CHAPTER 11

# Partial Differentiation

## Contents



Single-value differential calculus $\rightarrow$ functions of two or more independent variables.

## 11.1 Functions of several variables

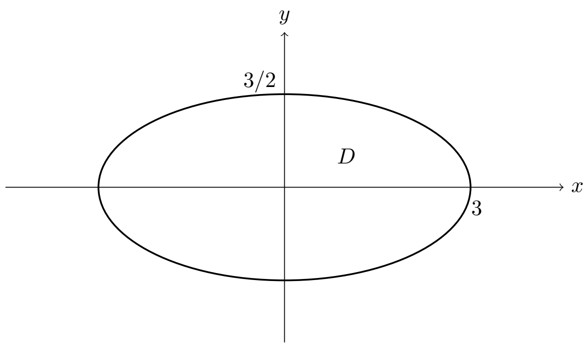
**Definition 11.1** (Function of 2 variables)**.** A function of two variables is a rule that assigns to each pair $(x, y)$ in a set $D$ a unique number $f(x, y)$. The set $D$ is called the **domain** of the function, and the corresponding values of $f(x, y)$ constitute the **range** of $f$.

**Note**: We may write $z - f(x, y)$ where $x$ and $y$ are the **independent variable** and $z$ is the **dependent variable**.

**Example 11.1.** Given the function $f(x, y) = \sqrt{9 - x^2 - 4y^2}$, evaluate $f(2, 1), f(2t, t^2)$, and the domain $D$ of $f$.

**Solution:**

- $f(2, 1) = \sqrt{9 - 4 - 4} = 1$
- $f(2t, t^2) = \sqrt{9 - (2t)^2 - 4(t^2)^2} = \sqrt{9 - 4t^2 - 4t^4}$
- $D : 9 - x^2 - 4y^2 \geq 0 \to x^2 + 4y^2 \leq 9$, that is satisfied by the internal points of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{9/4} = 1$.

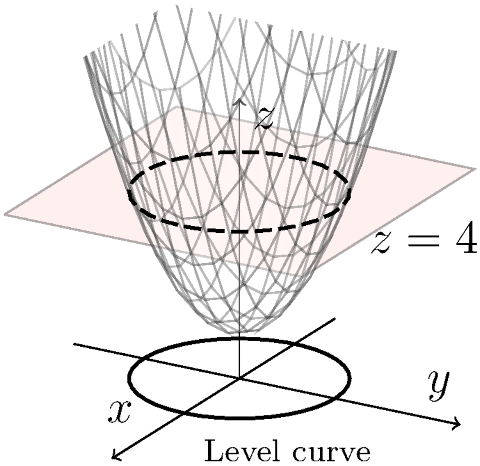



**Operations with functions of 2 variables.**

- Sum/difference: $(f \pm g)(x, y) = f(x, y) \pm g(x, y)$
- Product: $(f \cdot g)(x, y) = f(x, y) \cdot g(x, y)$

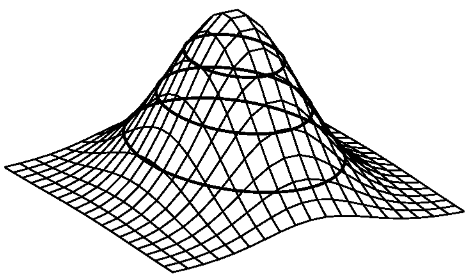
- Quotient: $\left(\dfrac{f}{g}\right)(x,y) = \dfrac{f(x,y)}{g(x,y)}$, with $g(x,y) \neq 0$

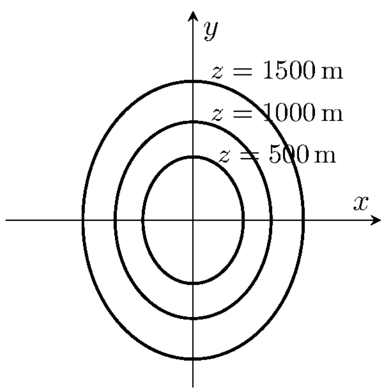
**Level curves and surfaces.** We define the graph of a function $f(x,y)$ as the collection of the points $(x,y,z)$ such that $z = f(x,y)$.




