CHAPTER 12

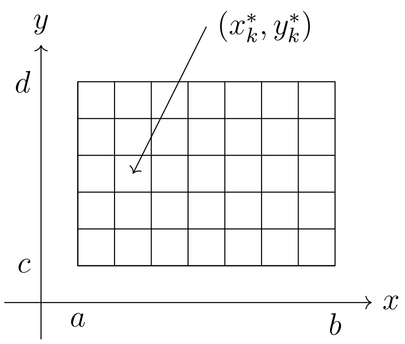
Multiple Integration

Contents

In this chapter, we will move from the single integrals $\int_a^b f(x)\,dx$, to multiple integrals where the integrand is a function of many variables.

12.1 Double integration over rectangular regions

Definition of double integral Recall the definition for single integrals: $\int_a^b f(x)\,dx$ is the limit of the Riemann sum $\sum_{k=1}^{n} f(x_k)\cdot\Delta x_k$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. Now we consider a rectangle:



Step 1: Partition the $[a, b]$ interval into m subintervals and $[c, d]$ into n subintervals $\Rightarrow\ n\cdot m$ cells.

Step 2: Choose a representative point for each cell (x_k^*, y_k^*) such that we have the sum $\sum_{k=1}^{N=n\cdot m} f(x_k^*, y_k^*)\cdot\Delta A_k$, where ΔA_k is the area of the cell.

Step 3: We define $||P||$ as the length of the diagonal of the cell. We can define $\lim_{||P||\to 0}\sum_{k=1}^{N} f(x_k^*, y_k^*)\cdot\Delta A_k$ (double integral).

Definition 12.1. If f is defined on a close, bounded rectangular region R in the xy-plane, then the double integral of f over R is

$$\iint\limits_R f(x,y)\,dA = \lim_{||P||\to 0}\sum_{k=1}^{N} f(x_k^*, y_k^*)\,\Delta A_k\,.$$

Remark: If $f(x, y)$ is continuous on a rectangle R then is integrable over R.

Properties of the double integrals

- **Linearity rule:** For constants a and b

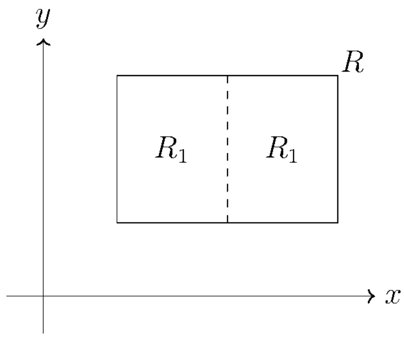
$$\iint\limits_R [a\,f(x,y) + b\,g(x,y)]\ dA = a\iint\limits_R f(x,y)\,dA + b\iint\limits_R g(x,y)\,dA\,.$$

- **Dominance rule:** If $f(x, y) \geq g(x, y)$ over R, then

$$\iint\limits_R f(x,y)\,dA \geq \iint\limits_R g(x,y)\,dA\,.$$

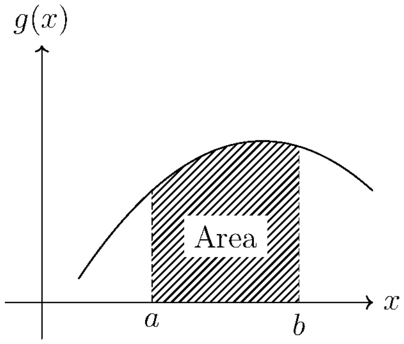
- **Subdivision rule:** If we divide R into two non overlapping rectangles R_1 and R_2, then

$$\iint\limits_R f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA\,.$$



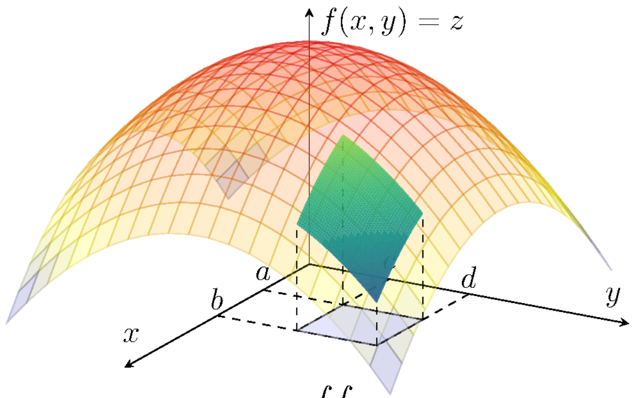
Volume interpretation

2D



$$\text{Area} = \int_a^b g(x)\,dx$$

3D



$$\text{Volume} = \iint\limits_R f(x,y)\,dA$$

For $f(x,y) \geq 0 \quad \forall\,(x,y) \in \mathbb{R}$, the double integral over R is equal to the volume between $f(x,y)$ and the xy- plane.

Iterated integration The iterated integration can be interpreted as a reversed partial differentiation.

Theorem 12.1 (Fubini's theorem over a rectangular region)**.** If $f(x,y)$ is continuous over the rectangle R: $a \leq x \leq b$, $c \leq y \leq d$, then the double integral

$$\iint\limits_R f(x,y)\,dA$$

may be evaluated by either iterated integrals, that is

$$\iint\limits_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx\,.$$

Note: If we are able to separate the variables as $f(x,y) = g(x) \cdot h(y)$ (not for $f(x,y) = g(x) + h(y)$), then

$$\iint\limits_R f(x,y)\,dA = \int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy\,.$$

Note: to apply this theorem, one should first compute the integral inside the inner brackets, considering the variable that is not integrated as a constant.

Example 12.1. Given the following integral

$$\int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

when integrate in dx we consider y as a constant. Then, the result should be integrated in dy.

Remark: Fubini's theorem also works for bounded functions that are not continuous on a subset S of R of area $= 0$.

Example 12.2. Compute

$$\iint\limits_R (2-y)\,dA\,,$$

where R is the rectangle with vertices $(0,0), (3,0), (3,2), (0,2)$.

Solution: The region is the rectangle $0 \le x \le 3$, $0 \le y \le 2$, so the integral is

$$\int_0^3 \left[\int_0^2 (2-y)\,dy\right] dx = \int_0^3 \left[2y - \frac{y^2}{2}\right]_0^2 dx = \int_0^3 \left[4 - \frac{4}{2} - 0\right] dx = \int_0^3 2\,dx = \Big[2x\Big]_0^3 = 6\,.$$

Example 12.3. Evaluate

$$\iint\limits_R x^2 y^5\,dA$$

over $R : 1 \le x \le 2$, $0 \le y \le 1$ with a) y-integration first b) x-integration first.

Solution:
a)

$$\int_1^2 \left[\int_0^1 x^2 y^5\,dy\right] dx = \int_1^2 \left[x^2 \cdot \frac{y^6}{6}\right]_0^1 dx = \int_1^2 \left[x^2 \cdot \frac{1}{6} - 0\right] dx = \left[\frac{1}{6} \cdot \frac{x^3}{3}\right]_1^2 = \left[\frac{x^3}{18}\right]_1^2 = \frac{7}{18}\,.$$

b)

$$\int_0^1 \left[\int_1^2 x^2 y^5\,dx\right] dy = \int_0^1 \left[\frac{x^3}{3} y^5\right]_1^2 dy = \int_0^1 \left[\frac{8}{3} y^5 - \frac{1}{3} y^5\right] dy = \int_0^1 \frac{7}{3} y^5\,dy = \left[\frac{7}{3}\frac{y^6}{6}\right]_0^1 = \frac{7}{18}\,.$$

Example 12.4 (Choosing the order of integration)**.** Evaluate

$$\iint\limits_R x\cos(xy)\,dA$$

for $R : 0 \le x \le \frac{\pi}{2}, 0 \le y \le 1$.

Solution: If I integrate with respect to x first, I would need to integrate by parts. Thus, we start with y-integration.

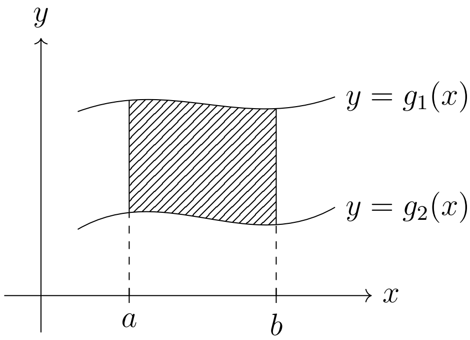
$$\int_0^{\frac{\pi}{2}} \left[\int_0^1 x\cos(xy)\,dy\right] dx = \int_0^{\frac{\pi}{2}} \left[x\,\frac{\sin(xy)}{x}\right]_0^1 dx = \int_0^{\frac{\pi}{2}} \Big[\sin(x) - \sin(0)\Big] dx = \Big[-\cos(x)\Big]_0^{\frac{\pi}{2}} = 1\,.$$

12.2 Double integration over nonrectangular regions

Double integrals over type I and type II regions. We start defining the type I and type II regions:

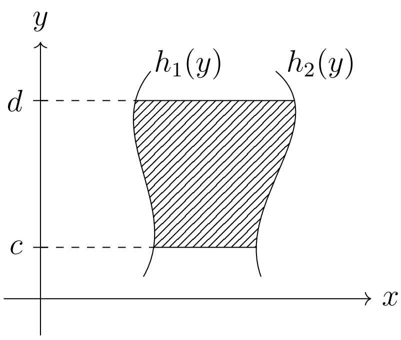
- **Type I**: $D_1: \quad a \leq x \leq b, \qquad g_1(x) \leq y \leq g_2(x).$

  This region is the set of all points (x, y) such that for each fixed x between $x = a$ and $x = b$ the vertical line segment $g_1(x) \leq y \leq g_2(x)$ lies in the region.



- **Type II:** $D_1: \quad c \leq y \leq d, \qquad h_1(y) \leq x \leq h_2(y).$

  This region is the set of all points (x, y) such that for each fixed y between $y = c$ and $y = d$ the horizontal line segment $h_1(y) \leq x \leq h_2(y)$ lies in the region.



Theorem 12.2 (Fubini's theorem for non rectangular regions)**.** If D_1 is a type I region, then

$$\iint_{D_1} f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx\,,$$

whenever both integrals exist.
Similarly, if D_2 is a type II region, then

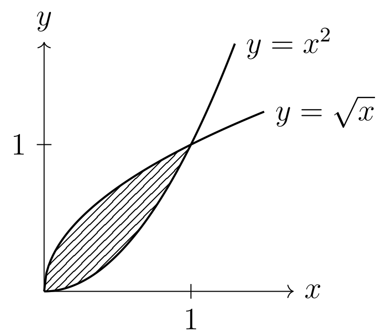
$$\iint\limits_{D_2} f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy\,.$$

Example 12.5. Evaluate

$$\int_0^1 \int_{x^2}^{\sqrt{x}} 160\,x\,y^3\,dy\,dx\,,$$

in the region between the functions $y=\sqrt{x}$ and $y=x^2$, with $0\le x\le 1$.

Solution:



$$\int_0^1 \int_{x^2}^{\sqrt{x}} 160\,x\,y^3\,dy\,dx = \int_0^1 \left[160\,x\cdot\frac{y^4}{4}\right]_{x^2}^{\sqrt{x}} dx = \int_0^1 160\,x\left(\frac{(\sqrt{x})^4}{4}-\frac{(x^2)^4}{4}\right)dx =$$

$$= \int_0^1 160\,x\left(\frac{x^2}{4}-\frac{x^8}{4}\right)dx = \int_0^1 \frac{160}{4}\left(x^3-x^9\right)dx =$$

$$= 40\left[\frac{x^4}{4}-\frac{x^{10}}{10}\right]_0^1 = 40\left(\frac{1}{4}-\frac{1}{10}\right) = 40\left(\frac{5-2}{20}\right) = 6\,.$$

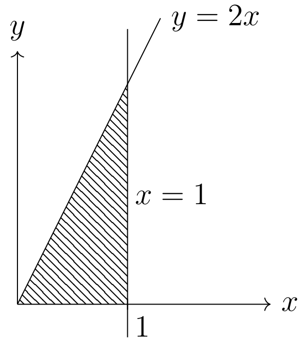
Example 12.6. Let T be the triangular region enclosed by the lines $y=0$, $y=2x$ and

$x = 1$. Evaluate the double integral

$$\iint_T (x+y)\, dA\,,$$

using an iterate integral with: a) y-integration first, b) x-integration first.

