partial y}(x, y, z) + \frac{\partial w}{\partial z}(x, y, z)\,.$$

Example 13.2. Find the divergence of each of the following vector fields:

a. $\mathbf{F}(x, y) = x^2 y\,\hat{\imath} + xy^3\,\hat{\jmath}$;

b. $\mathbf{G}(x, y, z) = x\,\hat{\imath} + y^3 z^2\,\hat{\jmath} + xz^3\,\hat{k}$.

Solution:

a. $\operatorname{div}\mathbf{F} = \dfrac{\partial}{\partial x}(x^2 y) + \dfrac{\partial}{\partial y}(xy^3) = 2xy + 3xy^2$;

b. $\operatorname{div}\mathbf{G} = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(y^3 z^2) + \dfrac{\partial}{\partial z}(xz^3) = 1 + 3y^2 z^2 + 3xz^2\,.$

Applied example: Let's consider a vector field $\mathbf{V}(x, y, z)$ representing the velocity field of a fluid with density ρ. It can be shown that, by calling $\mathbf{D} = \rho\mathbf{V}$ the flux density of the fluid (measure of the "mass flow" of the fluid), the following relation holds:

$$\operatorname{div}\mathbf{D} = \frac{\partial \rho}{\partial t} \qquad \text{Continuity equation}\,.$$

If $\operatorname{div}\mathbf{D} = 0$ the flow is incompressible.

Note: A useful operator for the definition of the divergence is the del operator:

$$\boldsymbol{\nabla} = \frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\,\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle .$$

So we can define, given a scalar function f and a vector function $\mathbf{F} = u\,\hat{\imath} + v\,\hat{\jmath} + w\,\hat{k}$:

- the gradient:

$$\nabla f = \frac{\partial f}{\partial x}\,\hat{\imath} + \frac{\partial f}{\partial y}\,\hat{\jmath} + \frac{\partial f}{\partial z}\,\hat{k}$$

- the divergence:

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \left(\frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\,\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}\right)\cdot\left(u\,\hat{\imath} + v\,\hat{\jmath} + w\,\hat{k}\right) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \operatorname{div}\mathbf{F}\,.$$

Curl We now define the curl operator. The curl of a differentiable vector field $\mathbf{V}(x,y,z) = u(x,y,z)\,\hat{\imath} + v(x,y,z)\,\hat{\jmath} + w(x,y,z)\,\hat{k}$ is defined by

$$\operatorname{curl}\mathbf{v} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\imath} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}\,.$$

Note that

$$\operatorname{curl}\mathbf{V} = \nabla\times\mathbf{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}.$$

A vector field such that curl $\mathbf{V} = 0$ is said to be **irrotational**. The field curl $\mathbf{V}$ is also called **vorticity** of the flow.

Example 13.3. Find the curl of each of the following vector fields:

a. $\mathbf{F} = x^2yz\,\hat{\imath} + xy^2z\,\hat{\jmath} + xyz^2\,\hat{k}$;

b. $\mathbf{G} = \big(x\cos(y)\big)\,\hat{\imath} + xy^2\,\hat{\jmath}$.

Solution:

a.

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} = \left[\frac{\partial}{\partial y}\left(xyz^2\right) - \frac{\partial}{\partial z}\left(xy^2z\right)\right]\hat{\imath} - \left[\frac{\partial}{\partial y}\left(xyz^2\right) - \frac{\partial}{\partial z}\left(x^2yz\right)\right]\hat{\jmath} +$$
$$+\left[\frac{\partial}{\partial y}\left(xy^2z\right) - \frac{\partial}{\partial z}\left(x^2yz\right)\right]\hat{k} = \left(xz^2 - xy^2\right)\hat{\imath} - \left(yz^2 - x^2y\right)\hat{\jmath} + \left(y^2z - x^2z\right)\hat{k}\,.$$

b.

$$\operatorname{curl}\mathbf{G} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x\cos(y) & xy^2 & 0 \end{vmatrix} = \left[0 - \frac{\partial}{\partial z}\left(xy^2\right)\right]\hat{\imath} - \left[0 - \frac{\partial}{\partial z}\left(x\cos(y)\right)\right]\hat{\jmath} +$$
$$+\left[\frac{\partial}{\partial y}\left(xy^2\right) - \frac{\partial}{\partial z}\left(x\cos(y)\right)\right]\hat{k} = \left(y^2 + x\sin(y)\right)\hat{k}\,.$$

The curl of a 2D motion is a vector perpendicular to the plane of the motion.

Laplacian operator Let $f(x, y, z)$ define a function with continuous first and second partial derivatives. Then, the laplacian of f is

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = f_{xx} + f_{yy} + f_{zz}\,.$$

The equation $\nabla^2 f = 0$ is called Laplace's equation, and a function that satisfies such an equation is a region D is said to be **harmonic** in D.

Example 13.4. Show that $f(x, y) = e^x \cos(y)$ is harmonic.

Solution:

$$f_x = e^x \cos(y)\,, \qquad f_{xx} = e^x \cos(y)\,,$$
$$f_y = -e^x \sin(y)\,, \qquad f_{yy} = -e^x \cos(y)\,.$$
$$\nabla^2 f = e^x \cos(y) - e^x \cos(y) = 0 \quad \rightarrow \quad f \text{ is harmonic}\,.$$

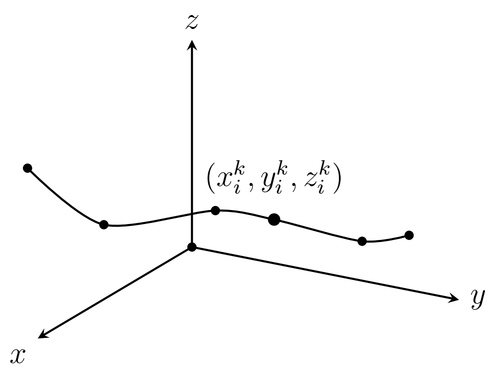
13.2 Line integrals

A line integral is an integral whose integrand is evaluated at points along a curve in $\mathbb{R}^2$ or in $\mathbb{R}^3$.

Definition 13.2 (Line integral)**.** If $f(x, y, z)$ is defined on the smooth curve C with parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$, then the line integral of f over C is given by

$$\int_C f(x, y, z)\, ds = \lim_{||\Delta s|| \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*) \Delta s_k \,,$$

provided that the limits exist. If C is a closed curve, we sometimes indicate the line integral of f around C by $\oint_C f\, ds$.



How to solve a line integral If the curve C can be represented as a function of the parameter t, with components $x(t), y(t), z(t)$ then

$$ds = \sqrt{(x'(t))\,2 + (y'(t))^2 + (z'(t))^2}\,,$$

so we can write the line integral in terms of t:

$$\int_C f(x, y, z)\, ds = \int_a^b f\left(x(t), y(t), z(t)\right) \sqrt{(x'(t))\,2 + (y'(t))^2 + (z'(t))^2}\, dt\,.$$

For two-dimensional functions, we have

$$\int_C f(x, y)\, ds = \int_a^b f\left(x(t), y(t)\right) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt\,.$$

Example 13.5. Evaluate the line integral $\int_C x^2 z\, ds$, where C is the helix $x = \cos(t), y = 2t, z = \sin(t)$ for $t \in [0, \pi]$.

Solution:

$$x'(t) = -\sin(t), \qquad y'(t) = 2, \qquad z'(t) = \cos(t)$$

$$\Rightarrow \int_C x^2 z\, ds = \int_0^\pi \cos^2(t)\sin(t)\sqrt{\sin^2(t) + 4 + \cos^2(t)}\, dt = \int_0^\pi \cos^2(t)\sin(t)\sqrt{5}\, dt =$$

$$\ldots \quad u = \cos(t) \to du = -\sin(t)dt \to -du = \sin(t)dt \quad \ldots$$

$$\ldots \quad t = 0 \to u = \cos(0) = 1 \quad \ldots$$

$$\ldots \quad t = \pi \to u = \cos(\pi) = -1 \quad \ldots$$

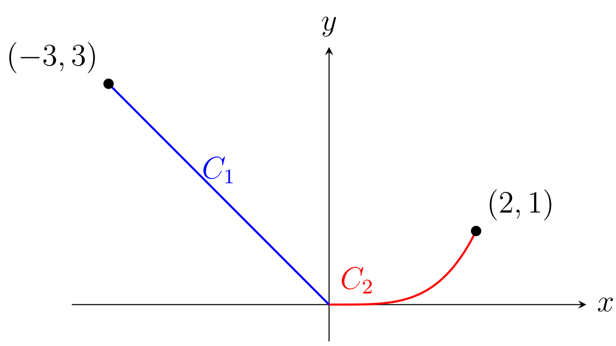
$$= \int_1^{-1} u^2\sqrt{5}(-du) = \sqrt{5}\int_{-1}^1 u^2\, du = \sqrt{5}\left[\frac{u^3}{3}\right]_{-1}^1 = \frac{2}{3}\sqrt{5}\,.$$

Integrals as the union of piecewise smooth curves The line integrals can be extended to curves that are piecewise smooth, in the sense that are the union of a finite number of smooth curves with only endpoints in common

$$\int_C f(x,y,z)\, ds = \int_{C_1} f(x,y,z)\, ds + \cdots + \int_{C_n} f(x,y,z)\, ds\,,$$

where $C_1, \ldots, C_n$ are all subarcs of C.