When the generic plane $z = K$ intersects the surface $z = f(x,y)$, the result is the two-variable equation $f(x,y) = K$, which is called *trace* or *level curve at* $K$. We can plot the level curves on the $x - y$ plane, obtaining the *contour curves.*

**Example 11.2.** If $f(x,y) = z$ we can represent the altitude of the mountains or isotherms.

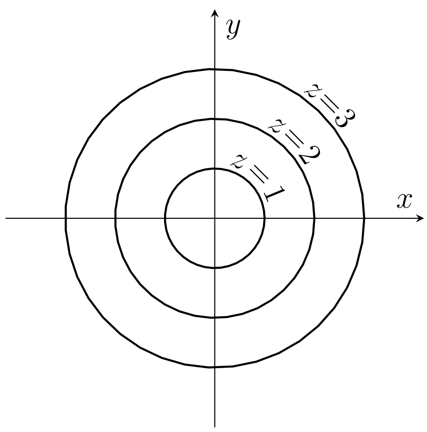




**Example 11.3.** Sketch the level curves of the function $f(x, y) = 10 - x^2 - y^2$ for $z = 1$, $z = 4$ and $z = 9$.

**Solution:**

- $z = 1 \to 10 - x^2 - y^2 = 1 \to x^2 + y^2 = 9 \to$ Circle with radius equal to 3
- $z = 6 \to 10 - x^2 - y^2 = 1 \to x^2 + y^2 = 4 \to$ Circle with radius equal to 2
- $z = 9 \to 10 - x^2 - y^2 = 1 \to x^2 + y^2 = 1 \to$ Circle with radius equal to 1

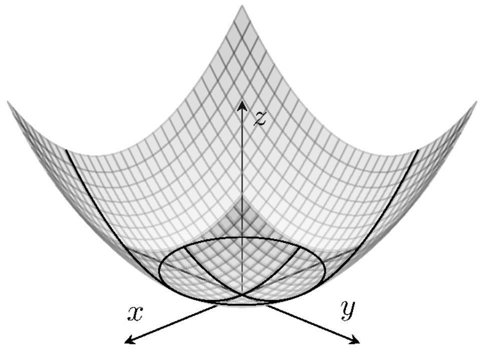



**Graph of a function of two variables.**

**Example 11.4.** Graph the function of $f(x, y) = x^2 + y^2$.

**Solution:** To graph it we consider the trace $z = k$. To have additional information we can also consider $x = A$ and $y = B$, with $A, B$ constant (traces along $x$-axis and $y$-axis respectively). Thus, with $z = x^2 + y^2$ we have:

- $z = k \to x^2 + y^2 = k \to$ Circle of radius $r$ equal to $\sqrt{k}$, with $k \geq 0$
- $x = A \to A^2 + y^2 = z \to$ Parabola
- $y = B \to x^2 + B^2 = z \to$ Parabola


z
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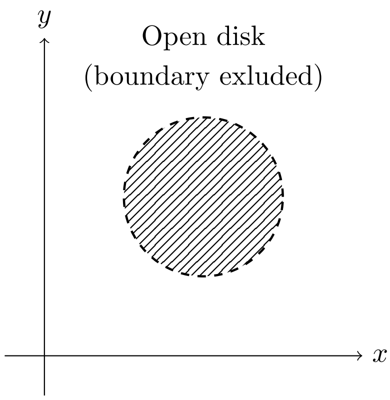
## 11.2 Limits and continuity




Open and closed sets in $\mathbb{R}^2$ and $\mathbb{R}^3$.

- Open disk: $\sqrt{(x-a)^2+(y-b)^2} < r$
- Closed disk: $\sqrt{(x-a)^2+(y-b)^2} \geq r$

Same analogy to open and closed set $]a, b[$ and $[a, b]$.

- Open set→ A set where the boundary is excluded from the domain
- Closed set→ A set where the boundary is included in the domain

**Definition 11.2.** Given the set $S$ define:

- *Interior point* of $S$: a point $P_0$ such that $\exists$ an open set centered in $P_0$ contained entirely within the set $S$.

- *Boundary point* $P_0$ of $S$: every open disk centered in $P_0$ contains both points of $S$ and points outside $S$.