Solution:



a) y-integration first: extremes are $y = 0$ and $y = 2x$.

$$\iint_T (x+y)\, dA = \int_0^1 \int_0^{2x} (x+y)\, dy\, dx = \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^{2x} dx =$$

$$= \int_0^1 \left(2x^2 + \frac{4x^2}{2}\right) dx = \int_0^1 4x^2\, dx = \left[\frac{4x^3}{3}\right]_0^1 = \frac{4}{3}\,.$$

b) x-integration first. If $y = 2x \to x = \dfrac{y}{2}$. The lower extreme for x- variable is $x = \dfrac{y}{2}$, the upper extreme is $x = 1$. The y variable goes from to 0 to 2.

$$\iint_T (x+y)\, dA = \int_0^2 \int_{\frac{y}{2}}^1 (x+y)\, dx\, dy = \int_0^2 \left[\frac{x^2}{2} + xy\right]_{\frac{y}{2}}^1 dy = \int_0^2 \left(\frac{1}{2} + y - \frac{y^2}{8} - \frac{y^2}{2}\right) dy =$$

$$= \int_0^2 \left(\frac{1}{2} + y - \frac{y^2 + 4y^2}{8}\right) dy = \int_0^2 \left(\frac{1}{2} + y - \frac{5}{8}y^2\right) dy =$$

$$= \left[\frac{1}{2}y + \frac{y^2}{2} - \frac{5y^3}{24}\right]_0^2 = 1 + 2 - \frac{5 \cdot 8}{24} = 3 - \frac{5}{3} = \frac{9-5}{3} = \frac{4}{3}\,.$$

Double integral as area and volume

- The **area** of a region D can be computed as

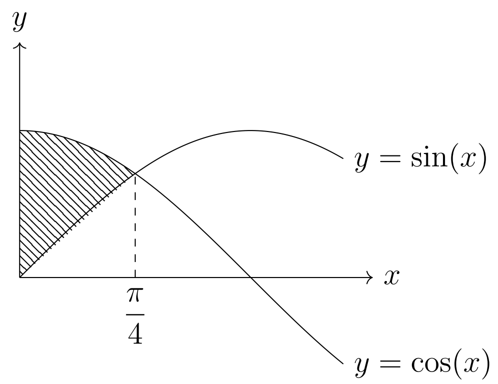
$$A = \iint_D dA\,.$$

- The **volume** of a solid under the surface $z = f(x, y)$ above the region D can be evaluated, if $f(x, y) \geq 0$ on D, as

$$V = \iint_D f(x, y)\, dA\,.$$

Example 12.7. Find the area of the region D between $y = \cos(x)$ and $y = \sin(x)$ over the interval $0 \leq x \leq \dfrac{\pi}{4}$. Do it a) with a single integral, b) with a double integral.

Solution:



a)

$$\int_0^{\frac{\pi}{4}} [\cos(x) - \sin(x)]\, dx = [\sin(x) + \cos(x)]_0^{\frac{\pi}{4}} = \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) - \sin(0) - \cos(0) =$$

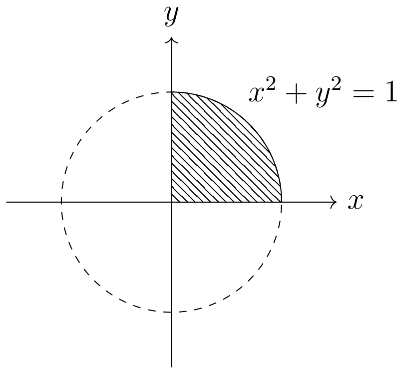
$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 = \sqrt{2} - 1\,.$$

b)

$$\iint\limits_D 1\,dx\,dy = \int_0^{\frac{\pi}{4}} \int_{\sin(x)}^{\cos(x)} 1\,dy\,dx = \int_0^{\frac{\pi}{4}} \Big[y\Big]_{\sin(x)}^{\cos(x)} dx = \int_0^{\frac{\pi}{4}} [\cos(x) - \sin(x)]dx = \sqrt{2} - 1\,.$$

Example 12.8. Find the volume of a solid bounded above by the plane $z = y$, and below in the xy-plane by the part of the disk $x^2 + y^2 \leq 1$ in the first quadrant.

Solution:



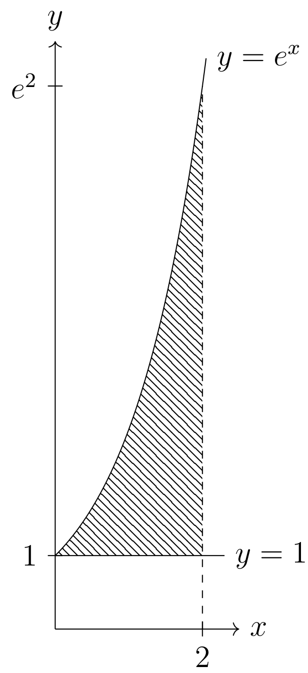
$$V = \iint\limits_D f(x,y)\,dx\,dy = \int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx = \int_0^1 \frac{1-x^2}{2}\,dx =$$
$$= \left[\frac{1}{2}x - \frac{x^3}{6}\right]_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{3-1}{6} = \frac{1}{3}\,.$$

Choosing the order of integration in a double integral

Example 12.9 (How to reverse the order of integration in a double integral)**.** Reverse the order of integration in the iterated integral

$$\int_0^2 \int_1^{e^x} f(x,y)\,dy\,dx\,.$$

Solution:

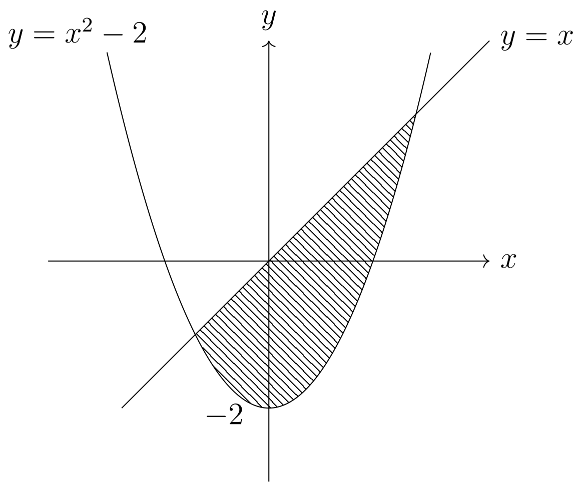


We first draw the domain of the integration $\Rightarrow \quad x \in [0, 2]$, $y \in [1, e^x]$.
Revers order: for fixed y-value, the x-value goes from $y = e^x$ (bottom line) and $x = 2$ (top line). Indeed, $y = e^x \quad \rightarrow \quad x = \ln(y)$.
Considering that now y varies from 1 to e^2, the new integral (reversed) is:

$$\int_1^{e^2} \int_{\ln(y)}^2 f(x, y)\, dx\, dy\,.$$

Example 12.10. The region D is bounded by the parabola $y = x^2 - 2$ and the line $y = x$ is vertically and horizontally simple. To find the area of D, would you prefer to use type I or type II?

Solution:



- **Type I**: We find the intersections :

$$\begin{cases} y = x \\ y = x^2 - 2 \end{cases} \quad \rightarrow \quad \begin{cases} y = x \\ x = x^2 - 2 \end{cases}$$

$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$

$$1) \begin{cases} x = 2 \\ y = 2 \end{cases} \qquad 2) \begin{cases} x = -1 \\ y = -1 \end{cases}$$

For a fixed $x \Rightarrow y$ is bounded by $y = x^2 - 2$ (lower) and $y = x$ (higher).

- **Type II**: it is not straightforward to compute: for a fixed $y \rightarrow x$ is bounded by $y = x^2 - 2 \Rightarrow x = \sqrt{y+2}$ (higher bound) and $x = y$ (lower bound). For $y < -1$, x is bounded by $x = -\sqrt{y+2}$ and $x = \sqrt{y+2} \quad \Rightarrow$ in this case is better to use Type 1.

$$A = \int_{-1}^{2}\int_{x^2-2}^{x} 1\,dy\,dx = \int_{-1}^{2}\Big[y\Big]_{x^2-2}^{x}dx = \int_{-1}^{2}\big(x - x^2 + 2\big)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_{-1}^{2} =$$
$$= \frac{4}{2} - \frac{8}{3} + 4 - \frac{1}{2} - \frac{1}{3} + 2 = \frac{12 - 16 + 36 - 3 - 2}{6} = \frac{27}{6} = \frac{9}{2}\,.$$

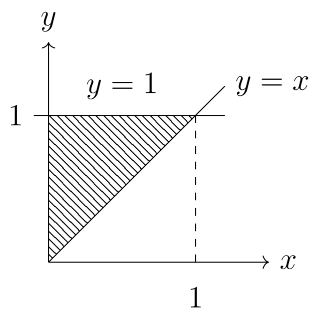
NB. Using Type II we have:

$$A = \int_{-2}^{-1} \int_{-\sqrt{y+2}}^{\sqrt{y+2}} dx\, dy + \int_{-1}^{2} \int_{y}^{\sqrt{y+2}} dx\, dy\,.$$

Example 12.11 (Evaluating an integral by reversing the order)**.** Evaluate

$$\int_0^1 \int_x^1 e^{y^2} dy\, dx\,.$$

Solution:



Note: Sometimes it is not possible to solve an integral (we do not know the antiderivative of e^{y^2}) $\Rightarrow$ we can try to reverse the order of the integral.
We first sketch the domain considering $x \in [0,1]$ and $y \in [x,1]$.
Reversing for a fixed y ($y \in [0,1]$) we have $x = y$ (upper bound) and $x = 0$ (lower bound).
Thus, we have

$$\int_0^1 \int_x^1 e^{y^2} dy\, dx = \int_0^1 \int_0^y e^{y^2} dx\, dy = \int_0^1 \left[e^{y^2} x \right]_0^y dy = \int_0^1 y e^{y^2} dy\,.$$

Now, consider the substitution: $u = y^2$ that leads to $y = \sqrt{u}$ and $dy = \dfrac{1}{2\sqrt{u}} du$.

$$\Rightarrow \int_0^1 \sqrt{u}\, e^u \frac{1}{2\sqrt{u}}\, du = \int_0^1 \frac{e^u}{2}\, du = \left[\frac{e^u}{2} \right]_0^1 = \frac{e}{2} - \frac{1}{2}\,.$$

12.3 Double integral in polar coordinates

Change of variables in polar form Using the polar coordinates for integration can be useful when the integrated or the region of integration (or both) has a simple polar description.
Polar description:

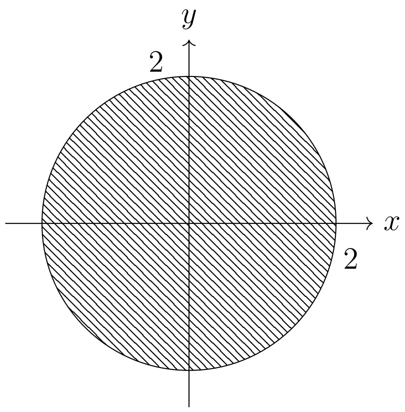
$$\begin{cases} x = r\,\cos(\theta) \\ y = r\,\sin(\theta) \end{cases} \quad \Rightarrow \quad \begin{aligned} r &= \sqrt{x^2+y^2} \\ \tan\theta &= \frac{y}{x}\,. \end{aligned}$$

Example 12.12. Consider the integral

$$\iint\limits_R \left(x^2+y^2+1\right)\,dA\,,$$

where R is the region (disk) in the xy-plane bounded by the circle $x^2+y^2=4$.

Solution:



We have a bound for x: $x \in [-2,2]$. If we fix $x \Rightarrow y$ is between $-\sqrt{4-x^2}$ (lower bound) and $\sqrt{4-x^2}$ (upper bound). Then, we obtain

$$\iint\limits_R \left(x^2+y^2+1\right)\,dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(x^2+y^2+1\right)\,dy\,dx \quad \rightarrow \quad \text{Not easy to solve!}$$

To apply a change of variable, we need to plug

$$x = r\ \cos(\theta)$$
$$y = r\ \sin(\theta)\,.$$

Extreme of integration:

$$r \quad \text{from } \ 0 \ \text{ to } 2 \ (= \text{radius})$$
$$\theta \quad \text{from } \ 0 \ \text{ to } 2\pi \ (\text{circle})$$

$$f\left(r\cos(\theta), r\sin(\theta)\right) = r^2\cos^2(\theta) + r^2\sin^2(\theta) + 1 = r^2 + 1\,.$$

$\rightarrow$ now we need to introduce a theorem to transform $dx\,dy$ into $dr\,d\theta$.