Example 13.6. Evaluate $\int_C xy\, ds$, where C consists of the line segment C_1 from $(-3,3)$ to $(0,0)$, followed by the portion of the curve C_2: $16y = x^4$ between $(0,0)$ and $(2,1)$.



Solution:

- The curve 1 can be parametrized as $x = t$, $y = -t$ over the interval $[-3, 0]$.

$$\begin{aligned} x'(t) &= 1 \\ y'(t) &= -1 \end{aligned} \quad \rightarrow \quad ds = \sqrt{1^2 + (-1)^2}\, dt = \sqrt{2}\, dt\,.$$

$$\int\limits_{C_1} xy\, ds = \int_{-3}^{0} -t^2\sqrt{2}\, dt = \sqrt{2}\left[-\frac{t^3}{3}\right]_{-3}^{0} = -9\sqrt{2}\,.$$

- The curve 2 can be parametrized as

$$x = t, \quad y = \frac{1}{16}t^4 \qquad \rightarrow \qquad x' = 1, \quad y' = \frac{t^3}{4} \qquad t \in [0, 2]\,.$$

$$\int\limits_{C_2} xy\, ds = \int_0^2 t\, \frac{1}{16}\, t^4 \sqrt{1 + \frac{t^6}{16}}\, dt = \int_0^2 \frac{1}{16}\, t^5\, \frac{\sqrt{16 + t^6}}{4}\, dt =$$

$$\ldots \quad 16 + t^6 = u \quad \rightarrow \quad 6t^5 dt = du \quad \rightarrow \quad t^5 dt = \frac{du}{6} \quad \ldots$$

$$\ldots \quad t = 0 \quad \rightarrow \quad u = 16 \quad \ldots$$

$$\ldots \quad t = 2 \quad \rightarrow \quad u = 80 \quad \ldots$$

$$= \int_{16}^{80} \frac{1}{16 \cdot 4}\sqrt{u}\frac{du}{6} = \frac{1}{16 \cdot 4}\left[\frac{u^{3/2}}{3/2}\right]_{16}^{80} = \frac{1}{24^2}\left(80^{3/2} - 16^{3/2}\right)\,.$$

$$\int\limits_{C} = \int\limits_{C_1} + \int\limits_{C_2}\,.$$

Theorem 13.1 (Properties of line integrals)**.** Let f be a given scalar function defined on a piecewise, smooth, orientable curve C. Then, for any constant k

- Constant multiple rule:

$$\int\limits_{C} kf\, ds = k \int\limits_{C} f\, ds\,;$$

- Sum rule:

$$\int_C (f_2 + f_2)\, ds = \int_C f_1\, ds + \int_C f_2\, ds\,;$$

- Opposite direction rule:

$$\int_{-C} f\, ds = -\int_C f\, ds\,;$$

- Subdivision rule:

$$\int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds + \cdots + \int_{C_n} f\, ds\,,$$

where C is the union of smooth orientable subarcs $C = C_1 \cup C_2 \cup \cdots \cup C_n$ with only endpoints in common.

Line integrals with respect to x, y and z In the definition of the line integral, if we replace Δs with Δx, we obtain the definition of the line integral $\int_C f(x, y, z)\, dz$. This line integral can be evaluated as follows,

$$\int_C f(x, y, z)\, dx = \int_a^b f\big(x(t), y(t), z(t)\big) x'(t)\, dt\,.$$

This definition also holds for dy and dz. By combining all the coordinate variables, we obtain a line integral in the form

$$\int_C \left[f(x, y, z)\, dx + g(x, y, z)\, dy + h(x, y, z)\, dz \right]\,.$$

Example 13.7. Evaluate the line integral $\int_C [y\, dx - z\, dy + x\, dz]$ where C is the curve with parametric equations $x = t^2$, $y = e^{-t}$, $z = e^t$ with $0 \le t \le 1$.

Solution: Since $x'(t) = 2t$, $y'(t) = -e^{-t}$, $z'(t) = e^t$, we have

$$\int_C [y\,dx - z\,dy + x\,dz] = \int_0^1 \left[e^{-t}\,2t\,dt - e^t(-e^{-t}dt) + t^2(e^t\,dt)\right] = \int_0^1 \left[2t\,e^{-t} + 1 + t^2\,e^t\right] dt =$$

$$= \left[-2e^{-t}(t+1) + t + e^t\left(t^2 - 2t + 2\right)\right]_0^1 = \cdots = e - 4e^{-1} + 1\,.$$

Line integrals of vector fields Let $\mathbf{F}(x,y,z) = u(x,y,z)\,\hat{\imath} + v(x,y,z)\,\hat{\jmath} + w(x,y,z)\,\hat{k}$ be a vector field, and let C be a piecewise smooth orientable curve with parametric representation

$$\mathbf{R}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k} \qquad \text{for } a \le t \le b\,.$$

Using $d\mathbf{R} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$, we define the line integral of $\mathbf{F}$ along C by

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_C (u\,dx + v\,dy + w\,dz) = \int_C \mathbf{F}(\mathbf{R}(t))\cdot\mathbf{R}'(t)\,dt =$$

$$= \int_a^b \left[u\big(x(t),y(t),z(t)\big)\frac{dx}{dt} + v\big(x(t),y(t),z(t)\big)\frac{dy}{dt} + w\big(x(t),y(t),z(t)\big)\frac{dz}{dt}\right] dt\,.$$

Example 13.8. Evaluate $\int_C \mathbf{F}\cdot\mathbf{R}$, where $\mathbf{F} = (y^2 - z^2)\,\hat{\imath} + (2yz)\,\hat{\jmath} - x^2\,\hat{k}$ and C is the curve defined parametrically by $x = t^2$, $y = 2t$, $z = t$, with $0 \le t \le 1$.

Solution: We rewrite $\mathbf{F}$ using the parameter t:

$$\mathbf{F}(t) = \left[(2t)^2 - (t)^2\right]\,\hat{\imath} + \left[2\cdot(2t)\cdot t\right]\,\hat{\jmath} - \left[\left(t^2\right)^2\right]\,\hat{k} = 3t^2\,\hat{\imath} + 4t^2\,\hat{\jmath} - t^4\,\hat{k}\,.$$

Moreover, because $\mathbf{R}(t) = t^2\,\hat{\imath} + 2t\,\hat{\jmath} + t\,\hat{k}$, we have

$$d\mathbf{R} = (2tdt)\,\hat{\imath} + (2dt)\,\hat{\jmath} + (dt)\,\hat{k}\,,$$

so

$$\mathbf{F}\cdot d\mathbf{R} = \left\langle 3t^2, 4t^2, -4t^4\right\rangle\cdot\langle 2t\,dt, 2\,dt, dt\rangle = \left(6t^3 + 8t^2 - t^4\right)\,dt\,.$$

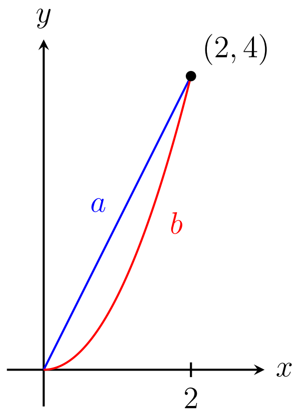
Thus,

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^1 \left(6t^3 + 8t^2 - t^4\right)\,dt = \left[\frac{3}{2}t^4 + \frac{8}{3}t^3 - \frac{1}{5}t^5\right]_0^1 = \frac{119}{30}\,.$$

Example 13.9 (Evaluating line integrals along different paths)**.** Let $\mathbf{F} = xy^2\,\hat{\imath} + x^2 y\,\hat{\jmath}$ and evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{R}$ between the points $(0,0)$ and $(2,4)$ along the following paths:

a. The line segment connecting the two points;

b. The parabolic arc $y = x^2$ connecting the points.



Solution:

a. $x = t$, $y = 2t$, with $0 \le t \le 2$.

$$\mathbf{R}(t) = t\,\hat{\imath} + 2t\,\hat{\jmath} \qquad\qquad d\mathbf{R} = (dt)\,\hat{\imath} + (2\,dt)\,\hat{\jmath}\,.$$

$$\mathbf{F}(t) = (t)\cdot(2t)^2\,\hat{\imath} + (t)^2\cdot(2t)\,\hat{\jmath} = 4t^3\,\hat{\imath} + 2t^3\,\hat{\jmath}\,.$$

$$\Rightarrow \mathbf{F}(t)\cdot d\mathbf{R} = 4t^3\,dt + 4t^3\,dt = 8t^3\,dt\,.$$

Thus,

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^2 8t^3\,dt = \left[2t^4\right]_0^2 = 32\,.$$

b. $y = x^2 \quad \rightarrow \quad x = t \quad \Rightarrow \quad y = t^2$, with $t \in [0,2]$.
Thus,

$$\mathbf{R}(t) = t\,\hat{\imath} + t^2\,\hat{\jmath} \quad \rightarrow \quad d\mathbf{R} = (dt)\,\hat{\imath} + (2t\,dt)\,\hat{\jmath}\,.$$

$$\mathbf{F} = xy^2\,\hat{\imath} + x^2y\,\hat{\jmath} = t^5\,\hat{\imath} + t^4\,\hat{\jmath}\,.$$

$$\mathbf{F}\cdot d\mathbf{R} = t^5\,dt + 2t^5\,dt = 3t^5\,dt\,.$$

$$\int\limits_C \mathbf{F}\cdot d\mathbf{R} = \int_0^2 3t^5\,dt = \left[\frac{1}{2}t^6\right]_0^2 = 32\,.$$

The integral value in this case does not depend on the path. This is not generally true, but when it is true, the line integral is said to be "path independent".