### 11.2.1 Limit of a function of two variables

The limit statement

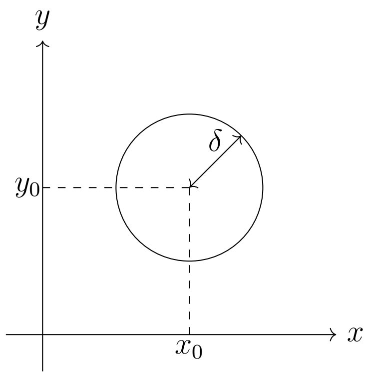
$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

means that for each given number $\varepsilon > 0$, there exists a number $\delta > 0$ so that whenever $(x, y)$ is a point in the domain $D$ of $f$ such that

$$0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta\,,$$

then

$$|f(x, y) - L| < \varepsilon\,.$$




**Example 11.5.**

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + x - xy - y}{x - y}\,.$$

**Solution:** Note that, for $x + y$ we have

$$\frac{x(x+1) - y(y+1)}{x - y} = \frac{(x - y)(x + 1)}{x - y} = x + 1\,.$$

Thus:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} x + 1 = 1\,.$$

**Note:** As in 2D, the limit can be examined from two different directions (left or right). In 3D the limit can be evaluated along **any line**.
If the limit

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

is not the same for all approaches (paths) then the **limit does not exist**.

**Example 11.6.** If $f(x,y) = \dfrac{2xy}{x^2 + y^2}$, show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist by evaluating the limit along the $x$-axis, the $y$-axis and the line $x = y$.

**Solution:**

- $x$-axis$\to$ $y = 0$ : $\displaystyle\lim_{(x,y)\to(0,0)} f(x,0) = \lim_{(x,y)\to(0,0)} \frac{2x(0)}{x^2} = 0$

- $y$-axis$\rightarrow$ $x = 0$ : $\lim\limits_{(x,y)\to(0,0)} f(0,y) = 0$

- $x - y$ line $x = y$ : $\lim\limits_{(x,y)\to(0,0)} f(x,x) = \lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2}{2x^2} = 1$

Therefore, the limit does not exist.

**Example 11.7.** General approach to show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

**Solution:** We consider the general line passing through $(0,0)$: $y = mx$

$$\lim_{(x,y)\to(0,0)} f(x,mx) = \lim_{(x,y)\to(0,0)} \frac{x^2mx}{x^4+(mx)^2} = \lim_{(x,y)\to(0,0)} \frac{mx^3}{x^4+x^2m^2} = \lim_{(x,y)\to(0,0)} \frac{mx}{x^2+m} = 0 \quad \forall m\,.$$

But if we consider the parabolic path $y = x^2$ we have

$$\lim_{(x,y)\to(0,0)} f(x,x^2) = \lim_{(x,y)\to(0,0)} \frac{x^2\cdot x^2}{x^4+(x^2)^2} = \lim_{(x,y)\to(0,0)} \frac{x^4}{2x^4} = \frac{1}{2}\,.$$

Depending on the path, we have different limits $\Rightarrow$ the limit does not exist.

**Basic formulas and rules of limits in two variables.** Suppose $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = M$. Then, for any constant $a \in \mathbb{R}$, we have

- Scalar multiple rule: $\lim\limits_{(x,y)\to(x_0,y_0)} [af](x,y) = aL$

- Sum rule: $\lim\limits_{(x,y)\to(x_0,y_0)} [f+g](x,y) = L + M$

- Product rule: $\lim\limits_{(x,y)\to(x_0,y_0)} [fg](x,y) = LM$

- Quotient rule: $\lim\limits_{(x,y)\to(x_0,y_0)} \left[\dfrac{f}{g}\right](x,y) = \dfrac{L}{M} \quad \text{if } M \neq 0$

**Example 11.8.** Assuming that these limits exist, evaluate

- $\lim_{(x,y)\to(4,3)} x^2 + xy + y^2 = 37\,.$
- $\lim_{(x,y)\to(1,2)} \dfrac{2xy}{x^2+y^2} = \dfrac{\lim_{(x,y)\to(1,2)} 2xy}{\lim_{(x,y)\to(1,2)} x^2+y^2} = \dfrac{4}{5}\,.$

### 11.2.2 Continuity of a function of 2 variables

The function $f(x,y)$ is **continuous** at the point $(x_0,y_0)$ if and only if

1. $f(x_0,y_0)$ is defined;
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists;
3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$;

A function is continuous on a set $S$ if it is continuous at each point in $S$.