Theorem 12.3 (Double integral in polar coordinates)**.** If f is continuous in the polar region D described by $0 \leq r_1(\theta) \leq r \leq r_2(\theta)$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint\limits_D f(r,\theta)\,dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\,r\,dr\,d\theta\,.$$

Note:

- We have the presence of r in the second integral.
- From this theorem, we get the transformation from a cartesian integral to a polar one

$$\iint\limits_R f(x,y)\,dA = \iint_D f(r\ \cos(\theta), r\ \sin(\theta))\,r\,dr\,d\theta\,.$$

Preview of section 12.8: In general, the change of variable $x = x(u,v)$, $y = y(u,v)$ transforms the integral $\iint f(x,y)\,dA$ into $\iint f(u,v)\ |J(u,v)|\ du\,dv$, where

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \qquad \rightarrow \qquad \begin{array}{c} |J(u,v)| \\ \text{Jacobian of the transformation.} \end{array}$$

In the polar coordinates case, we have (with $x = r\cos(\theta)$, $y = r\sin(\theta)$)

$$J(r,\theta) = \begin{vmatrix} \frac{\partial}{\partial r}(r\cos(\theta)) & \frac{\partial}{\partial \theta}(r\cos(\theta)) \\ \frac{\partial}{\partial r}(r\sin(\theta)) & \frac{\partial}{\partial \theta}(r\sin(\theta)) \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r\,.$$

This confirms the theorem statement:

$$\iint_R f(x,y)\,dA = \iint_D f(r,\theta)\,r\,dr\,d\theta = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f\left(r\cos(\theta), r\sin(\theta)\right)\,r\,dr\,d\theta\,.$$

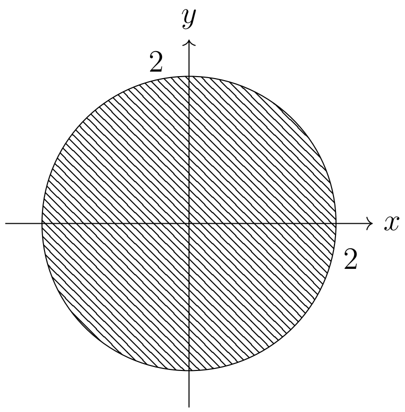
Area and volume in polar form

Example 12.13 (Double integral in polar form)**.** Evaluate

$$\iint_D \left(x^2 + y^2 + 1\right)\,dA\,,$$

where D is the region inside the circle $x^2 + y^2 = 4$.

Solution:

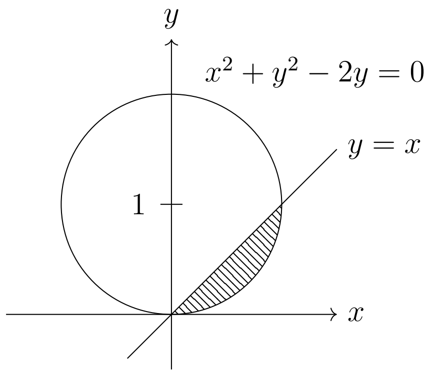


The given circle, in polar coordinates, is represented by $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$ (circle of radius 2). Thus:

$$\iint_D \left(x^2+y^2+1\right)\, dA = \int_0^{2\pi}\int_0^2 \left[(r\cos(\theta))^2+(r\sin(\theta))^2+1\right] r\,dr\,d\theta =$$
$$= \int_0^{2\pi}\int_0^2 \left[r^2\cos^2(\theta)+r^2\sin^2(\theta)+1\right] r\,dr\,d\theta =$$
$$= \int_0^{2\pi}\int_0^2 \left(r^3+r\right) dr\,d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}+\frac{r^2}{2}\right]_0^2 d\theta =$$
$$= \int_0^{2\pi}\left[\frac{16}{4}+\frac{4}{2}\right] d\theta = \int_0^{12}(4+2)\,d\theta = \Big[6\,\theta\Big]_0^{2\pi} = 12\pi\,.$$

Example 12.14 (Computing area in polar form using a double integral)**.** Compute the area of the region D bounded above by the function $y=x$ and below by the circle $x^2+y^2-2y=0$.

Solution:



$$x^2+y^2-2y+1-1=0 \qquad \rightarrow \qquad x^2+(y-1)^2=1$$

Therefore, we have a circle with center in $(0,1)$ and radius equal to 1.
Transform in polar coordinates: $y=x$ is represented by $\theta=\dfrac{\pi}{4}$. The circle is:

$$x^2+y^2-2y=0 \quad \rightarrow \quad (r\cos(\theta))^2+(r\sin(\theta))^2-2\,r\sin(\theta)=0$$

$$r^2-2\,r\,\sin(\theta)=0\,, \qquad r(r-2\sin(\theta))=0\,.$$

$$r-2\sin(\theta) \quad \rightarrow \quad r=2\sin(\theta)\,.$$

Thus, the angle θ varies from 0 to $\frac{\pi}{4}$, the radius from 0 to $2\sin(\theta)$

$$A = \iint_D dA = \int_0^{\frac{\pi}{4}} \int_0^{2\sin(\theta)} r\,dr\,d\theta = \int_0^{\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_0^{2\sin(\theta)} d\theta = \int_0^{\frac{\pi}{4}} \left(\frac{4\sin^2(\theta)}{2} - 0\right) d\theta =$$

$$= \int_0^{\frac{\pi}{4}} 2\sin^2(\theta) d\theta = \dots \text{ Recalling that } \sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \dots =$$

$$= \int_0^{\frac{\pi}{4}} 2\frac{1-\cos(2\theta)}{2}\,d\theta = \left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4} - \frac{\sin\left(\frac{\pi}{2}\right)}{2} - 0 + \frac{\sin(0)}{2} = \frac{\pi}{4} - \frac{1}{2}.$$

Example 12.15 (Volume in polar form)**.** Show that a sphere of radius a has volume $\frac{4}{3}\pi a^3$.

Solution:
Consider the sphere domain: $x^2 + y^2 + z^2 \le a^2$ (the inner domain). If $z \ge 0$ (half sphere), we can write $z = \sqrt{a^2 - x^2 - y^2}$. As a domain, we consider the circle $x^2 + y^2 = a^2$.

$$z = \sqrt{a^2 - x^2 - y^2} \xrightarrow{\text{Polar form}} z = \sqrt{a^2 - r^2} \qquad (r^2 = x^2 + y^2)$$

$$\text{Disk} \quad x^2 + y^2 = a^2 \xrightarrow{\text{Polar form}} r \in [0, a], \quad \theta \in [0, 2\pi]$$

$$V^* = \frac{V}{2} = \iint_{\text{Disk}} z\,dA = \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2}\, r\,dr\,d\theta = \dots$$

Consider now: $u = a^2 - r^2, \quad du = -2\,r\,dr$

$$\dots = \int_0^{2\pi} \int_0^a -\frac{1}{2}\sqrt{a^2 - r^2}\,(-2r)\,dr\,d\theta = \int_0^{2\pi} -\frac{1}{2}\left[\frac{(a^2 - r^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a d\theta =$$

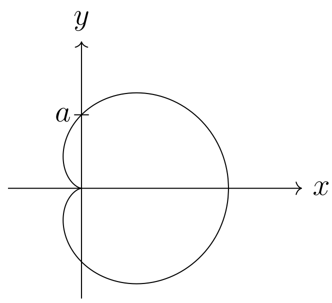
$$= \int_0^{2\pi} -\frac{1}{2}\left[-\frac{a^3}{\frac{3}{2}}\right] d\theta = \frac{1}{3}a^3 \int_0^{2\pi} d\theta = \frac{2\pi}{3}a^3$$

To have the complete volume: $V = 2V^* = \frac{4\pi}{3}a^3$.

Example 12.16 (Region of integration between two polar curves)**.** Evaluate $\iint_D \frac{1}{x}\,dA$, where D is the region that lies inside the circle $r = 3\cos(\theta)$ and outside the cardioid

$r = 1 + \cos(\theta)$.

Solution:



Sketch of the generic cardioid $r = a + a\cos(\theta)$.

[*See solution on page 946 of the textbook (not covered in class).*]

Example 12.17 (Converting an integral to polar form)**.** Evaluate

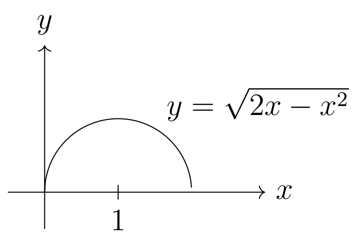
$$\int_0^2 \int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\, dy\, dx$$

by converting it in polar coordinates.

Solution: The integration region is $x \in [0, 2]$, $y \in [0, \sqrt{2x - x^2}]$. Note that:

$$y = \sqrt{2x - x^2}$$
$$y^2 = 2x - x^2 \quad \rightarrow \quad x^2 + y^2 - 2x = 0$$
$$x^2 - 2x + 1 + y^2 - 1 = 0$$
$$(x-1)^2 + y^2 = 1 \qquad \text{(Circle of radius = 1 and center in (1,0))}$$

Thus, $y = \sqrt{2x - x^2}$ is the semicircle.



Let's put the semicircle in polar form:

$$\begin{aligned}
&y = \sqrt{2x - x^2}\\
&y^2 = 2x - x^2\\
&x^2 + y^2 = 2x\\
&(r\sin(\theta))^2 + (r\cos(\theta))^2 = 2\,r\cos(\theta)\\
&r^2 = 2\,r\cos(\theta) \quad \rightarrow \quad r = 2\cos(\theta)\,, \quad r = 0\,.
\end{aligned}$$

The angle θ in this semicircle varies from 0 to $\dfrac{\pi}{2}$

$$\begin{aligned}
\int_0^2 \int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\,dy\,dx &= \iint\limits_D y\sqrt{x^2+y^2}\,dA = \int_0^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} r\ \sin(\theta)\sqrt{r^2}\,r\,dr\,d\theta =\\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} r^3 \sin(\theta)\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^{2\cos(\theta)} \sin(\theta)\,d\theta =\\
&= \int_0^{\frac{\pi}{2}} \frac{16\cos^4(\theta)}{4} \sin(\theta)\,d\theta = 4\left[-\frac{\cos^5(\theta)}{5}\right]_0^{\frac{\pi}{2}} = \frac{4}{5}.
\end{aligned}$$

12.4 Surface area

For two-dimensional computations, the length L of the portion of the graph $y = f(x)$ between $x = a$ and $x = b$ is

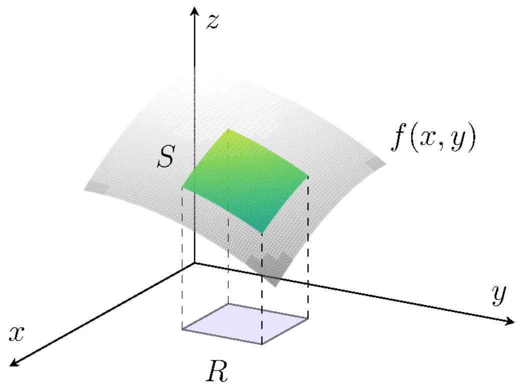
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx\,.$$

We now want to extend this definition to surfaces.

Definition of surface area Assume that the function $f(x, y)$ has continuous partial derivatives f_x and f_y in a region R of the xy-plane. Then, the portion of the surface $z = f(x, y)$ that lies over R has surface area

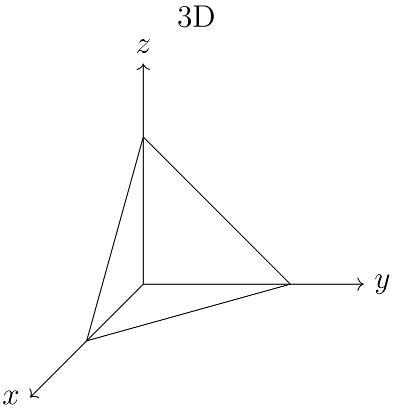
$$S = \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA\,.$$

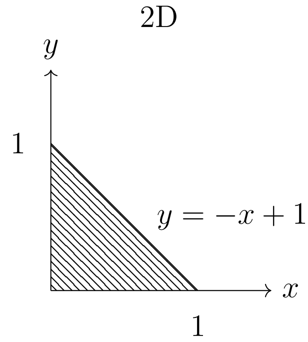
Note that $dS = \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$ and this is called surface area element. The region R can also be seen as the projection ("shadow") of S on the xy-plane.



Example 12.18. Find the area of the portion of the plane $x + y + z = 1$ that lies in the first octant (where $x \geq 0$, $y \geq 0$, $z \geq 0$).