Applications: Mass and Work Consider a thin wire of the shape of a curve C, and let $\rho(x, y, z)$ be the density at each point $P(x, y, z)$ of the wire. Then, the **mass** of the wire is equal to

$$m = \int\limits_C \rho(x, y, z)\,ds\,.$$

The **center of mass** of the wire is the point

$$\bar{x} = \frac{1}{m}\int\limits_C x\,\rho(x, y, z)\,ds\,, \qquad \bar{y} = \frac{1}{m}\int\limits_C y\,\rho(x, y, z)\,ds\,, \qquad \bar{z} = \frac{1}{m}\int\limits_C z\,\rho(x, y, z)\,ds\,.$$

Example 13.10. A wire has the shape $x = \sqrt{2}\sin(t)$, $y = \cos(t)$, $z = \cos(t)$, for $0 \le t \le \pi$. The wire has density $\rho(x, y, z) = xyz$. Find the mass.

Solution:

$$x'(t) = \sqrt{2}\,\cos(t)\,, \qquad y'(t) = -\sin(t)\,, \qquad z'(t) = -\sin(t)\,.$$

$$
\begin{aligned}
m = \int\limits_C \rho(x,y,z)\,dt = \int\limits_C xyz\sqrt{(x')^2+(y')^2+(z')^2}\,dt = \\
&= \int\limits_C \sqrt{2}\,\sin(t)\,\cos^2(t)\sqrt{2\cos^2(t)+2\sin^2(t)}\,dt \\
&= \int_0^\pi \sqrt{2}\,\sin(t)\,\cos^2(t)\,\sqrt{2}\,dt = 2\int_0^\pi \sin(t)\,\cos^2(t)\,dt = \\
&\ldots \quad \cos(t)=u \quad \to \quad -\sin(t)\,dt = du \quad \to \quad \sin(t)\,dt = -du \quad \ldots \\
&\ldots \quad t=0 \to u=1 \qquad\qquad t=\pi \to u=-1 \quad \ldots \\
&= 2\int_1^{-1} u^2(-du) = 2\int_{-1}^1 u^2\,du = 2\left[\frac{u^3}{3}\right]_{-1}^1 = \frac{4}{3}\,.
\end{aligned}
$$

- The work: Let $\mathbf{F}$ be a continuous force field over a domain D. Then the work W performed as an object moves along a smooth curve C in D is given by the integral

$$W = \int\limits_C \mathbf{F}\cdot\mathbf{T}\,ds\,,$$

where $\mathbf{T}$ is the unit tangent at each point on C. Remembering that $\mathbf{T} = \dfrac{d\mathbf{R}}{ds}$, we have

$$W = \int\limits_C \mathbf{F}\cdot d\mathbf{R}\,.$$

Example 13.11 (Problem 11, HW 13.1-13.4)**.** Let $\mathbf{F}$ be the radial force $\mathbf{F} = x\,\hat{\imath} + y\,\hat{\jmath}$. Find the work done by this force along the parabola $x=t$, $y=t^2$, with $0 \le t \le 1$.

Solution: The position vector $\mathbf{R}$ is given by $\mathbf{R} = \langle t, t^2\rangle$. Then, $\mathbf{R}' = \langle 1, 2t\rangle$.

$$\mathbf{F} = \langle x, y\rangle = \left\langle t, t^2\right\rangle \qquad \to \qquad \mathbf{F}\cdot\mathbf{R}' = t + 2t^3\,.$$

$$\Rightarrow \quad W = \int_0^1 \left(t+2t^3\right)\,dt = \left[\frac{t^2}{2}+\frac{t^4}{2}\right]_0^1 = 1\,.$$

Example 13.12 (Problem 12, HW 13.1-13.4)**.** Find the work done by the force field $\mathbf{F}(x,y,z) = 3x\,\hat{\imath} + 3y\,\hat{\jmath} + 5\hat{k}$ on a particle that moves along the helix $\mathbf{r}(t) = 7\cos(t)\,\hat{\imath} + 7\sin(t)\,\hat{\jmath} + 2t\,\hat{k}$, for $0 \le t \le 2\pi$.

Solution:

$$\mathbf{r}'(t) = \langle -7\sin(t), 7\cos(t), 2\rangle \qquad \mathbf{F}(t) = \langle 21\cos(t), 21\sin(t), 5\rangle$$

$$\mathbf{r}' \cdot \mathbf{F} = -147\cos(t)\sin(t) + 147\cos(t)\sin(t) + 10\,.$$

$$W = \int_0^{2\pi} \mathbf{r}' \cdot \mathbf{F}\,dt = \int_0^{2\pi} 10\,dt = [10\,t]_0^{2\pi} = 20\,\pi\,.$$

13.3 The fundamental theorem and path independence

Fundamental theorem for line integrals Let C be a piecewise smooth curve that is parametrized by the vector function $\mathbf{R}(t)$ for $a \leq t \leq b$, and let $\mathbf{F}(t)$ be a vector field that is continuous on C. If $\mathbf{F}$ is a scalar function such that $\mathbf{F} = \nabla f$, then

$$\int_C \mathbf{F} \cdot d\mathbf{R} = f(Q) - f(P)\,,$$

where $Q = \mathbf{R}(b)$ and $P = \mathbf{R}(a)$ are the endpoints of C.

Example 13.13. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{R}$, where $\mathbf{F} = \nabla\left(e^x \sin(y) - xy - 2y\right)$ and C is the path described by $\mathbf{R}(t) = \left[t^3 \sin\left(\frac{\pi}{2}t\right)\right]\hat{\imath} - \left[\frac{\pi}{2}\cos\left(\frac{\pi}{2}t + \frac{\pi}{2}\right)\right]\hat{\jmath}$, for $0 \leq t \leq 1$.

Solution: The hypotheses of the fundamental theorem are satisfied since $f(x, y) = e^x \sin(y) - xy - 2y$ has continuous partial derivatives. Endpoints:

- $t = 0 \quad \rightarrow \quad \mathbf{R}(0) = \langle 0, 0\rangle \qquad f(0, 0) = 0\,,$
- $t = 1 \quad \rightarrow \quad \mathbf{R}(1) = \left\langle 1, \frac{\pi}{2}\right\rangle \qquad f\left(1, \frac{\pi}{2}\right) = e - \frac{3\pi}{2}\,.$

Thus, using the fundamental theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{R} = f(Q) - f(P) = f\left(1, \frac{\pi}{2}\right) - f(0, 0) = \left(e - \frac{3\pi}{2}\right) - 0 = e - \frac{3\pi}{2}\,.$$

Conservative vector fields A vector field $\mathbf{F}$ is said to be **conservative** in a region D if $\mathbf{F} = \nabla f$ for some scalar function f in D. The function f is called a **scalar potential** of $\mathbf{F}$ in D. That is,

$$\mathbf{F} = \nabla f\,, \qquad \text{for } (x, y) \in D\,,$$

where $\mathbf{F}$ is a conservative vector field, and f is the scalar potential.

Example 13.14. Verify that the vector field $\mathbf{F} = 2xy\,\hat{\imath} + x^2\,\hat{\jmath}$ is conservative, with scalar potential $f = x^2 y$.

Solution: $\nabla f = 2xy\,\hat{\imath} + x^2\,\hat{\jmath} = \mathbf{F} \quad \Rightarrow \quad$ conservative.

Usually, the function f is not given, so we need a theorem to understand if a vector field is conservative:

Theorem 13.2 (Cross-partial test for a conservative vector field in the plane)**.** Consider the vector field $\mathbf{F}(x,y) = u(x,y)\,\hat{\imath} + v(x,y)\,\hat{\jmath}$, where u and v have continuous first partials in the open, simply connected region D in the plane. Then $\mathbf{F}(x,y)$ is a conservative in D if and only if

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}\,, \qquad \text{troughout } D\,.$$

Example 13.15. Show that the vector field $\mathbf{F} = (e^x \sin(y) - y)\;\hat{\imath} + (e^x \cos(y) - x - 2)\;\hat{\jmath}$ is conservative and then find a scalar potential function f of $\mathbf{F}$.