**Example 11.9.** Test the continuity of $f(x,y) = \dfrac{1}{y-x^2}$.

**Solution:** The function is a rational function in $x$ and $y$, so it is not continuous when the denominator is $= 0$.
$y - x^2 = 0 \Rightarrow$ not continuous along the parabola $y = x^2\,.$

### 11.2.3 Limits and continuity for functions with 3 variables

The extension from functions with 2 variables is straightforward (same rules).
- Limit:

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L$$

means that for each number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x,y,z) - L| < \varepsilon$$

whenever $(x, y, z)$ is a point of the domain of $f$ such that

$$0 \leq \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} < \delta\,.$$

- Continuity: the function is continuous in $P(x_0, y_0, z_0)$ if

1. $f(x_0, y_0, z_0)$ is defined;
2. $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z)$ exists;
3. $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y) = f(x_0, y_0, z_0)$.

**Example 11.10.** For what points $f(x, y, z) = \dfrac{3}{\sqrt{x^2+y^2-2z}}$ is continuous?

**Solution:** The function is continuous if $x^2 + y^2 - 2z > 0$. Therefore, the continuity is present for the points outside the paraboloid $z < \dfrac{x^2}{2} + \dfrac{y^2}{2}$.

## 11.3 Partial derivatives

The process of differentiating a function of several variables with respect to one of its variables while keeping the other variables fixed is called **partial differentiation** and the resulting derivative is a **partial derivative** of the function.

**Definition 11.3.** If $z = f(x, y)$, then the partial derivatives of $f$ with respect to $x$ and $y$ are the functions $f_x$ and $f_y$, respectively, defined by

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$
$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

provided the limits exist.

**Note:** to compute $f_x(x, y)$ we can consider $y$ as a constant and vice-verse.

**Example 11.11.** Given $f(x, y) = x^3 y + x^2 y^2$, find $f_x, f_y$.

**Solution:**

- $f_x$: $y$ as a constant $\to f_x = 3x^2 y + 2xy^2$
- $f_y$: $x$ as a constant $\to f_y = x^3 + 2x^2 y$

**Alternative notation for partial derivatives.**

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x,y) = z_x = D_x(f)$$
$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x,y) = z_y = D_y(f)$$

The values of $f(x, y)$ derivatives at the point $(a, b)$ are denoted by $\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a, b)$ and $\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a, b)$.

**Example 11.12.** Given $z = x^2 \sin(3x + y^3)$, evaluate $\left.\frac{\partial z}{\partial x}\right|_{(\frac{\pi}{3},0)}$ and $z_y$ at (1,1).

**Solution:**

1.

$$\frac{\partial z}{\partial x} = 2x\ \sin(3x + y^3) + x^2\, 3\ \cos(3x + y^3)$$
$$\left.\frac{\partial z}{\partial x}\right|_{(\frac{\pi}{3},0)} = 2\,\frac{\pi}{3}\ \sin\left(3\,\frac{\pi}{3}\right) + \left(\frac{\pi}{3}\right)^2\ 3\ \cos\left(3\,\frac{\pi}{3}\right) =$$
$$= \frac{2\pi}{3}\ \sin(\pi) + \frac{\pi^2}{3}\ \cos(\pi) = -\frac{\pi^2}{3}\,.$$

2.

$$\frac{\partial z}{\partial y} = x^2\, 3y^2\ \cos\left(3x + y^3\right)$$
$$\left.\frac{\partial z}{\partial y}\right|_{(1,1)} = 3\ \cos(4)\,.$$

**Example 11.13.** Given the 3 variables function $f(x, y, z) = x^2 + 2xy^2 + yz^3$, we have

$$f_x = 2x + 2y^2\,, \qquad f_y = 4xy + z^3\,, \qquad f_z = 3yz^2\,.$$

**Example 11.14** (Partial derivative of an implicitly defined function)**.** Let $z$ be an implicitly defined function

$$x^2 z + yz^3 = x\,.$$

Determine 1) $\frac{\partial z}{\partial x}$ and 2) $\frac{\partial z}{\partial y}$.