Solution:





Consider the plane $x + y + z = 1$, with its normal vector $\langle 1, 1, 1\rangle$ and the point $(0, 0, 1)$. For $z = 0$ we have $x + y = 1 \quad \Rightarrow \quad y = -x + 1$. We recall that

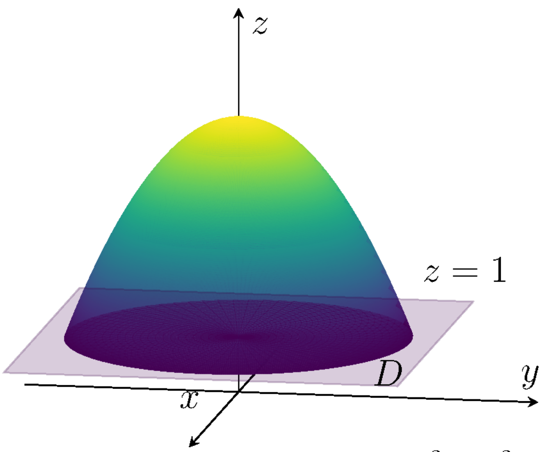
$$S = \iint \sqrt{f_x^2 + f_y^2 + 1}\, dA\,,$$

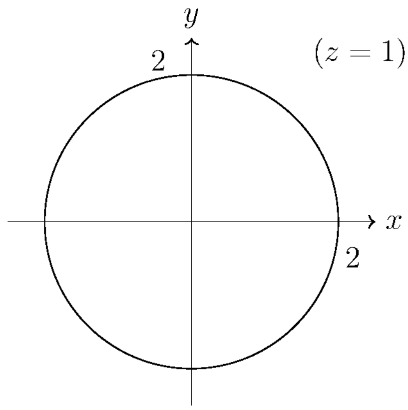
and given the function $f(x, y) = z = 1 - x - y$ we have $f_x = -1$ and $f_y = -1$. Therefore,

$$S = \iint \sqrt{1+1+1}\, dA = \int_0^1 \int_0^{-x+1} \sqrt{3}\, dy\, dx = \int_0^1 \left[\sqrt{3x}\right]_0^{-x+1} dx =$$
$$= \int_0^1 \sqrt{3}\,(-x+1)\, dx = \sqrt{3}\left[-\frac{x^2}{2} + x\right]_0^1 = \sqrt{3}\left(-\frac{1}{2} + 1\right) = \frac{\sqrt{3}}{2}\,.$$

Example 12.19 (Surface area by changing to polar coordinates.)**.** Find the surface area of that part of the paraboloid $x^2 + y^2 + z = 5$ that lies above the plane $z = 1$.

Solution:





From the paraboloid $z = 5 - x^2 - y^2 = f(x, y)$ we obtain $f_x = -2x$ and $f_y = -2y$. Therefore,

$$S = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA = \iint_D \sqrt{4x^2 + 4y^2 + 1}\, dA\,.$$

Now we consider the domain of the integration

$$\begin{cases} z = 1 \\ z = 5 - x^2 - y^2 \end{cases} \qquad \begin{cases} z = 1 \\ x^2 + y^2 = 4 \end{cases} \quad \to \quad \text{circle}$$

Having a circular domain, we move to polar coordinates:

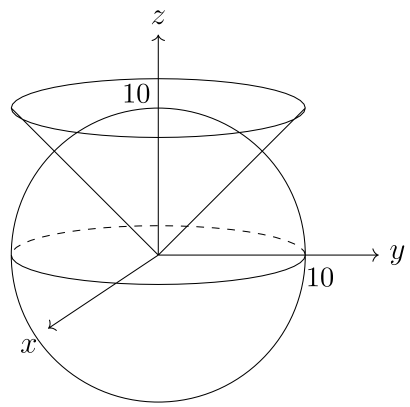
$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad \theta \in [0, 2\pi], \quad r \in [0, 2]\,.$$

$$S = \int_0^{2\pi}\int_0^2 \sqrt{4r^2\cos^2(\theta) + 4r^2\sin^2(\theta) + 1}\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 \sqrt{4r^2+1}\, r\, dr\, d\theta = \dots$$

Consider: $u = 4r^2 + 1 \to du = 8r\, dr \quad \Rightarrow \quad \text{if } r = 0 \to u = 1, \;\; \text{if } r = 2 \to u = 17$

$$\dots = \int_0^{2\pi}\int_1^{17} \sqrt{u}\,\frac{du}{8}\, d\theta = \int_0^{2\pi} \frac{1}{8}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^{17} d\theta = 2\pi\,\frac{1}{8}\,\frac{2}{3}\left(17^{\frac{3}{2}} - 1\right) = \frac{\pi}{6}\left(17^{\frac{3}{2}} - 1\right)\,.$$

Example 12.20 (11 HW 12.3-12.5)**.** Find the surface area of the sphere $x^2+y^2+z^2 = 100$ that lies above the cone $z = \sqrt{x^2 + y^2}$.



Solution: Consider the function $z = \sqrt{100 - x^2 - y^2} = f(x, y)$ (from the sphere), and the intersection between the curves (domain of the integration).

$$\begin{aligned} z = \sqrt{x^2+y^2} \quad &\rightarrow \quad x^2 + y^2 + (\sqrt{x^2+y^2})^2 = 100 \\ 2x^2 + 2y^2 = 100 \quad &\rightarrow \quad x^2 + y^2 = 50 \quad \text{(circle of radius } \sqrt{50}) \end{aligned}$$

Recalling that $\displaystyle\iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA$, we have

$$\begin{aligned} f_x &= \frac{-2x}{2\sqrt{100 - x^2 - y^2}} = -\frac{x}{\sqrt{100 - x^2 - y^2}} \\ f_y &= \frac{-2y}{2\sqrt{100 - x^2 - y^2}} = -\frac{y}{\sqrt{100 - x^2 - y^2}} \,. \end{aligned}$$

Therefore,

$$\iint_D \sqrt{\frac{x^2 + y^2}{100 - x^2 - y^2} + 1}\, dA \,.$$

Switching to polar coordinates with $r \in [0, \sqrt{50}]$, $\theta \in [0, 2\pi]$ we have

$$\iint_D \sqrt{\frac{x^2 + y^2 + (100 - x^2 - y^2)}{100 - x^2 - y^2}}\, dA = \iint_D \sqrt{\frac{100}{100 - x^2 - y^2}} = \int_0^{2\pi} \int_0^{\sqrt{50}} \frac{10r}{\sqrt{100 - r^2}}\, dr\, d\theta \,.$$

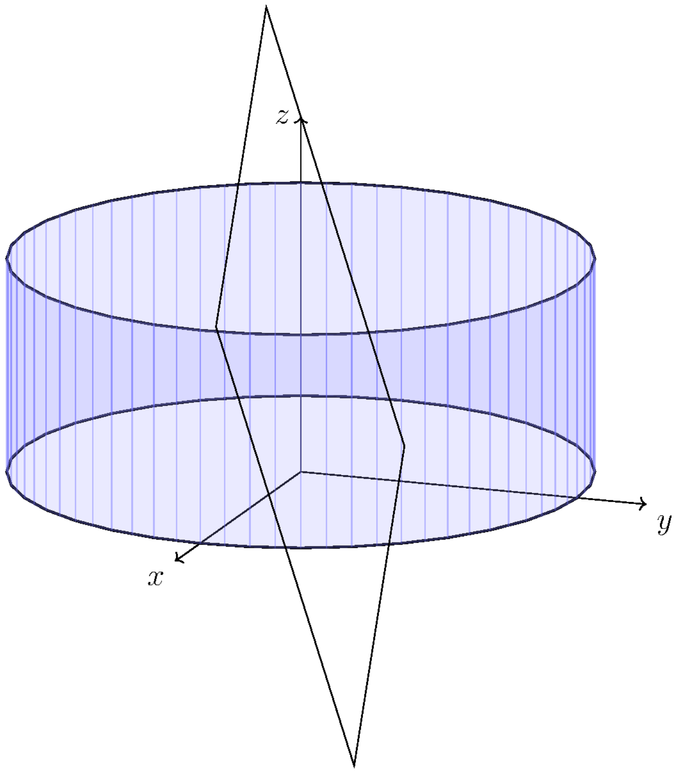
Now considering $100 - r^2 = u^2 \to -2r\,dr = 2u\,du$ we have that

$$r = \sqrt{50} \quad \to \quad u = \sqrt{50}$$
$$r = 0 \quad \to \quad u = 10\,.$$

Finally we have

$$\int_0^{2\pi}\int_{10}^{\sqrt{50}} \frac{10r}{u}\left(-\frac{2u}{2r}\right)du = \int_0^{2\pi}\int_{10}^{\sqrt{50}} -10\,du\,d\theta = \int_0^{2\pi} [-10\,u]_{10}^{\sqrt{50}}\,d\theta = -20\,\pi\left(\sqrt{50}-10\right)\,.$$

Example 12.21 (12 HW 12.3-12.5)**.** Find the surface area of the part of the plane $2x + 3y + z = 5$ that lies inside the cylinder $x^2 + y^2 = 16$.

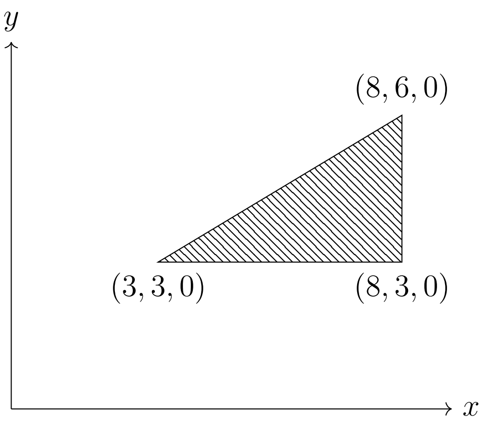


Solution: Given the plane, the normal is $\langle 2, 3, 1\rangle$ and the point $(0, 0, 5)$.
We have to find the intersection cylinder-plane: $\Rightarrow$ all the cylinder bases $x^2 + y^2 = 16$.
Then, given the function $f(x, y) = z = 5 - 2x - 3y$ we have $f_x = -2$ and $f_y = -3$.

We switch to polar coordinates for the domain: $\theta \in [0, 2\pi]$ and $r \in [0, 4]$.

$$\iint_D \sqrt{4+9+1}\, dA = \int_0^{2\pi} \int_0^4 \sqrt{14}\, r\, dr\, d\theta = \int_0^{2\pi} \sqrt{14} \left[\frac{r^2}{2}\right]_0^4 d\theta = \sqrt{14} \cdot 8 \cdot 2\pi = 16\pi\sqrt{14}\,.$$

Example 12.22 (16 HW 12.3-12.5)**.** Find the surface area of the part of the plane $2x + 2y + z = 4$ that lies above the triangle formed by the three points $(3, 3, 0)$, $(8, 3, 0)$ and $(8, 6, 0)$.



Solution: We start considering that $x \in [3, 8]$ and $y \in \left[3, \frac{3}{5}x + \frac{6}{5}\right]$ (line between two points (3,3) and (8,6)).

$$\frac{x-3}{8-3} = \frac{y-3}{6-3} \quad \rightarrow \quad 3x - 9 = 5y - 15 \quad \rightarrow \quad y = \frac{3}{5}x + \frac{6}{5}\,.$$

The function is $f(x, y) = z = 4 - 2x - 2y \quad \rightarrow \quad f_x = -2$ and $f_y = -2$.

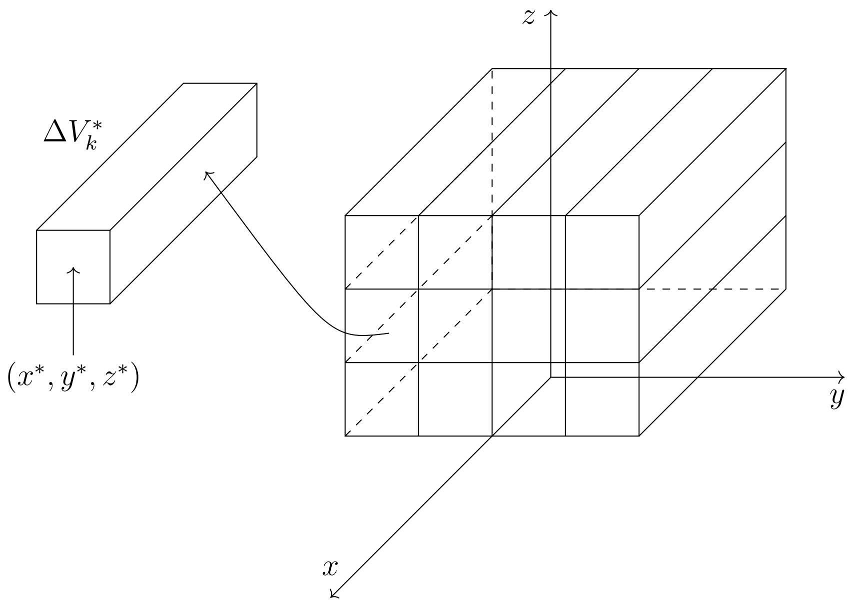
$$S = \iint_D \sqrt{4+4+1}\, dA = \int_3^8 \int_3^{\frac{3}{5}x+\frac{6}{5}} \sqrt{9}\, dy\, dx = \int_3^8 [3y]_3^{\frac{3}{5}x+\frac{6}{5}} dx = \int_3^8 \left(\frac{9}{5}x + \frac{18}{5} - 9\right) dx =$$

$$= \int_3^8 \left(\frac{9}{5}x - \frac{27}{5}\right) dx = \frac{9}{5}\int_3^8 (x-3)\, dx = \frac{9}{5}\left[\frac{x^2}{2} - 3x\right]_3^8 =$$

$$= \frac{9}{5}\left(\frac{64}{2} - 24 - \frac{9}{2} + 9\right) = \frac{9}{5}\left(32 - 24 + 9 - \frac{9}{2}\right) = \frac{9}{5}\,\frac{25}{2} = \frac{45}{2}\,.$$

12.5 Triple integrals

While a double integral $\iint\limits_D f(x,y)\,dA$ is evaluated over a two-dimensional domain D, a triple integral $\iiint\limits_D f(x,y,z)\,dV$ is evaluated over a closed, bounded region (solid) $D \in \mathbb{R}^3$. Similarly to the double integrals, we partition the domain into boxes.