Solution:

$$\frac{\partial u}{\partial y} = e^x \cos(y) - 1\,, \qquad\qquad \frac{\partial v}{\partial x} = e^x \cos(y) - 1\,.$$

$\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x} \quad \Rightarrow \quad \mathbf{F}$ is conservative. Now, since the field is conservative;

$$f(x,y) = \int u(x,y)\,dx = \int (e^x \sin(y) - y)\;dx = e^x \sin(y) - xy + C(y)\,.$$

The function f is now defined with a constant $C(y)$. This function must also satisfy

$$\left.\begin{aligned} f_y(x,y) = v = e^x \cos(y) - x - 2 \\ f_y = e^x \cos(y) - x + \frac{dC(y)}{dy} \end{aligned}\right\} \quad \Rightarrow \quad \frac{dC(y)}{dy} = -2\,,$$

$$C(y) = \int (-2)\,dy = -2y + C_1\,.$$

Thus:

$$f(x,y) = e^x \sin(y) - xy - 2y + C_1\,,$$

where all numbers are allowed in C_1.

Theorem 13.3 (The curl criterion for a conservative vector field in $\mathbb{R}^3$)**.** Suppose the vector field $\mathbf{F}$ and the curl $(\mathbf{F})$ are continuous in the simply connected region D of $\mathbb{R}^3$. Then $\mathbf{F}$ is conservative in D if and only if curl $(\mathbf{F}) = \mathbf{0}$.

Example 13.16. Show that the vector field $\mathbf{F} = \langle 20x^3z + 2y^2, 4xy, 5x^4 + 3z^2\rangle$ is conservative in $\mathbb{R}^{\nVdash}$ and find a scalar potential function for $\mathbf{F}$.

Solution:

$$\text{curl }\mathbf{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 20x^3z + 2y^2 & 4xy & 5x^4 + 3z^2 \end{vmatrix} = \left(\frac{\partial\,(5x^4 + 3z^2)}{\partial y} - \frac{\partial\,(4xy)}{\partial z}\right)\hat{\imath} +$$
$$- \left(\frac{\partial\,(5x^4 + 3z^2)}{\partial x} - \frac{\partial\,(20x^3z + 2y^2)}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial(4xy)}{\partial x} - \frac{\partial\,(20x^3z + 2y^2)}{\partial y}\right)\hat{k} =$$
$$= 0\,\hat{\imath} - \left(20x^3 - 20x^3\right)\hat{\jmath} + (4y - 4y)\,\hat{k} = \langle 0, 0, 0\rangle\,.$$

Thus, $\mathbf{F}$ is conservative.

Independence on path The line integral $\int\limits_C \mathbf{F}\cdot d\mathbf{R}$ is **independent of path** in a region D if for any two points P and Q in D the line integral along every piecewise smooth curve in D from P to Q has the same value.

Theorem 13.4 (Equivalent conditions for path independence)**.** If $\mathbf{F}$ is a continuous vector field on the open connected set D, then the following three conditions are either all true or all false:

1. $\mathbf{F}$ is conservative on D; that is, $\mathbf{F} = \nabla f$ for some functions defined on D;
2. $\oint\limits_C \mathbf{F}\cdot d\mathbf{R} = 0$ for every piecewise closed curve C in D;
3. $\int\limits_C \mathbf{F}\cdot d\mathbf{R}$ is independent of path within D.

Example 13.17. Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{R}$, where

$$\mathbf{F} = \left[\left(2x - x^2 y\right) e^{-xy} + \tan^{-1}(y)\right]\,\hat{\imath} + \left[\frac{x}{y^2+1} - x^3 e^{-xy}\right]\,\hat{\jmath}\,,$$

for each of the following curves:

a. C_1 : the ellipse $9x^2 + 4y^2 = 36$;

b. C_2 : the curve with parametric equations

$$x = t^2\cos(\pi\, t)\,, \qquad y = e^{-t}\sin(\pi\, t)\,, \qquad \text{for } 0 \le t \le 1\,.$$

Solution: First, we verify if $\mathbf{F}$ is conservative:

$$\frac{\partial v}{\partial x} = \frac{1}{y^2+1} - 3x^2 e^{-xy} + x^3 y e^{-xy}$$

$$\frac{\partial u}{\partial y} = \frac{1}{y^2+1} - x^2 e^{-xy} + (-x)(2x - x^2 y)e^{-xy} = \frac{1}{y^2+1} - 3x^2 e^{-xy} + x^3 y e^{-xy}\,.$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \quad \to \quad \mathbf{F} \text{ is conservative}\,.$$

a. Since $\mathbf{F}$ is conservative and C is closed, then $\oint_C \mathbf{F}\cdot d\mathbf{R} = 0$;

b. The curve C_2 starts from $t = 0 \to P(0,0)$ and ends at $t = 1 \to Q(-1,0)$. Since $\mathbf{F}$ is conservative, then $\int_C \mathbf{F}\cdot d\mathbf{R}$ is path independent. Thus, to simplify the integral, we can use another parametric curve connecting P and Q. We use the segment $PQ:\ x = -t\,,\quad y = 0\,,\qquad t\in[0,1]$.

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^1 \mathbf{F}(\mathbf{R}(t))\cdot\mathbf{R}'(t)\,dt =$$

$$= \int_0^1 \left[\left(2(-t) - t^2\cdot 0\right) - e^0 + \tan^{-1}(0)\right](-1) + \left[\frac{-t}{0+1} - (-t)^3\right](0)\,dt =$$

$$= \int_0^1 2t\,dt = 1\,.$$

Example 13.18. *(#19 HW 13.1-13.4)* Consider the vector field

$$\mathbf{F} = y\sin(z)\,\hat{\imath} + (x\sin(z) + 2y)\,\hat{\jmath} + (xy\cos(z))\,\hat{k}.$$

Show that $\mathbf{F}$ is conservative and find the potential function f.

Then compute

$$\int_C \mathbf{F}\cdot d\mathbf{r},$$

where C goes from $(1,1,0)$ to $(2,1,\frac{\pi}{2})$.

Solution. To verify that $\mathbf{F}$ is conservative, compute

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y\sin z & x\sin z + 2y & xy\cos z \end{vmatrix}.$$

This gives

$$\nabla\times\mathbf{F} = \hat{\imath}(x\cos z - x\cos z) - \hat{\jmath}(y\cos z - y\cos z) + \hat{k}(\sin z - \sin z) = \mathbf{0},$$

so $\mathbf{F}$ is conservative.

To find the potential f, integrate:

$$f_x = y\sin z \quad\Rightarrow\quad f = \int y\sin z\,dx = xy\sin z.$$

$$f_y = x\sin z + 2y \quad\Rightarrow\quad f = \int (x\sin z + 2y)\,dy = xy\sin z + y^2.$$

$$f_z = xy\cos z \quad\Rightarrow\quad f = \int xy\cos z\,dz = xy\sin z.$$

Hence,

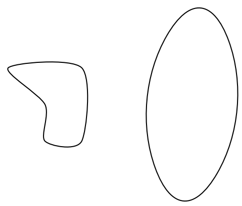
$$f(x,y,z) = xy\sin z + y^2.$$

Finally, the line integral is path independent:

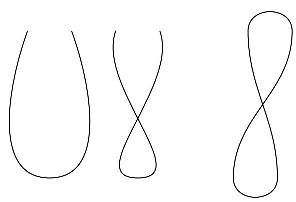
$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(2,1,\tfrac{\pi}{2}) - f(1,1,0) = [(2)(1)\sin(\tfrac{\pi}{2}) + 1^2] - [(1)(1)\sin(0) + 1^2] = 2.$$

13.4 Green's theorem

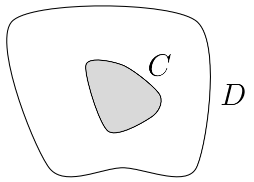
Green's theorem relates a line integral around a closed curve to a double integral over the region contained by the curve. A **Jordan curve** is a closed curve that does not intersect itself. A **simply connected region** D in the plane has the property that it is connected and the interior of every Jordan curve C in D also lies in D.



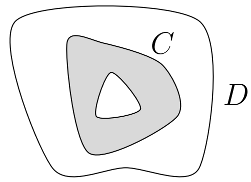
Jordan curves



Non–Jordan curves



Simply connected region



Not-simply connected region

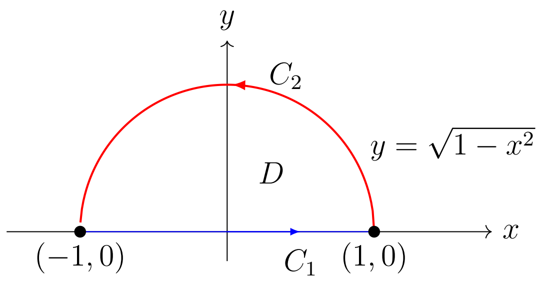
Theorem 13.5 (Green's theorem)**.** Let D be a simply connected region that is bounded by the positively oriented piecewise smooth Jordan curve C. Then if the vector field $\mathbf{F}(x, y) = M(x, y)\,\hat{\imath} + N(x, y)\,\hat{\jmath}$ is continuously differentiable on D, we have

$$\oint_C (M\,dx + N\,dy) = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA\,.$$

Example 13.19. Show that Green's theorem is true for the line integral

$$\int_C (-y\,dx + x\,dy)\,,$$

where C is the closed path in the figure.