**Solution:**

1) We differentiate everything treating $y$ as a constant.

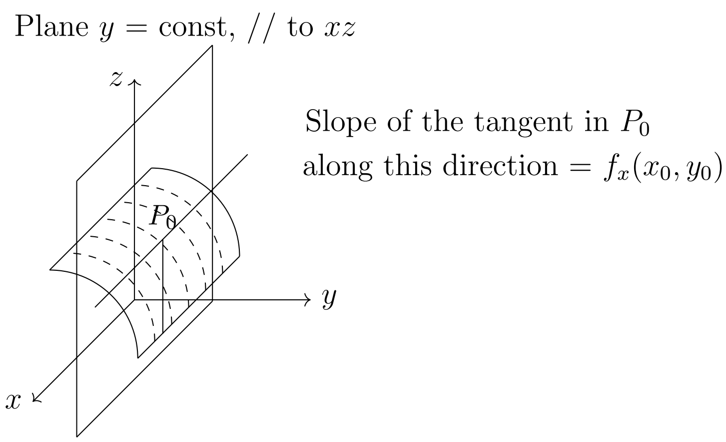
Example: $\dfrac{\partial(x^2 z)}{\partial x} = \dfrac{\partial(x^2)}{\partial x} z + x^2 \dfrac{\partial z}{\partial x}$. So, we have

$$
\begin{aligned}
&2xz + x^2\,\frac{\partial z}{\partial x} + y\,\frac{\partial(z^3)}{\partial x} = 1\\
&2xz + x^2\,\frac{\partial z}{\partial x} + y\,3z^2\,\frac{\partial z}{\partial x} = 1\\
&\frac{\partial z}{\partial x}(x^2 + 3yz^2) = 1 - 2xz \to \frac{\partial z}{\partial x} = \frac{1 - 2xz}{x^2 + 3yz^2}\,.
\end{aligned}
$$

2) We treat $x$ as a constant, and we differentiate in the $y$ variable:

$$
\begin{aligned}
&x^2\,\frac{\partial z}{\partial y} + z^3 + y\,\frac{\partial(z^3)}{\partial y} = 0\\
&x^2\,\frac{\partial z}{\partial y} + z^3 + 3yz^2\,\frac{\partial z}{\partial y} = 0\\
&\frac{\partial z}{\partial y}(x^2 + 3yz^2) \to \frac{\partial z}{\partial y} = -\frac{z^3}{x^2 + 3yz^2}\,.
\end{aligned}
$$

**Partial derivative as a slope** The line parallel to the $xz$-plane and tangent to the surface $z = f(x, y)$ at the point $P_0(x_0, y_0, z_0)$ has slope $f_x(x_0, y_0)$. Likewise, the tangent line to the surface at $P_0$ that is parallel to the $yz$-plane has slope $f_y(x_0, y_0)$.




**Partial derivatives as rate** As the point $(x, y)$ moves from the fixed point $P_0(x_0, y_0)$, the function $f(x, y)$ changes at a rate given by $f_y(x_0, y_0)$ in the direction of the positive $x$-axis and by $f_y(x_0, y_0)$ in the direction of the positive $y$-axis.

**Higher order partial derivatives**

- Second order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx}$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = (f_y)_y = f_{yy}$$

- Mixed second order partial derivatives

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx}$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}$$

**Theorem 11.1** (Equality of mixed partials)**.** If the function $f(x,y)$ has mixed second order partial derivatives $f_{xy}$ and $f_{yx}$ that are continuous in an open disk containing $(x_0, y_0)$, then

$$f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)\,.$$

**Example 11.15.** Given $z = f(x,y) = 5x^2 - 2xy + 3y^3$, determine $\dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y \partial x}, \dfrac{\partial^2 z}{\partial x^2}$.