If f is a function defined over a closed, bounded, solid region D, then the triple integral of f over D is defined to be the limit

$$\iiint\limits_D f(x,y,z)\,dV = \lim_{||P|| \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

provided this limit exists. $||P||$ is the longest diagonal of ΔV_k. The basic rules for double integrals are still valid:

- Linearity rule:

$$\iiint\limits_D af(x,y,z) + bg(x,y,z)\,dV = a\iiint\limits_D f(x,y,z)\,dV + b\iiint\limits_D g(x,y,z)\,dV\,.$$

- Dominance rule: if $f > g$ on D then

$$\iiint\limits_D f(x,y,z)\,dV \geq \iiint_D g(x,y,z)\,dV\,.$$

- Subdivision rule:

$$\iiint\limits_D = \iiint\limits_{D_1} + \iiint\limits_{D_2}$$

where $D = D_1 \cup D_2$, $D_1 \cap D_2 \neq 0$.

Iterated integration

Theorem 12.4 (Fubini's theorem over a parallepiped in space)**.** If $f(x,y,z)$ is continuous over a rectangular box $B : a \leq x \leq b$, $c \leq y \leq d$, $r \leq z \leq s$, then the triple integral may be evaluated by iterated integral

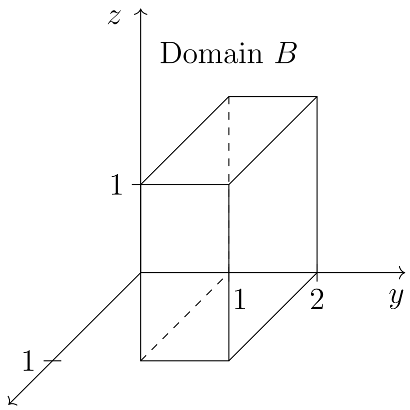
$$\iiint\limits_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz\,.$$

The iterated integration can be performed in any order.

Remark: As in the case of double integrals, if $f(x,y,z) = f_1(x)\,f_2(y)\,f_3(z)$ (separation of variables), then the integration can be written as

$$\iiint\limits_B f(x,y,z)\,dV = \int_a^b f_1(x)\,dx \int_c^d f_2(y)\,dy \int_r^s f_3(z)\,dz\,.$$

Example 12.23. Evaluate $\displaystyle\iiint\limits_B z^2\,y\,e^x\,dV$, where B is the box given by $0 \leq x \leq 1$, $1 \leq y \leq 2$, $-1 \leq z \leq 1$.



Solution:

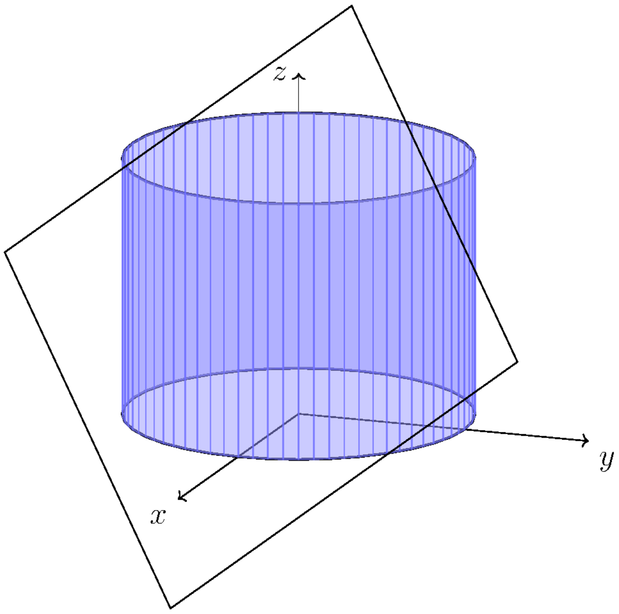
$$\iint_B f(x,y,z)\,dV = \int_{-1}^{1}\int_{1}^{2}\int_{0}^{1} z^2\,y\,e^x\,dx\,dy\,dz = \int_{-1}^{1}\int_{1}^{2} z^2\,y\,[e^x]_0^1\,dy\,dz =$$
$$= \int_{-1}^{1}\int_{1}^{2} z^2\,y\,(e-1)\,dy\,dz = \int_{-1}^{1} z^2\,(e-1)\left[\frac{y^2}{2}\right]_1^2 dz =$$
$$= \int_{-1}^{1} z^2\,(e-1)\left(2-\frac{1}{2}\right) dz = \left[\frac{z^3}{3}\right]_{-1}^{1}(e-1)\,\frac{3}{2} =$$
$$= \frac{2}{3}\,\frac{3}{2}\,(e-1) = e-1\,.$$

Triple integrals over z-simple solid region

Theorem 12.5 (Triple integral over z-simple solid region)**.** Suppose D is a solid region bounded below by the surface $z = u(x,y)$ and above by $z = v(x,y)$, that projects onto the region A in the xy-plane. If A is either type I or type II, then the integral of the continuous function $f(x,y,z)$ over D is

$$\iiint_D f(x,y,z)\,dV = \iint_A \left[\int_{u(x,y)}^{v(x,y)} f(x,y,z)\,dz\right] dA\,.$$

Example 12.24. Evaluate $\iiint_D x\,dV$, where D is the solid in the first octant bounded by the cylinder $x^2+y^2=4$ and the plane $2y+z=4$.



Solution: Consider the plane $2y + z = 4$. For $x = 0$ and $y = 0$ we have $z = 4$. The normal is $\langle 0, 2, 1\rangle$.
The domain is bounded (in z- direction) below by the circle $x^2 + y^2 = 4$ and $z = 0$, and above by the plane $z = 4 - 2y$. Thus, we have

$$\iiint\limits_D x\,dV = \iint\limits_A \int_0^{4-2y} x\,dz\,dA$$

where A is the quarter of circle $x^2 + y^2 = 4$, thus $y = \sqrt{4 - x^2}$. Therefore we have

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-2y} x\,dz\,dy\,dx = \int_0^2 \int_0^{\sqrt{4-x^2}} x\Big[z\Big]_0^{4-2y} dy\,dx = \int_0^2 \int_0^{\sqrt{4-x^2}} x\,(4-2y)\,dy\,dx =$$

$$= \int_0^2 \left[x\,\left(4y - y^2\right)\right]_0^{4-x^2} dx = \int_0^2 x\,\left(4\sqrt{4-x^2} - 4 + x^2\right)\,dx = \int_0^2 \left(4x\sqrt{4-x^2} - 4x + x^3\right)\,dx =$$

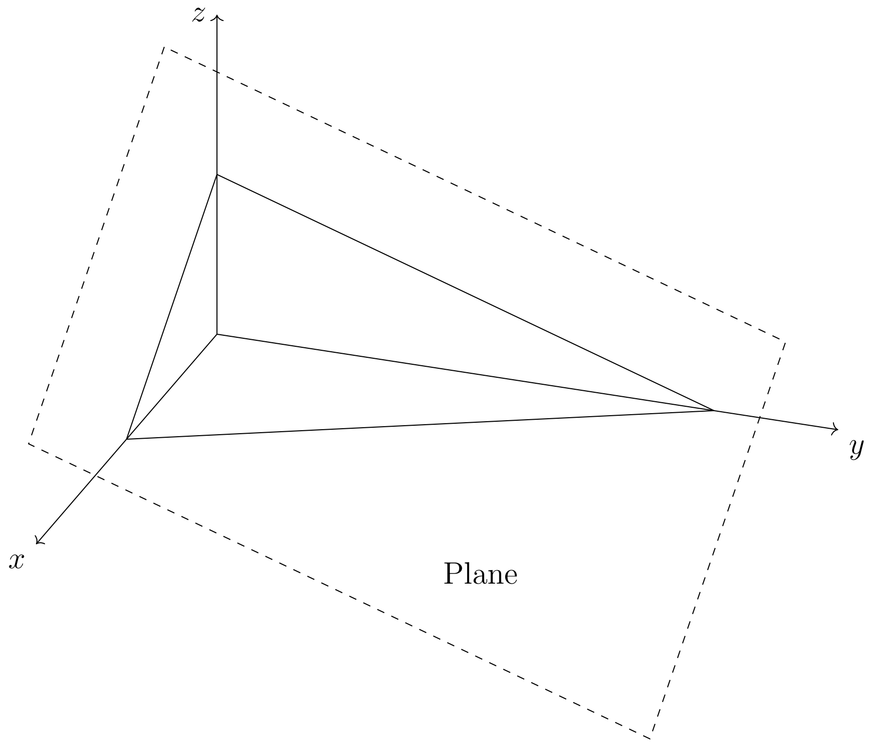
$$= -2\int_0^2 \left(-2x\sqrt{4-x^2}\right)\,dx - \Big[2x^2\Big]_0^2 + \left[\frac{x^4}{4}\right]_0^2 = -2\left[\frac{(4-x^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 - \Big[2x^2\Big]_0^2 + \left[\frac{x^4}{4}\right]_0^2 =$$

$$= -2\left(0 - \frac{2}{3}\cdot 4^{\frac{3}{2}}\right) - (2\cdot 4 - 0) + \left(\frac{2^4}{4} - 0\right) = \frac{4\cdot 8}{3} - 8 + 4 = \frac{20}{3}.$$

Volume by triple integrals As a double integral can be interpreted as the area of the region of integration, a triple integral may be interpreted as the volume of a solid:

$$V = \iiint\limits_D dV\,.$$

Example 12.25 (Volume of a tetrahedron)**.** Find the volume of the tetrahedron T bounded by the plane $2x + y + 3z = 6$ and the coordinate planes $x = 0$, $y = 0$ and $z = 0$.



Solution: The z coordinate is bounded above by the plane $3z = 6 - 2x - y \quad \rightarrow \quad z = 2 - \frac{2}{3}x - \frac{1}{3}y$, and below by the plane $z = 0$.