Solution: We first evaluate the line integral:

C_1: $x = t\,, \quad y = 0\,,$ with $-1 \le t \le 1 \quad \to \quad dx = dt\,, \quad dy = 0\,;$

C_2: $x = u\,, \quad y = \sqrt{1-u^2}\,,$ with $-1 \le u \le 1 \quad \to \quad dx = du\,, \quad dy = \dots$

Better use the polar coordinates for C_2:

$$x = \cos(u)\,, \qquad y = \sin(u)\,, \qquad \text{with } 0 \le u \le \pi$$

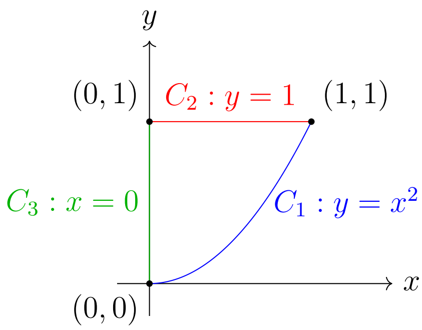
$$dx = -\sin(u)\,du\,, \qquad dy = \cos(u)\,du\,.$$

$$\begin{aligned} \oint_C (-y\,dx + x\,dy) &= \int_{C_1} (-y\,dx + x\,dy) + \int_{C_2} (-y\,dx + x\,dy) = \\ &= \int_{-1}^{1} 0 \cdot dt + t \cdot 0 \,+ \int_0^{\pi} -\sin(u)(-\sin(u)\,du) + \cos^2(u)\,du = \\ &= \int_0^{\pi} \left(\sin^2(u) + \cos^2(u)\right)\,du = \int_0^{\pi} du = \pi\,. \end{aligned}$$

Now, we evaluate

$$\iint_D \left(\frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y} \right) dA = \iint_D 2\,dA = 2 \cdot \text{Area of } D = 2\frac{\pi\,(1)^2}{2} = \pi\,.$$

Example 13.20. A closed path is defined in the figure:



Find the work done on an object moving along C in the force field

$$\mathbf{F} = \left(x + xy^2\right)\,\hat{\imath} + 2\left(x^2y - y^2\sin(y)\right)\,\hat{\jmath}\,.$$

Solution: Remembering that $W = \oint_C \mathbf{F}\cdot d\mathbf{R}$, we can apply the Green's theorem:

$$\begin{aligned} W = \oint_C \mathbf{F}\cdot d\mathbf{R} &= \iint_D \left[\frac{\partial}{\partial x}\left(2x^2y - 2y^2\sin(y)\right) - \frac{\partial}{\partial y}\left(x + xy^2\right)\right] dA = \\ &= \iint_D (4xy - 2xy)\, dA \overset{\text{Type I}}{=} 2\int_0^1\int_{x^2}^1 xy\,dy\,dx = 2\int_0^1 x\left[\frac{x^2}{2}\right]_{x^2}^1 dx = \\ &= 2\int_0^1 x\left(\frac{1}{2} - \frac{x^4}{2}\right) dx = \int_0^1 \left(x - x^5\right) dx = \left[\frac{x^2}{2} - \frac{x^6}{6}\right]_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}\,. \end{aligned}$$

Theorem 13.6 (Area as a line integral)**.** Let D be a simply connected region in the plane with piecewise smooth, positively oriented closed boundary C. Then, the area A of region D is given by each of the following integrals

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C [x\,dy - y\,dx]\,.$$

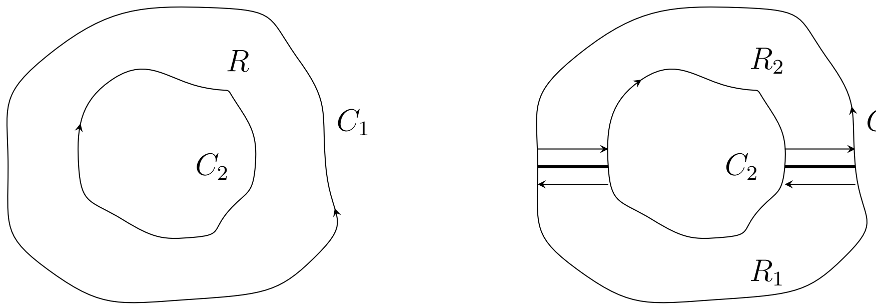
Example 13.21. Show that the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has $A = \pi\,a\,b$.

Solution: Parametrization: $x = a\cos(\theta)\,,\ y = b\sin(\theta)\,,$ for $\theta \in [0, 2\pi]$. Then we have

$dx = -a\,\sin(\theta)\,d\theta\,,\; dy = b\,\cos(\theta)\,d\theta.$

$$A = \frac{1}{2}\oint_C (-y\,dx + x\,dy) = \frac{1}{2}\int_0^{2\pi} [-b\,\sin(\theta)(-a)\,\sin(\theta) + a\cos(\theta)\,b\cos(\theta)]\;d\theta = \pi\,a\,b\,.$$

Green's theorem for multiply connected regions To state Green's theorem, we required simply connected regions. We can now extend it to multiply-connected regions.

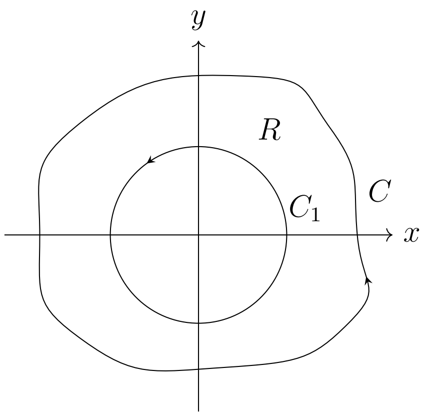


Theorem 13.7 (Green's theorem for doubly connected regions)**.** Let R be a doubly connected region (one hole) in the plane, with outer boundary C, oriented counter-clockwise, and boundary C_2, the hole oriented clockwise. If the boundary curves and $\mathbf{F}(x,y) = M(x,y)\,\hat{\imath} + N(x,y)\,\hat{\jmath}$ satisfy the hypotheses of Green's theorem, then

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,dA = \oint_{C_1} (M\,dx + N\,dy) + \oint_C (M\,dx + N\,dy)\;.$$

Example 13.22. Show that $\displaystyle\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = 2\pi$, where C is any piecewise smooth Jordan curve enclosing the origin $(0,0)$.

Solution: Let $M(x,y) = \dfrac{-y}{x^2+y^2}$ and $N(x,y) = \dfrac{x}{x^2+y^2} \quad\rightarrow\quad \dfrac{\partial N(x,y)}{\partial x} = \dfrac{\partial M(x,y)}{\partial y} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$. We consider a generic curve C and an inner circle oriented clockwise.



Since $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$ on R, we have

$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} + \oint_{C_1} \frac{-y\,dx + x\,dy}{x^2+y^2} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 0\,.$$

Therefore

$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = -\oint_{C_1} \frac{-y\,dx + x\,dy}{x^2+y^2} = +\oint_{-C_1} \frac{-y\,dx + x\,dy}{x^2+y^2}\,.$$

We are interested in the line $-C_1$ since it is counterclockwise and we can apply the polar coordinates:

$$\oint_{-C_1} \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^{2\pi} \frac{-r\,\sin(\theta)(-r\,\sin(\theta)\,d\theta) + r\,\cos(\theta)(r\,\cos(\theta)\,d\theta)}{r^2} = \int_0^{2\pi} \frac{r^2}{r^2}\,d\theta = 2\,\pi\,.$$

Alternate forms for Green's theorem Let D be a simply connected region with a positively oriented boundary C. Then if the vector field $\mathbf{F} = M\,\hat{i} + N\,\hat{j}$ is continuously differentiable on D, we have

- Tangential component of $\mathbf{F}$:

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \oint (M\,dx + N\,dy) = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_D \left(\text{curl }\mathbf{F}\cdot\hat{k}\right) dA\,;$$

- Normal component of $\mathbf{F}$:

$$\oint_C \mathbf{F}\cdot\mathbf{N}\,ds = \iint_D \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)\,dA = \iint_D \operatorname{div}\,\mathbf{F}\,dA\,.$$

Normal derivatives The normal derivative of f, denoted by $\dfrac{\partial f}{\partial n}$, is the directional derivative of f in the direction pointing to the exterior of the domain of f. In other words,

$$\frac{\partial f}{\partial n} = \nabla f\cdot\mathbf{N}\,,$$

where $\mathbf{N}$ is the unit normal vector.

Example 13.23 (Green's formula for the integral of the Laplacian)**.** Suppose f is a scalar function with continuous first and second partial derivatives in the simply connected region D. If the piecewise smooth, positively oriented closed curve C bounds D, show that

$$\iint_D \nabla^2 f\,dx\,dy = \oint_C \frac{\partial f}{\partial n}\,ds\,,$$

where $\nabla^2 f = f_{xx}+f_{yy}$ and $\dfrac{\partial f}{\partial n} = \nabla f\cdot\mathbf{N}$.