**Solution:**

1. $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial x}(-2x + 9y^2) = -2\,.$

2. $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial y}(10x - 2y) = -2\,.$

3. $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial}{\partial x}(10x - 2y) = 10\,.$

**Example 11.16.** Given $f(x, y) = x^2ye^y$, find $f_{xy}$, $f_{yx}$, $f_{xx}$, $f_{xxy}$.

**Solution:**

$$\begin{aligned}
f_{xy} &= (f_x)_y = (2xye^y)_y = 2xe^y + 2xye^y = 2xe^y(1+y)\,.\\
f_{yx} &= (f_y)_x = (x^2e^y + x^2ye^y)_x = 2xe^y + 2xye^y = 2xe^y(1+y)\,.\\
f_{xx} &= (f_x)_x = (2xye^y)_x = 2ye^y\,.\\
f_{xxy} &= ((f_x)_x)_y = ((2xye^y)_x)_y = (2ye^y)_y = 2e^y + 2ye^y = 2e^y(1+y)\,.
\end{aligned}$$

**Example 11.17** (Partial differential equation: the Heat Equation)**.**

$$\frac{\partial T}{\partial t} = c^2 \frac{\partial^2 T}{\partial x^2} \qquad T(x,t) \to \text{temperature}$$

Equation for the study of the temperature in a thin rod at position $x$ at time $t$.

Verify that $T(x,t) = e^{-t}\cos\left(\frac{x}{c}\right)$ satisfies the heat equation.

**Solution:**

$$\begin{aligned}
\frac{\partial T(x,t)}{\partial t} &= \frac{\partial\left[e^{-t}\cos\left(\frac{x}{c}\right)\right]}{\partial t} = -e^{-t}\cos\left(\frac{x}{c}\right)\\
\frac{\partial T(x,t)}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial\left[e^{-t}\cos\left(\frac{x}{c}\right)\right]}{\partial x}\right] = \frac{\partial}{\partial x}\left[e^{-t}\frac{1}{c}\left(-\sin\left(\frac{x}{c}\right)\right)\right] =\\
&= \frac{e^{-t}}{c}\frac{1}{c}\left(-\cos\left(\frac{x}{c}\right)\right) = \frac{-e^{-t}}{c^2}\cos\left(\frac{x}{c}\right)\\
\frac{\partial T}{\partial t} = c^2\frac{\partial^2 T}{\partial x^2} \qquad &\Rightarrow \qquad -e^{-t}\cos\left(\frac{x}{c}\right) = c^2\left(\frac{-e^{-t}}{c^2}\cos\left(\frac{x}{c}\right)\right).
\end{aligned}$$

□

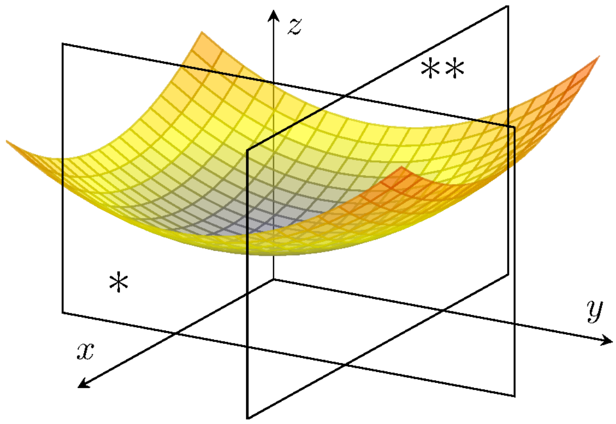
**Example 11.18.** Show that $f_{xyz} = f_{yzx} = f_{zyx}$ for the function $f(x,y,z) = xyz + x^2y^3z^4$.