The domain of integration along the xy plane is a triangle

$$\begin{cases} z = 0 \\ z = 2 - \frac{2}{3}x - \frac{1}{3}y \end{cases} \qquad \begin{cases} z = 0 \\ \frac{2}{3}x + \frac{1}{3}y - 2 = 0 \end{cases}$$

$$\begin{cases} \frac{1}{3}y = -\frac{2}{3}x + 2 \quad \rightarrow \quad y = -2x + 6 \quad \text{(upper bound in } y\text{-direction)} \\ y = 0 \quad \text{(lower bound in } x\text{-direction)} \end{cases}$$

$$z \in \left[0, 2 - \frac{2}{3}x - \frac{1}{3}y\right], \qquad y \in [0, -2x+6].$$

For $y = 0$, $y = -2x + 6 \quad \rightarrow \quad -2x + 6 = 0 \quad \rightarrow \quad x = 3$. Therefore, $x \in [0, 3] \quad \Rightarrow$

$$\int_0^3 \int_0^{-2x+6} \int_0^{2-\frac{2}{3}x-\frac{1}{3}y} 1\, dz\, dy\, dx = \int_0^3 \int_0^{-2x+6} \Big[z\Big]_0^{2-\frac{2}{3}x-\frac{1}{3}y} dy\, dx =$$

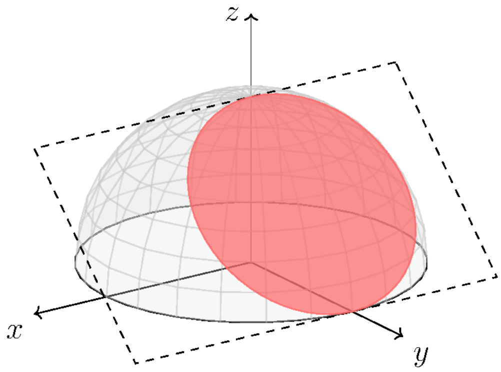
$$= \int_0^3 \int_0^{-2x+6} \left(2 - \frac{2}{3}x - \frac{1}{3}y\right) dy\, dx = \int_0^2 \left[2y - \frac{2}{3}xy - \frac{1}{3}\frac{y^2}{2}\right]_0^{-2x+6} dx =$$

$$= \int_0^3 \left[-4x + 12 - \frac{2}{3}x(-2x+6) - \frac{1}{6}(-2x+6)^2\right] dx =$$

$$= \int_0^3 \left(-4x + 12 + \frac{4}{3}x^2 - 4x - \frac{2}{3}x^2 + 4x - 6\right) =$$

$$= \int_0^3 \left(\frac{2}{3}x^2 - 4x + 6\right) dx = \left[\frac{2x^3}{9} - 2x^2 + 6x\right]_0^3 = 6.$$

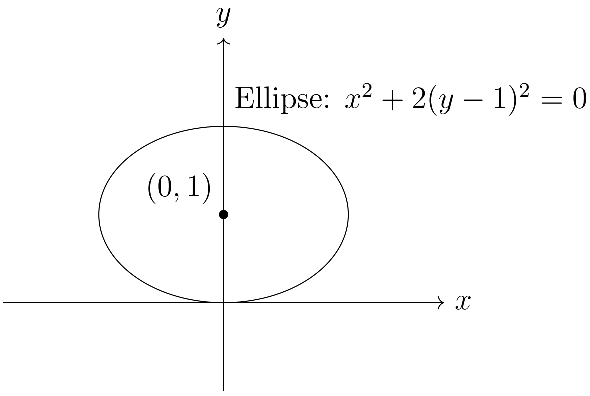
Example 12.26. Set up (but do not evaluate) a triple integral for the volume of the solid D that is bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the plane $y + z = 2$.



Solution: Since the intersection between the plane and the sphere occurs above the xy-plane, the sphere can be represented as $z = \sqrt{4 - x^2 - y^2}$. Now we find the intersection between the sphere and the plane.

$$\begin{cases} z = \sqrt{4 - x^2 - y^2} \\ z = 2 - y \end{cases} \Rightarrow \quad \begin{aligned} \sqrt{4 - x^2 - y^2} &= 2 - y \\ 4 - x^2 - y^2 &= (2 - y)^2 \\ 4 - x^2 - y^2 &= 4 - 4y + y^2 \\ x^2 + 2y^2 - 4y &= 0 \quad \text{(Ellipse)} \\ x^2 + 2(y^2 - 2y + 1) - 2 &= 0 \\ x^2 + 2(y - 1)^2 &= 0\,. \end{aligned}$$

Although this intersection occurs in $\mathbb{R}^3$, its equation does not contain z $\Rightarrow$ we can use it as a projection of the volume of integration on the xy plane.



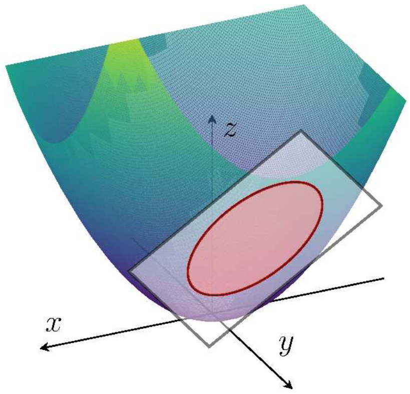
It is better to consider this as a type II region, since

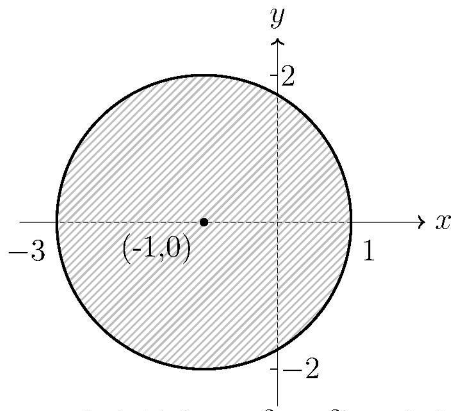
$$x^2 = -2(y-1)^2 + 2 = 4y - 2y^2 \quad \rightarrow \quad x = \pm\sqrt{4y - 2y^2}\,.$$

Thus $x \in \left[-\sqrt{4y - 2y^2}, +\sqrt{4y - 2y^2}\right]$, $y \in [0, 2]$, $z \in \left[2 - y, \sqrt{4 - x^2 - y^2}\right]$ (the plane and the sphere).

$$V = \int_0^2 \int_{-\sqrt{4y-2y^2}}^{\sqrt{4y-2y^2}} \int_{2-y}^{\sqrt{4-x^2-y^2}} dz\, dx\, dy\,.$$

Example 12.27. Find the volume of the solid D bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $2x + z = 3$ (do not solve, only find the integral).





Solution: We find the intersection between the paraboloid ($z = x^2 + y^2$) and the plane ($z = 3 - 2x$).

$$\begin{aligned} x^2 + y^2 &= 3 - 2x \\ (x^2 + 2x + 1) - 1 + y^2 &= 3 \\ (x+1)^2 + y^2 &= 4 \end{aligned}$$

that is a circle of radius equal to 2 and center $(-1, 0)$ int the xy-plane.
Volume between the paraboloid and the plane:

$$\begin{array}{llll} z: & \text{Lower bound:} & \text{Paraboloid} & z = x^2 + y^2 \\ & \text{Upper bound:} & \text{Plane} & z = -2x + 3 \end{array}$$

For the xy-plane domain $\rightarrow$ circle: $(x+1)^2 + y^2 = 4$.

$$y = \pm\sqrt{4 - (x+1)^2} = \pm\sqrt{3 - 2x - x^2}$$

x-range: $x \in [-3, 1]$ (see figure). So the volume is:

$$V = \int_{-3}^{1} \int_{-\sqrt{3-2x-x^2}}^{\sqrt{3-2x-x^2}} \int_{x^2+y^2}^{-2x+3} dz\, dy\, dx\,.$$

12.6 .

12.7 Cylindrical and spherical coordinates

12.7.1 Cylindrical coordinates

The cylindrical coordinates are an extension of the polar coordinates in $\mathbb{R}^3$.

Conversion:

- Cylindrical to rectangular

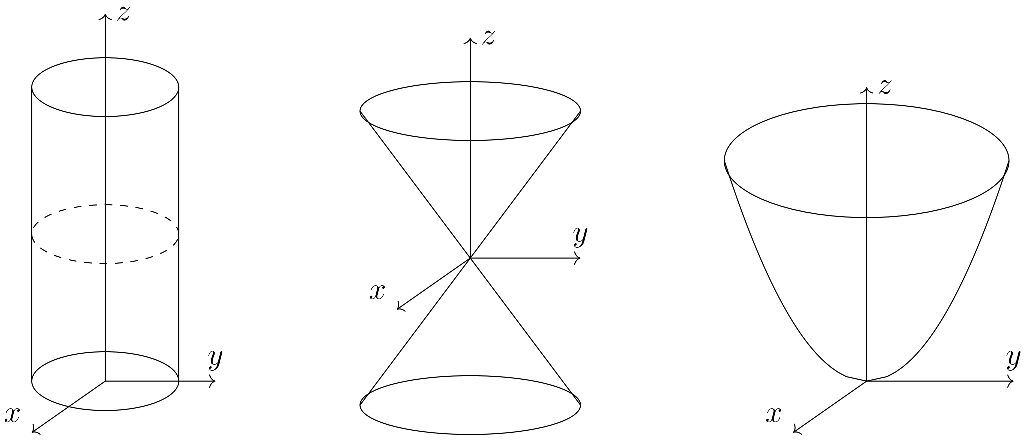
$$\begin{cases} x = r\ \cos(\theta) \\ y = r\ \sin(\theta) \\ z = z \end{cases}$$

- Rectangular to cylindrical

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \tan(\theta) = \dfrac{y}{x} \\ z = z \end{cases}$$

Main examples:

	Cylinder	**Cone**	**Paraboloid**
Rect.	$x^2 + y^2 = a^2$	$x^2 + y^2 = z^2$	$x^2 + y^2 = az$
Cyl.	$r = a$	$r = z$	$r^2 = az$



Example 12.28. Find an equation in cylindrical coordinates for the elliptic paraboloid $z = x^2 + 3y^2$.

Solution:

$$\begin{aligned} x &= r\ \cos(\theta) & & z = r^2\cos^2(\theta) + 3r^2\sin^2(\theta) \\ y &= r\ \sin(\theta) & \rightarrow\quad & z = r^2\left(1 - \sin^2(\theta) + 3\sin^2(\theta)\right) \\ z &= z & & z = r^2\left(1 + 2\sin^2(\theta)\right) \end{aligned}$$

Integration with cylindrical coordinates For a z-simple integral, if the projection in xy-plane can be described better with polar coordinates, then

$$\iiint_D f(x,y,z)\,dV = \iint_A \left[\int_{u(x,y)}^{v(x,y)} f(x,y,z)\,dz\right] dA = \iint_A \left[\int_{u(x,y)}^{v(x,y)} f\left(r\,\cos(\theta), r\,\sin(\theta)\right)\,dz\right] dA\,,$$

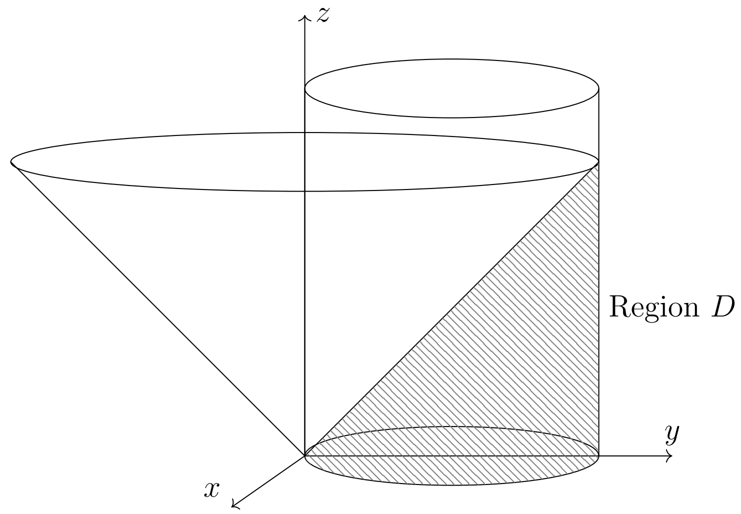
where $u(x,y) \le z \le v(x,y)$. Then, we should also transform the boundaries in the z-direction as a function of r and θ.

- Let D be a solid with upper surface $z = v(r,\theta)$ and lower surface $z = u(r,\theta)$, and let A be the projection of D in the xy-plane. Then,

$$\iiint_D f(x,y,z)\,dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{u(r,\theta)}^{v(r,\theta)} f\left(r\,\cos(\theta), r\,\sin(\theta), z\right)\,r\,dz\,dr\,d\theta\,,$$

where $A = \Big\{(r,\theta) \quad \text{such that} \quad \alpha \le \theta \le \beta \quad \text{and} \quad g_1(\theta) \le r \le g_2(\theta)\Big\}$.

Example 12.29. Find the volume of the solid in the first octant that is bounded by the cylinder $x^2 + y^2 = 2y$, the half cone $z = \sqrt{x^2+y^2}$ and the xy-plane.