Solution: Let $u = -\dfrac{\partial f}{\partial y}$ and $v = \dfrac{\partial f}{\partial x}$.

$$\begin{aligned}\iint_D \nabla^2 f\,dx\,dy &= \iint_D \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\,dx\,dy \overset{\text{Green}}{=} \oint_C (u\,dx+v\,dy) = \oint_C \left(u\frac{dx}{ds}+v\frac{dy}{ds}\right)\,ds = \\ &= \oint_C \left(-f_y\frac{dx}{ds}+f_x\frac{dy}{ds}\right)\,ds = \oint_C \nabla f\cdot\left(\frac{dy}{ds}\,\hat{\imath}-\frac{dx}{ds}\,\hat{\jmath}\right)\,ds = \\ &= \oint_C \nabla f\cdot\mathbf{N}\,ds = \oint_C \frac{\partial f}{\partial n}\,ds\,.\end{aligned}$$

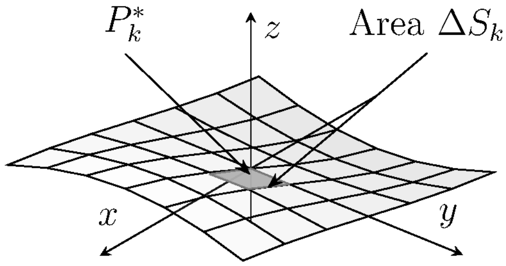
Because $\mathbf{N} = \left\langle \dfrac{dy}{ds}, -\dfrac{dx}{ds}\right\rangle$ by definition.

13.5 Surface integrals

Surface integration Let S be a surface defined by $z = f(x, y)$ and R is its projection in the xy-plane. If f, f_x and f_y are continuous in R and g is a continuous function of three variables on S, then the surface integral of g over S is

$$\iint_S g(x,y,z)\,dS = \iint_R g(x,y,f(x,y))\sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\,dA\,.$$

Note: This integration can again be interpreted as the Riemann sum $\displaystyle\sum_{k=1}^{n} g(P_k^*)\Delta S_k\,.$



Example 13.24. Evaluate the surface integral $\displaystyle\iint_S g\,dS$, where $g(x,y,z) = xz + 2x^2 - 3xy$ and S is the portion of the plane $2x - 3y + z = 6$ that lies over the unit square $2 \le x \le 3\,,\ 2 \le y \le 3$.

Solution: The plane has equation $z = f(x,y) = 6 - 2x + 3y \quad \to \quad f_x = -2\,, \quad f_y = 3\,.$

$$g(x,y,f(x,y)) = x(6 - 2x + 3y) + 2x^2 - 3xy = 6x - 2x^2 + 3xy + 2x^2 - 3xy = 6x\,.$$

$$\iint_S g\,dS = \int_2^3\int_2^3 6x\sqrt{(-2)^2 + (3)^2 + 1}\,dx\,dy = \int_2^3\int_2^3 6\sqrt{14}\,x\,dx\,dy =$$

$$= 6\sqrt{14}\int_2^3 \left[\frac{x^2}{2}\right]_2^3 dy = 6\sqrt{14}\,[y]_2^3\left(\frac{9}{2} - 2\right) = 15\sqrt{14}\,.$$

Surface area, mass and center of mass of a lamina

- Surface area formula: $A = \displaystyle\iint_S dS\,;$

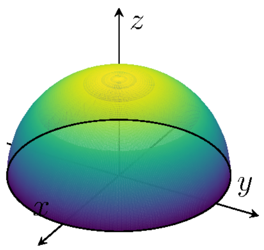
- If $\rho(x,y,z)$ is the density of the lamina, then the total mass of the lamina is given by

$$m = \iint_S \rho(x,y,z)\,dS\,,$$

and the center of mass is

$$\bar{x} = \frac{1}{m}\iint_S x\,\rho\,dS\,, \qquad \bar{y} = \frac{1}{m}\iint_S y\,\rho\,dS\,, \qquad \bar{z} = \frac{1}{m}\iint_S z\,\rho\,dS\,.$$

Example 13.25. Find the mass of a lamina of density $\rho(x,y,z) = z$ in the shape of the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$.



Solution:

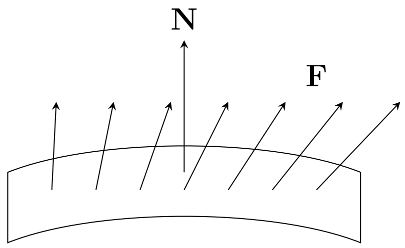
$$z_x = \frac{1}{2}\left(a^2 - x^2 - y^2\right)^{-\frac{1}{2}}(-2x) = -x\left(a^2 - x^2 - y^2\right)^{-\frac{1}{2}}$$
$$z_y = -y\left(a^2 - x^2 - y^2\right)^{-\frac{1}{2}}\,.$$

$$dS = \sqrt{1 + z_x^2 + z_y^2}\,dA = \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}}\,dA =$$
$$= \sqrt{\frac{a^2 - x^2 - y^2 + x^2 + y^2}{a^2 - x^2 - y^2}}\,dA = \sqrt{\frac{a^2}{a^2 - x^2 - y^2}}\,dA = \frac{a}{\sqrt{a^2 - x^2 - y^2}}\,dA\,.$$
$$m = \iint_S \rho(x,y,z)\,dS = \iint_S z\,dS = \iint_S \left(a^2 - x^2 - y^2\right)^{\frac{1}{2}} \frac{a}{(a^2 - x^2 - y^2)^{\frac{1}{2}}}\,dA =$$
$$= a\iint_S dA = \pi\,a^2 \cdot a = \pi\,a^3\,.$$

Flux integrals Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on the surface S, which is oriented by the unit normal field $\mathbf{N}$. Then the flux of $\mathbf{F}$

across S is given by the surface integral

$$\text{Flux} = \iint_S \mathbf{F}\cdot\mathbf{N}\,dS\,.$$



In general, if we consider the surface S described as $z = f(x,y)$, then we can define $G(x,y,z) = z - f(x,y)$. The upward normal (unit) to S is

$$\mathbf{N} = \frac{\nabla G}{||\nabla G||} \quad\Rightarrow\quad \mathbf{F}\cdot\mathbf{N}\,dS = \mathbf{F}\cdot\left(\frac{\nabla G}{||\nabla G||}\right)\sqrt{f_x^2+f_y^2+1}\,dA\,.$$

Thus, since $||\nabla G|| = \sqrt{f_x^2+f_y^2+1}$ we have

- Upward normal:

$$\iint_S \mathbf{F}\cdot\mathbf{N} = \iint_R \mathbf{R}(x,y,f(x,y))\cdot\langle -f_x, -f_y, 1\rangle\ dA\,;$$

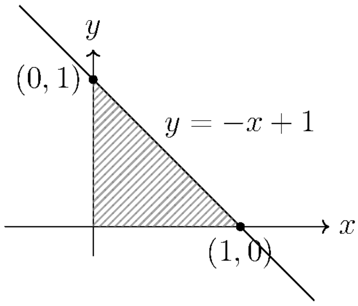
- Downward normal:

$$\iint_S \mathbf{F}\cdot\mathbf{N} = \iint_R \mathbf{R}(x,y,f(x,y))\cdot\langle f_x, f_y, -1\rangle\ dA\,.$$

Example 13.26. Compute the flux integral

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS\,,$$

where $\mathbf{F} = xy\,\hat{i} + z\,\hat{j} + (x+y)\,\hat{k}$, and S is the triangular surface cut off from the plane $x+y+z=1$. Assume $\mathbf{N}$ is the upward unit normal.



Solution:

$$f(x,y) = z = 1 - x - y \qquad \to \qquad f_x = -1\,, \qquad f_y = -1\,.$$

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iint_D \mathbf{F}(x,y,f(x,y))\cdot\langle -f_x, -f_y, 1\rangle\; dA =$$

$$\overset{\text{Type I}}{=} \int_0^1 \int_0^{-x+1} \langle xy, 1-x-y, x+y\rangle \cdot \langle 1,1,1\rangle\, dA =$$

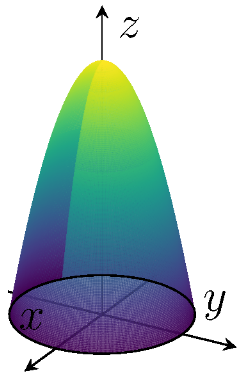
$$= \int_0^1 \int_0^{-x+1} (-xy + 1 - x - y + x + y)\, dy\, dx = \int_0^1 \left[-x\frac{-y^2}{2} + y \right]_0^{-x+1} dx =$$

$$= \int_0^1 \frac{-x(-x+1)^2}{2} + (-x+1)\, dx = \int_0^1 \frac{-x^3 + 2x^2 - x - 2x + 2}{2}\, dx =$$

$$= \int_0^1 \frac{-x^3 + 2x^2 - 3x + 2}{2}\, dx = \frac{1}{2}\left[-\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2} + 2x \right]_0^1 =$$

$$= \frac{1}{2}\left(-\frac{1}{4} + \frac{2}{3} - \frac{3}{2} + 2 \right) = \frac{13}{24}\,.$$

Example 13.27. Let R be the region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the xy-plane. The heat flow is the vector field $\mathbf{H} = -k\nabla T$, where $T(x,y,z) = 2x + y - 3z^2$, and k is constant. Find the total heat flow $\iint_R \mathbf{H}\cdot\mathbf{N}\,dS$ out of the region ($\mathbf{N}$ outer unit normal).



Solution: We have the temperature gradient equal to $\nabla T = \langle 2, 1-6z\rangle$. Then, we should consider two surfaces: the paraboloid (S_1) $z = 9 - x^2 - y^2$ and the plane (S_2) $z = 0$.

$$S_1\colon\ G = z + x^2 + y^2 - 9 \qquad \to \qquad \nabla G = \langle 2x, 2y, 1\rangle$$

$$\iint_{S_1} \mathbf{H}\cdot\mathbf{N}_1\, dS = \iint_{S_1} -k\nabla T\cdot\nabla G\, dS = \iint_{S_1} -k\langle 2,1,-6z\rangle\cdot\langle 2x,2y,1\rangle\, dA =$$

$$= \iint_{S_1} -k\left(4x + dy - 6(9 - x^2 - y^2)\right)\, dA =$$

$$= -k\int_0^{2\pi}\int_0^3 \left[4r\cos(\theta) + 2r\sin(\theta) - 6(9-r^2)\right]\, dr\, d\theta =$$

$$= -k\int_0^{2\pi}\left[36\cos(\theta) + 18\sin(\theta) - \frac{243}{2}\right]\, d\theta = 243\,\pi\, k\,.$$

$$S_2\colon\ G = z \qquad \to \qquad \nabla G = \langle 0,0,-1\rangle \text{ (Downwards)}$$

$$\iint_{S_2} \mathbf{H}\cdot\mathbf{N}_2\, dS = \iint_D k\langle 2,1,-6z\rangle\cdot\langle 0,0,-1\rangle\, dA = -k\iint_D 6z\, dA = -k\iint_D 0\, dA = 0\,,$$

since $z = 0$ on S_2.

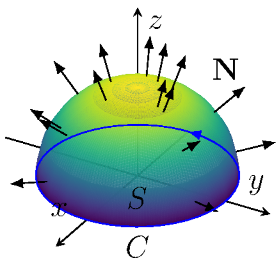
$$\text{Total heat flow} \quad\Rightarrow\quad \iint_S \mathbf{H}\cdot\mathbf{N} = 243\,\pi\, k.$$

13.6 Stoke's theorem and applications

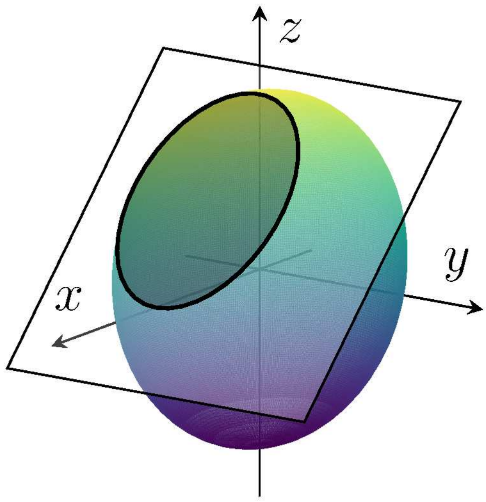
Definition 13.3 (Compatible orientation)**.** We say that a closed path C is **compatible with the orientation** on the surface S, if the positive direction on C is counterclockwise with the outward normal vector $\mathbf{N}$ of the surface.

Theorem 13.8 (Stoke's theorem)**.** Let S be an oriented surface with unit normal vector field $\mathbf{N}$, and assume that S is bounded by a piecewise smooth Jordan curve C whose orientation is compatible with that of S. If $\mathbf{F}$ is a vector field that is continuously differentiable on S, then

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_S (\text{curl } \mathbf{F} \cdot \mathbf{N}) \, dS \, .$$



Example 13.28. Evaluate $\oint_C \left(\frac{1}{2}y^2 \, dx + z \, dy + x \, dz\right)$ where C is the curve of intersection between the plane $x + z = 1$ and the ellipsoid $x^2 + 2y^2 + z^2 = 1$ oriented counterclockwise from above.



Solution: Given $\mathbf{F} = \left\langle \frac{1}{2}y^2, z, x \right\rangle$, we have

$$\text{curl } \mathbf{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{2}y^2 & z & x \end{vmatrix} = \hat{\imath}(0-1) - \hat{\jmath}(1-0) + \hat{k}\left(0 - \frac{2y}{2}\right) = \langle -1, -1, -y \rangle .$$

Normal vector to the plane: $\mathbf{N} = \langle 1, 0, 1 \rangle$.

Unit normal $\mathbf{N} = \dfrac{\langle 1,0,1\rangle}{||\mathbf{N}||} = \dfrac{\langle 1,0,1\rangle}{\sqrt{1+1}} = \left\langle \dfrac{1}{\sqrt{2}}, 0, \dfrac{1}{\sqrt{2}} \right\rangle$. Thus,

$$\text{curl } \mathbf{F} \cdot \mathbf{N} = \langle -1, -1, -y \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle = -\frac{1}{\sqrt{2}}(1+y) .$$

We aim to do a surface integral on the plane $\rightarrow$ $z_x = 1\,,$ $z_y = 0$ $\rightarrow$ $dS = \sqrt{(-1)^2 + 0^2 + 1} = \sqrt{2}$.

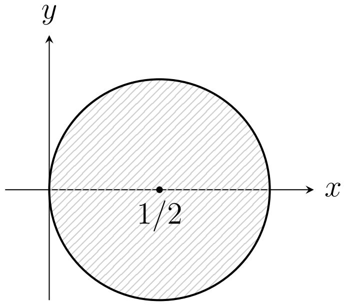
$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_S \text{curl } \mathbf{F} \cdot \mathbf{N}\, dS = \iint_A -\frac{1}{\sqrt{2}}\sqrt{2}(1+y)\, dA \,,$$

where A is the intersection between the plane and the ellipsoid:

$$\begin{cases} z = 1 - x \\ z^2 = 1 - x^2 - 2y^2 \end{cases} \rightarrow \quad (1-x)^2 = 1 - x^2 - 2y^2$$

$$
\begin{aligned}
&1 - 2x + x^2 = 1 - x^2 - 2y^2 \\
&2x^2 - 2x + 2y^2 = 0 \\
&\left(x^2 - x + \frac{1}{4}\right) + y^2 = \frac{1}{4} \qquad \rightarrow \quad x^2 - x + y^2 = 0 \\
&\left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2
\end{aligned}
$$

So we have a circle, with center $c\left(\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$.



$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Plug into $x^2 - x + y^2 = 0$:

$$
\begin{aligned}
&r^2 - r\cos(\theta) = 0 \\
&r(r - \cos(\theta)) = 0
\end{aligned}
\quad \rightarrow \quad r = 0 \text{ and } r = \cos(\theta) \quad \Rightarrow \quad r \in [0, \cos(\theta)]
$$

$$
\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{R} &= \int_{-\pi/2}^{\pi/2} \int_0^{\cos(\theta)} -(1 + r\sin(\theta)) r \, dr \, d\theta = -\int_{-\pi/2}^{\pi/2} \left[\frac{r^2}{2} + \frac{r^3 \sin(\theta)}{3}\right]_0^{\cos(\theta)} d\theta = \\
&= -\int_{-\pi/2}^{\pi/2} \left[\frac{\cos^2(\theta)}{2} + \frac{\cos^3(\theta)\sin(\theta)}{3}\right] d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{1 + \cos(2\theta)}{4} + \frac{\cos^3(\theta)\sin(\theta)}{3}\right] d\theta = \\
&= -\left[\frac{1}{4}\theta + \frac{\sin(2\theta)}{8} + \frac{\cos^4(\theta)}{12}\right]_{-\pi/2}^{\pi/2} = -\frac{\pi}{4}.
\end{aligned}
$$

Example 13.29 (Maxwell's current density equation)**.** In physics, we have that if I is the current crossing a surface S bounded by a closed curve C, then:

$$\oint_C \mathbf{H} \cdot d\mathbf{R} = I \qquad \text{and} \qquad \iint_S \mathbf{J} \cdot \mathbf{N}\, dS = I\,,$$

where $\mathbf{H}$ is the magnetic intensity and $\mathbf{J}$ is the electric current density. Use this information to derive Maxwell's current density equation curl $\mathbf{H} = \mathbf{J}$.