Solution: The domain in the xy- plane can be recognized as a half circle $\Rightarrow$ we transform in cylindrical coordinates.

$$\text{- Cylinder:} \quad x^2+y^2=2y \quad \to \quad \begin{aligned} &r^2\cos^2(\theta)+r^2\sin^2(\theta)=2\,r\sin(\theta)\\ &r^2=2r\,\sin(\theta)\\ &r=2\sin(\theta)\end{aligned}$$

$$\text{- Half cone:} \quad z=\sqrt{x^2+y^2} \quad \to \quad \begin{aligned} &z=\sqrt{r^2\cos^2(\theta)+v^2\sin^2(\theta)}\\ &z=\sqrt{r^2}\\ &z=r\end{aligned}$$

The region D has the following bounds:

$$z\in[0,r]\quad\text{(half cone)},\qquad r\in[0,2\sin(\theta)],\qquad \theta\in\left[0,\frac{\pi}{2}\right]\quad\text{(first octant)}.$$

$$V=\iiint\limits_D dV=\iint\limits_A\int_{u(r,\theta)}^{v(r,\theta)} r\,dz\,dA=\int_0^{\frac{\pi}{2}}\int_0^{2\sin(\theta)}\int_0^r r\,dz\,dr\,d\theta=\int_0^{\frac{\pi}{2}}\int_0^{2\sin(\theta)} r\Big[z\Big]_0^r dr\,d\theta=$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{2\sin(\theta)} r^2dr\,d\theta=\int_0^{\frac{\pi}{2}}\left[\frac{r^3}{3}\right]_0^{2\sin(\theta)}d\theta=\int_0^{\frac{\pi}{2}}\frac{8}{3}\sin^3(\theta)\,d\theta=\frac{8}{3}\int_0^{\frac{\pi}{2}}\sin^3(\theta)\,d\theta=$$

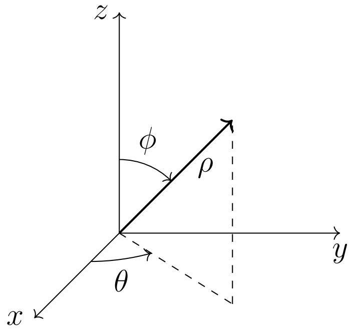
$$=\frac{8}{3}\int_0^{\frac{\pi}{2}}\sin^2(\theta)\cdot\sin(\theta)\,d\theta=\frac{8}{3}\int_0^{\frac{\pi}{2}}\big(1-\cos^2(\theta)\big)\sin(\theta)\,d\theta=\dots$$

$$\begin{aligned}&\text{let}\quad u=\cos(\theta)\quad\to\quad du=-\sin(\theta)\,d\theta\quad\to\quad -du=\sin(\theta)\,d\theta\\ &\text{for}\quad \theta=0\quad\to\quad u=\cos(0)=1\\ &\text{for}\quad \theta=\frac{\pi}{2}\quad\to\quad u=\cos\left(\frac{\pi}{2}\right)=0\end{aligned}$$

$$\dots=\frac{8}{3}\int_1^0\left(1-u^2\right)(-du)=\frac{8}{3}\int_0^1\left(1-u^2\right)\,du=\frac{8}{3}\left[u-\frac{u^3}{3}\right]_0^1=\frac{8}{3}\cdot\frac{2}{3}=\frac{16}{9}\,.$$

12.7.2 Spherical coordinates

In spherical coordinates we label a point P by a triple (ρ,θ,ϕ):



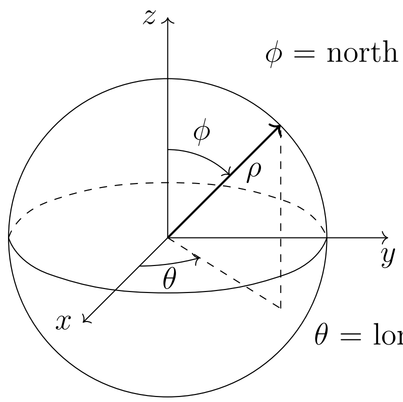
$\rho =$ distance from the origin of the point P ($\rho \geq 0$).

$\theta =$ polar angle (as in polar coordinates) $0 \leq \theta \leq 2\pi$.

$\phi =$ angle between z-axis and the vector $\overline{OP}$, $0 \leq \phi \leq \pi$.

Note: the spherical coordinates can be led back to the latitude and the longitude of the earth.

$\rho =$ radius of the earth



$\phi =$ north latitude

$\theta =$ longitude east

- Spherical to rectangular

$$\begin{cases} x = \rho \sin(\phi) \cos(\theta) \\ y = \rho \sin(\phi) \sin(\theta) \\ z = \rho \cos(\theta) \end{cases}$$

- Rectangular to spherical

$$\begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} \\ \tan(\theta) = \dfrac{y}{x} \\ \phi = \cos^{-1}\left(\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \end{cases}$$

- Spherical to cylindrical

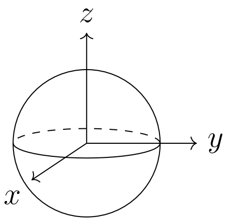
$$\begin{cases} r = \rho \sin(\phi) \\ \theta = \theta \\ z = \rho \cos(\phi) \end{cases}$$

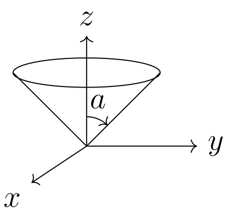
- Cylindrical to spherical

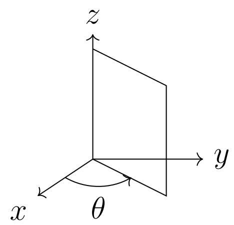
$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = \theta \\ \phi = \cos^{-1}\left(\dfrac{z}{\sqrt{r^2 + z^2}}\right) \end{cases}$$

Main examples:

	Sphere	**Half-Cone**	**Vertical half-plane**
Rect.	$x^2 + y^2 + z^2 = a^2$	$\tan(a) \cdot z = \sqrt{x^2 + y^2}$	$\theta = a$
Sph.	$\rho = a$	$\phi = a, \quad 0 < a < \dfrac{\pi}{2}$	$0 \leq \theta \leq 2\pi$







Example 12.30. Convert in spherical coordinates the paraboloid $z = x^2 + y^2$.

Solution:

$$\begin{aligned}
\rho\cos(\phi) &= \big(\rho\sin(\phi)\cos(\theta)\big)^2 + \big(\rho\sin(\phi)\sin(\theta)\big)^2 \\
\rho\cos(\phi) &= \big(\rho^2\sin^2(\phi)\big) \cdot \big(\cos^2(\theta) + \sin^2(\theta)\big) \\
\rho\cos(\phi) &= \rho^2\sin^2(\phi) \\
\rho\cos(\phi) &= \rho\sin^2(\phi) \quad \to \quad \rho = \frac{\cos(\phi)}{\sin^2(\phi)} = \frac{\cot(\phi)}{\sin(\phi)}\,.
\end{aligned}$$

Integration with spherical coordinates If f is continuous on the bounded solid region D, then the triple integral of f over D is given by

$$\iiint_D f(x,y,z)\,dV = \iiint_{\overline{D}} f\big(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)\big)\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi\,,$$

where $\overline{D}$ is the region D expressed in spherical coordinates. Note that $dV = \rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$.

Example 12.31. Verify that a sphere of radius R has volume $V = \frac{4}{3}\pi R^3$.

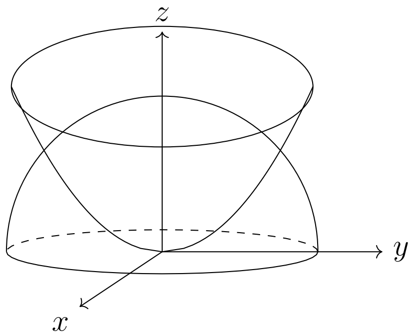
Solution: we have the following extremes:

$$\rho \in [0,R], \qquad \theta \in [0,2\pi], \qquad \phi \in [0,\pi]\,.$$

$$V = \iiint_D dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \sin(\phi) \left[\frac{\rho^3}{3}\right]_o^R d\phi\, d\theta =$$

$$= \int_0^{2\pi} \Big[-\cos(\phi)\Big]_0^{\pi} \frac{R^3}{3}\, d\theta = \big[\theta\big]_0^{2\pi} (1+1)\, \frac{R^3}{3} = \frac{4}{3} R^3\, \pi\,.$$

Example 12.32. Evaluate the integral $I = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} z\, dz\, dy\, dx.$



Solution: Consider the z boundaries:
Lower: $z = x^2 + y^2$ (paraboloid)
Upper: $z = \sqrt{2 - x^2 - y^2}$ (half sphere) $\to z^2 = 2 - x^2 - y^2 \quad \to \quad x^2 + y^2 + z^2 = 2\,.$
In spherical coordinates:
Lower: $\rho \cos(\phi) = \rho^2 \sin^2(\phi) \cos^2(\theta) + \rho^2 \sin^2(\phi) \sin^2(\theta) \quad \to \quad \rho\cos(\phi) = \rho^2 \sin^2(\phi)$

$$\Rightarrow \rho = \frac{\cos(\phi)}{\sin^2(\phi)}$$

Upper: $x^2 + y^2 + z^2 = 2 \quad \Rightarrow \quad \rho = \sqrt{2}$.
Note that, depending on the value of ϕ, the maximum value for ρ can be either $\rho = \sqrt{2}$ or $\rho = \dfrac{\cos(\phi)}{\sin^2(\phi)} \quad \Rightarrow \quad$ we need to find the intersection between the two curves:

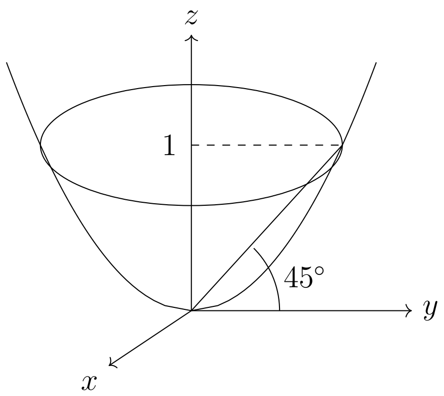
$$\begin{cases} \rho = \sqrt{2} \\ \rho = \dfrac{\cos(\phi)}{\sin^2(\phi)} \end{cases} \qquad \to \qquad \frac{\cos(\phi)}{\sin^2(\phi)} = \sqrt{2} \qquad \text{(no answer)}$$

$$\begin{cases} x^2+y^2+z^2=2 \\ z=x^2+y^2 \end{cases} \quad \rightarrow \quad \begin{array}{l} z+z^2=2 \\ z^2+z-2=0 \quad \rightarrow \quad (z+2)(z-1)=0\,. \end{array}$$

$z=-2$ (impossible, $z \geq 0$)

$z=1 \quad \rightarrow \quad x^2+y^2=1$ (circle of radius =1, at $z=1$)

$$\Rightarrow \quad \text{For:} \quad \begin{array}{lll} \phi \leq \frac{\pi}{4} & \rightarrow & \rho_{max}=\sqrt{2} \\ \phi > \frac{\pi}{4} & \rightarrow & \rho_{max}=\dfrac{\cos(\phi)}{\sin^2(\phi)} \end{array}$$



So we need to split the ϕ integral in two parts:

$$I=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}}\big(\rho\cos(\phi)\big)\rho^2\sin(\phi)d\rho d\phi d\theta+\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\frac{\cos(\phi)}{\sin^2(\phi)}}\big(\rho\cos(\phi)\big)\rho^2\sin(\phi)d\rho d\phi d\theta=$$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\left[\frac{\rho^4}{4}\right]_0^{\sqrt{2}}\cos(\phi)\sin(\phi)\,d\phi\,d\theta+\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\left[\frac{\rho^4}{4}\right]_0^{\frac{\cos(\phi)}{\sin^2(\phi)}}\cos(\phi)\sin(\phi)\,d\phi\,d\theta=\ldots$$

$\Rightarrow$ not easy to solve!

However, in this case, the simplest approach is the cylindrical coordinates:

$$\iint_A\int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} z\,dz\,dA\,,$$

where A is the circle of radius =1 ("shadow" of the area found before, $\theta \in [0, 2\pi]$ and $r \in [0, 1]$).

$$\iint_A \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} z\, dz\, dA = \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} z\, r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^1 \left[\frac{z^2}{2} r \right]_{r^2}^{\sqrt{2-r^2}} dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(\frac{(2-r^2)r}{2} - \frac{r^4 \cdot r}{2} \right) dr\, d\theta = \int_0^{2\pi} \int_0^1 \left(\frac{2r - r^3 - r^5}{2} \right) dr\, d\theta =$$

$$= \int_0^{2\pi} \frac{1}{2} \left[\frac{2r^2}{2} - \frac{r^4}{4} - \frac{r^6}{6} \right]_0^1 d\theta = \int_0^{2\pi} \frac{1}{2} \left(1 - \frac{1}{4} - \frac{1}{6} \right) d\theta =$$

$$= 2\pi \cdot \frac{1}{2} \cdot \frac{12 - 3 - 2}{12} = \frac{7\pi}{12}.$$

Example 12.33. Evaluate $\iiint_D \sqrt{x^2 + y^2 + z^2}\, dV$, where D is the portion of the ball $x^2 + y^2 + z^2 \leq 4$ in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$).