Solution: By Stoke's theorem, we have

$$\oint_C \mathbf{H} \cdot d\mathbf{R} = \iint_S (\text{curl } \mathbf{H} \cdot \mathbf{N})\; dS\,.$$

Since

$$\oint_C \mathbf{H} \cdot d\mathbf{R} = \iint_S \mathbf{J} \cdot \mathbf{N}\, dS \quad \Rightarrow \quad \iint_S \text{curl } \mathbf{H} \cdot \mathbf{N}\, dS = \iint_S \mathbf{J} \cdot \mathbf{N}\, dS$$

$$\iint_S (\text{curl } \mathbf{H} - \mathbf{J}) \cdot \mathbf{N}\, dS = 0 \quad \to \quad \text{curl } \mathbf{H} - \mathbf{J} = 0 \quad \to \quad \text{curl } \mathbf{H} = \mathbf{J}\,.$$

Example 13.30. Given $\mathbf{F}(x, y, z) = \langle z + \cos(x), x + y^2, y + e^z \rangle$, find $\oint_C \mathbf{F} \cdot d\mathbf{R}$, where C is the intersection of the sphere $x^2 + y^2 + z^2 = 4$, and the cone $z = \sqrt{x^2 + y^2}$.

Solution: For the Stoke's theorem, $\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_C \text{curl } \mathbf{F} \cdot \mathbf{N}\, dS$.

- First, we compute the curve C:

$$\begin{cases} x^2 + y^2 + z^2 = 4 \\ z = \sqrt{x^2 + y^2} \end{cases} \quad \to \quad \begin{aligned} x^2 + y^2 + (\sqrt{x^2 + y^2})^2 &= 4 \\ 2x^2 + 2y^2 &= 4 \\ x^2 + y^2 &= 2 \quad \text{Circle of radius } \sqrt{2} \end{aligned}$$

- Then, we evaluate curl $\mathbf{F}$:

$$\text{curl } \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z + \cos(x) & x + y^2 & y + e^z \end{vmatrix} = \hat{i}(1 - 0) - \hat{j}(0 - 1) + \hat{k}(1 - 0) = \langle 1, 1, 1 \rangle\,.$$

- We note that a normal to the circle of radius $\sqrt{2}$ is $\mathbf{N} = \langle 0, 0, 1 \rangle$.

Thus, the integral can be evaluated as

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint\limits_{\substack{S \\ \text{(Circle)}}} \langle 1, 1, 1 \rangle \cdot \langle 0, 0, 1 \rangle \, dS = \int_0^{2\pi} \int_0^{2\pi} (1 - r) \, dr \, d\theta = \int_0^{2\pi} \left[\frac{r^2}{2} \right]_0^{\sqrt{2}} d\theta = 2\pi \, .$$

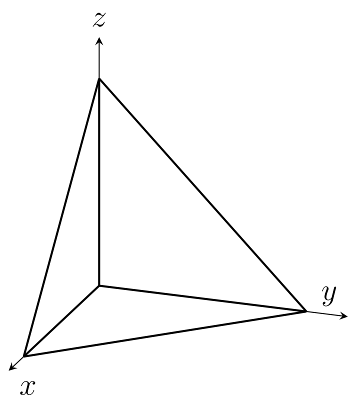
13.7 Divergence theorem and applications

Theorem 13.9 (Divergence theorem)**.** Let S be a smooth, orientable surface that encloses a solid region R in $\mathbb{R}^3$. If $\mathbf{F}$ is a continuous vector field whose components have continuous partial derivatives, in an open set containing R, then

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_R \operatorname{div}\,\mathbf{F}\,dV\,,$$

where $\mathbf{N}$ is the unit outward normal field for the surface S.

Example 13.31. Evaluate $\iint_S \mathbf{F}\cdot\mathbf{N}\,dS$, where $\mathbf{F} = x^2\,\hat{\imath} + xy\,\hat{\jmath} + x^3y^3\,\hat{k}$ and S is the surface of the tetrahedron bounded by the plane $x+y+z=1$ and the coordinate planes.



Solution: First, we evaluate div $\mathbf{F}$:

$$\operatorname{div}\,\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 2x + x + 0 = 3x\,.$$

Thus

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_R 3x\,dV\,,$$

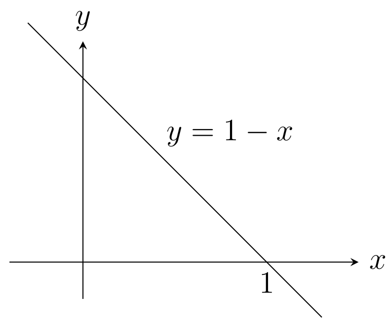
where R is the tetrahedron volume.

In z-direction:

- Upper bound: Plane $x+y+z=1 \quad \rightarrow \quad z=1-x-y$;
- Lower bound: xy-plane $\quad \rightarrow \quad z=0$.

$$\iiint\limits_V 3x\, dV = \iint\limits_A \int_0^{1-x-y} 3x\, dz\, dA\,,$$

where A is the projection of the tetrahedron on the xy-plane (triangle).



$$\begin{cases} z=1-x-y \\ z=0 \end{cases} \rightarrow \quad \begin{array}{l} 1-x-y=0 \\ y=1-x \end{array} \qquad \begin{array}{l} \text{Type I} \\ x\in[0,1]\,, \\ y\in[0,1-x]\,. \end{array}$$

$$\iint\limits_A \int_0^{1-x-y} 3x\, dz\, dA = \int_0^1\int_0^{1-x}\int_0^{1-x-y} 3x\, dz\, dy\, dx = \int_0^1 3x \int_0^{1-x} \left[z\right]_0^{1-x-y} dy\, dx =$$

$$= \int_0^1 3x \int_0^{1-x} (1-x-y)\, dy\, dx = \int_0^1 3x \left[y - xy - \frac{y^2}{2}\right]_0^{1-x} dx =$$

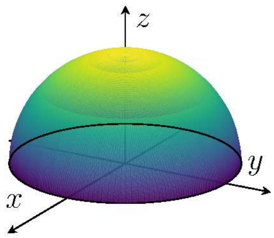
$$= \int_0^1 3x \left(1-x-x+x^2-\frac{(1-x)^2}{2}\right) dx =$$

$$= \int_0^1 3x \left(1-2x+x^2-\frac{1-2x+x^2}{2}\right) dx = 3\int_0^1 \left(\frac{1}{2}x - x^2 + \frac{1}{2}x^3\right) dx =$$

$$= \frac{3}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{3}{2}\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right) = \frac{3}{2}\frac{6-8+3}{12} = \frac{1}{8}\,.$$

Example 13.32. Let $\mathbf{F} = 2x\,\hat{\imath} - 3y\,\hat{\jmath} + 5z\,\hat{k}$, and let S be the hemisphere $z=\sqrt{9-x^2-y^2}$

together with the disk $x^2 + y^2 \leq 9$ in the xy-plane. Evaluate $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{N}\,dS$.



Solution: Using the divergence theorem, we can compute a triple integral over the considered volume (sphere of radius 3, $z \geq 0$).

$$\text{div}\,\mathbf{F} = \frac{\partial(2x)}{\partial x} + \frac{\partial(-3y)}{\partial y} + \frac{\partial(5z)}{\partial z} = 2 - 3 + 5 = 4\,.$$

Thus

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_R 4\,dV\,,$$

where R can be identified in spherical coordinates as

$$\rho \in [0,3]\,, \qquad \theta \in [0,2\pi]\,, \qquad \phi \in \left[0, \frac{\pi}{2}\right]\,.$$

$$\iiint_R 4\,dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^3 4\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi = \int_0^{\pi/2}\int_0^{2\pi}\sin(\phi)\,4\left[\frac{\rho^3}{3}\right]_0^3 d\theta\,d\phi =$$

$$= 36\int_0^{\pi/2}\sin(\phi)\int_0^{2\pi} d\theta\,d\phi = 72\,\pi\big[-\cos(\theta)\big]_0^{\pi/2} = 72\,\pi\,.$$

Example 13.33. Use the divergence theorem to evaluate

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS, \qquad \text{where } \mathbf{F} = \frac{x^3}{3}\,\hat{\imath} + z\,\hat{\jmath} + y^2 z\,\hat{k},$$

and S is the external surface of the solid bounded above by $z = 1 - x^2 - y^2$ and below by $z = 0$.

Solution. Divergence theorem:

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV.$$

Compute the divergence:

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^3/3) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(y^2 z) = x^2 + 0 + y^2 = x^2 + y^2.$$

Region V: $z \in [0, 1 - x^2 - y^2]$, with bounds in the xy plane given by the intersection:

$$\begin{cases} z = 1 - x^2 - y^2 \\ z = 0 \end{cases} \quad \Rightarrow\; x^2 + y^2 = 1 \text{ (circle of radius } = 1)$$

Use cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$: In cylindrical, $x^2 + y^2 = r^2$ and $dV = r\, dz\, dr\, d\theta$. Hence

$$\iiint_V (x^2 + y^2)\, dV = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} r^2\, r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^1 r^3(1 - r^2)\, dr\, d\theta.$$

Compute:

$$\int_0^{2\pi} d\theta = 2\pi, \qquad \int_0^1 r^3(1 - r^2)\, dr = \int_0^1 (r^3 - r^5)\, dr = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

Therefore

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_V (x^2 + y^2)\, dV = 2\pi \cdot \frac{1}{12} = \frac{\pi}{6}.$$