Solution:

$$x^2 + y^2 + z^2 = \rho^2 \cos^2(\theta) \sin^2(\phi) + \rho^2 \sin^2(\theta) \sin^2(\phi) + \rho^2 \cos^2(\phi) =$$

$$= \rho^2 \sin^2(\phi) \left(\cos^2(\theta) + \sin^2(\theta) \right) + \rho^2 \cos^2(\phi) = \rho^2.$$

Sphere, first octant $\quad \rightarrow \quad \phi \in \left[0, \frac{\pi}{2}\right], \quad \theta \in \left[0, \frac{\pi}{2}\right], \quad \rho \in [0, 2] \Rightarrow$

$$\iiint_D \sqrt{x^2 + y^2 + z^2}\, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho \cdot \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi = \int_0^{\pi/2} \int_0^{\pi/2} \left[\frac{\rho^4}{4} \right]_0^2 \sin(\phi)\, d\theta\, d\phi =$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} 4 \sin(\phi)\, d\theta\, d\phi = \int_0^{\pi/2} 4 \sin(\phi) \left[\theta \right]_0^{\pi/2} d\phi =$$

$$= \int_0^{\pi/2} 2\pi \sin(\phi)\, d\phi = -2\pi \left[\cos(\phi) \right]_0^{\pi/2} = 2\pi.$$

12.8 Jacobian: change of variables

12.8.1 Change of variables in double integrals

In a single integral, the change of variables works as follows:

- $x = g(u)$
- $dx = g'(u)\,du$

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

- Extremes: $a, b \to g(c) = a, g(d) = b$. In general, this process involves a "mapping factor", called **Jacobian**.

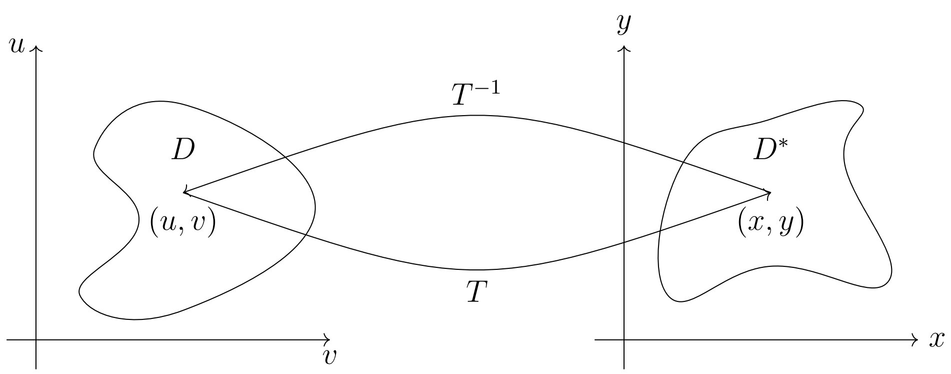
Theorem 12.6 (Change of variables in a double integral)**.** Let f be a continuous function on the interior of a region D in the xy-plane and bounded on that region, and let T be a one-to-one transformation except possibly on the boundary that maps the region D^* in the uv-plane onto D under the change of variables $x = g(u, v)$, $y = h(u, v)$, where g and h are continuously differentiable functions in D^*. Then,

$$\iint_D f(x, y)\,dy\,dx = \iint_{D^*} f\left(g(u, v), h(u, v)\right) \cdot |J(u, v)|\,du\,dv\,,$$

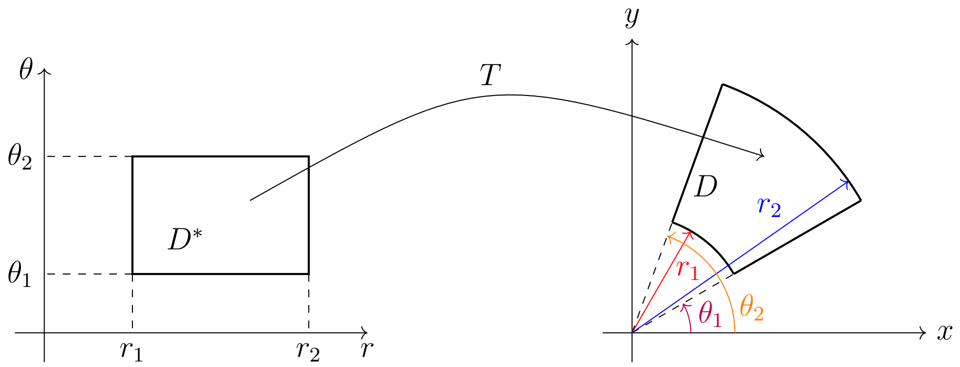
where

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}\,.$$

Note: The transformation T is a function that transforms the uv-plane onto the xy-plane. Its inverse T^{-1} maps the xy-plane onto the uv-plane.



Example 12.34. Find the Jacobian for the change of variables from rectangular to polar coordinates, namely $x = r\cos(\theta)$, $y = r\sin(\theta)$.



Solution:

$$J = \begin{vmatrix} \dfrac{\partial\big(r\cos(\theta)\big)}{\partial r} & \dfrac{\partial\big(r\cos(\theta)\big)}{\partial\theta} \\ \dfrac{\partial\big(r\sin(\theta)\big)}{\partial r} & \dfrac{\partial\big(r\sin(\theta)\big)}{\partial\theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r\,cos^2(\theta) + r\sin^2(\theta) = r\,.$$

Example 12.35. If $u = xy$ and $v = x^2 - y^2$, express the Jacobian in terms of u and v.

Solution: It is not easy to express x, y as a function of u, v.

$$\Rightarrow J^{-1} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} y & x \\ 2x & 2y \end{bmatrix} = -2y^2 - 2x^2 = -2(x^2+y^2)\,.$$

Since $\quad (x^2+y^2)^2 = (x^2-y^2)^2 - 4x^2y^2 = v^2 - 4u^2 \quad \to \quad (x^2+y^2) = \sqrt{v^2-4u^2}.$

$$J^{-1} = -2\sqrt{v^2-4u^2} \quad \to \quad J = \frac{-1}{2\sqrt{v^2-4u^2}}\,.$$

Example 12.36. Compute $\displaystyle\iint_D \left(\frac{x-y}{x+y}\right)^4 dy\,dx$, where D is the triangular region bounded by the line $x + y = 1$ and the coordinate axis.

Solution: A good substitution for the given integral is: $u = x - y$ and $v = x + y$.

$$\Rightarrow \quad x = u + y \quad \Rightarrow \quad v = u + y + y \quad \Rightarrow \quad 2y = v - u \quad \Rightarrow \quad y = \frac{v-u}{2}$$

$$x = u + \frac{v-u}{2} = \frac{u+v}{2}$$

$$\begin{cases} x = \dfrac{u+v}{2} \\ y = \dfrac{v-u}{2} \end{cases}$$

Thus, the Jacobian reads

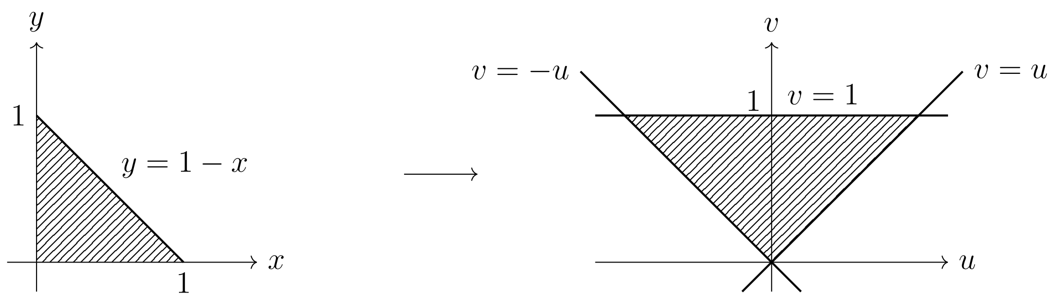
$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Boundaries:

- line $x = 0 \quad \to \quad \dfrac{u+v}{2} = 0 \quad \to \quad u + v = 0 \quad \to \quad u = -v$

- line $y = 0 \quad \to \quad \dfrac{v-u}{2} = 0 \quad \to \quad u - v = 0 \quad \to \quad u = v$

- line $x + y = 1 \quad \to \quad \dfrac{u+v}{2} + \dfrac{v-u}{2} = 1 \quad \to \quad v = 1$



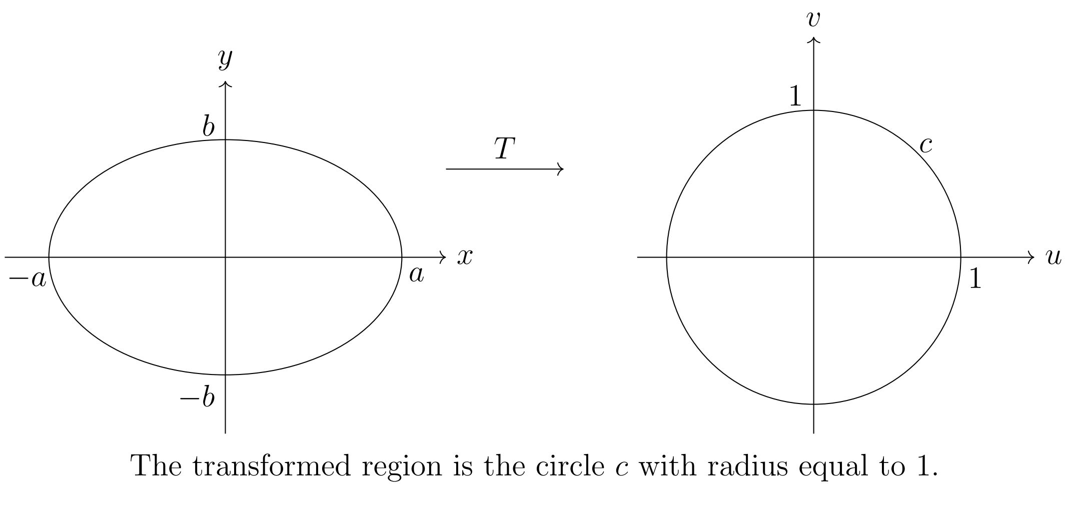
Type II integral

$$\int_0^1 \int_{-v}^{v} \left(\frac{u}{v}\right)^4 J\,du\,dv = \int_0^1 \int_{-v}^{v} \frac{1}{2}\left(\frac{u}{v}\right)^4 du\,dv = \int_0^1 \frac{1}{2}\frac{1}{v^4}\left[\frac{u^5}{5}\right]_{-v}^{v} dV =$$
$$= \int_0^1 \frac{1}{2v^4}\left(\frac{v^5}{5}+\frac{v^5}{5}\right) dV = \int_0^1 \frac{1}{2v^4}\,\frac{2v^5}{5}\,dv = \int_0^1 \frac{v}{5}\,dV = \left[\frac{v^2}{10}\right]_0^1 = \frac{1}{10}\,.$$

Example 12.37 (Change of variables to simplify a region)**.** Find the area of the region E bounded by the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Solution: The area is given by $A = \displaystyle\iint_D 1\,dy\,dx$. To solve a simpler integral, we can consider the transformation

$$\begin{array}{ccccc} u=\dfrac{x}{a} & v=\dfrac{y}{b} & \Rightarrow & \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 & \rightarrow \quad u^2+v^2=1\,. \\ x=au & y=bv & & & \end{array}$$



The transformed region is the circle c with radius equal to 1.

$$J = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = ab$$

$$A = \iint_C 1\cdot J\,du\,dv = \int_0^{2\pi}\int_0^1 a\,b\,r\,dr\,d\theta = a\,b\,\pi\,.$$

(Area of the circle of radius =1 $\quad\rightarrow\quad A = \pi\,(1)^2 = \pi$).

12.8.2 Change of variables in a triple integral

The change of variables formula for triple integrals is similar to the formula for double integrals:

$$T: \quad x = x(u,v,w), \quad y = y(u,v,w), \quad z = z(u,v,w)\,.$$

Then, the Jacobian is the determinant of

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}\,.$$

And the change of variables yields

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_{R^*} f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw\,.$$

Example 12.38 (HW set 12.7-12.8 Ex. 16)**.** Given

$$x = 8u - 10uv$$
$$y = 7uv - uvw$$
$$z = -6uvw$$

find J.

Solution:

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{bmatrix} 8-10v & -10u & 0 \\ 7v - vw & 7u - uw & -uv \\ -6vw & -6uw & -6uv \end{bmatrix} =$$

$$=(8-10v)(7u-uw)(-6uv) + (-10u)(-uv)(-6vw) + 0 - 0-$$
$$+(-6uw)(-uv)(8-10uv) - (7v-vw)(-10u)(-6uv) = \ldots$$