**Solution:**

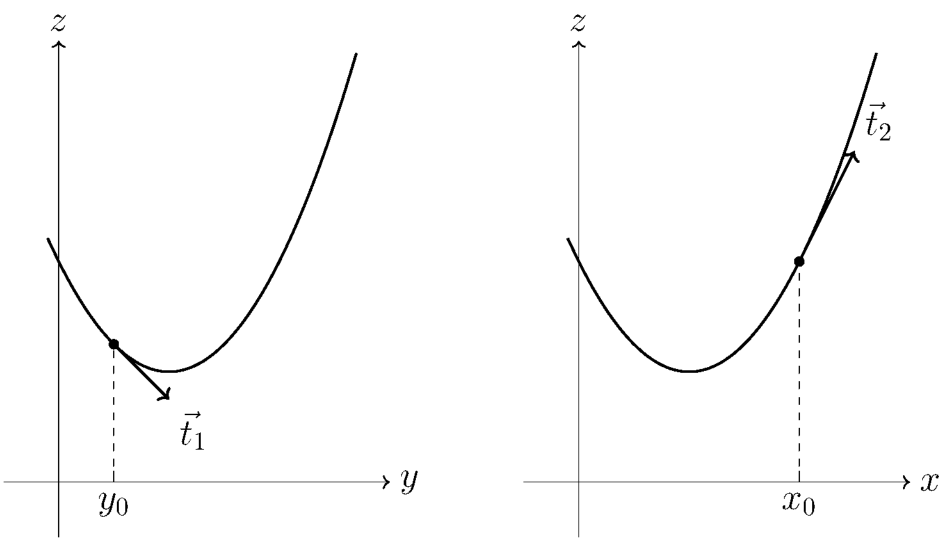
$$f_{xyz} = ((f_x)_y)_z = ((yz + 2xy^3z^4)_y)_z = (z + 6xy^2z^4)_z = 1 + 24xy^2z^3\,.$$
$$f_{yzx} = ((f_y)_z)_x = ((xz + 3x^2y^2z^4)_z)_x = (x + 12x^2y^2z^3)_x = 1 + 24xy^2z^3\,.$$
$$f_{zyx} = \dots$$

## 11.4 Tangent planes, approximations and differentiability

**Tangent planes** Suppose $S$ is a surface with equation $z = f(x, y)$. We consider two planes, $x = x_0$ (*) and $y = y_0$ (**).




On (*) and (**), the surface $S$ reduces to a line with tangent $\vec{t}_1$ and $\vec{t}_2$ respectively, in the point $P_0(x_0, y_0)$.



General equation of a plane passing through $P_0(x_0, y_0, z_0)$:

$$\begin{aligned} A(x-x_0)+B(y-y_0)+C(z-z_0)=0 \\ C(z-z_0)=-A(x-x_0)-B(y-y_0) \\ z-z_0=-\frac{A}{C}(x-x_0)-\frac{B}{C}(y-y_0)\,. \end{aligned}$$

We introduce $a=\frac{A}{C}$ and $b=\frac{B}{C}$ and we obtain

$$z-z_0=a(x-x_0)+b(y-y_0)\,.$$

On the plane (*), the slope of the tangent is the derivative of $f$ along $y$. In (**) the slope of the tangent is the derivative along $x$. Thus we have:

$$a=f_x(x_0,y_0)=\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \qquad b=f_y(x_0,y_0)=\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}\,.$$

**Definition 11.4.** Suppose $S$ is a surface with equation $z=f(x,y)$ and $P_0(x_0,y_0,z_0)$ be a point of $S$ at which a tangent plane exists. Then, the tangent plane to $S$ at $P_0$ is:

$$z-z_0=\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x-x_0)+\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y-y_0)\,.$$

**Example 11.19.** Find the tangent plane to $z=f(x,y)=\tan^{-1}\left(\frac{y}{x}\right)$ at the point $P_0(1,\sqrt{3},\frac{\pi}{3})$.

**Solution:**

$$\begin{aligned} \frac{\partial f}{\partial x}&=\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}=\frac{-\frac{y}{x^2}}{\frac{x^2+y^2}{x^2}}=\frac{-y}{x^2+y^2}\to\left.\frac{\partial f}{\partial x}\right|_{(1,\sqrt{3})}=\frac{-\sqrt{3}}{1+3}=-\frac{\sqrt{3}}{4}\,. \\ \frac{\partial f}{\partial y}&=\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x}=\frac{1}{\frac{x^2+y^2}{x^2}x}=\frac{x}{x^2+y^2}\to\left.\frac{\partial f}{\partial y}\right|_{(1,\sqrt{3})}=\frac{1}{4}\,. \end{aligned}$$

Therefore we have

$$z-\frac{\pi}{3}=-\frac{\sqrt{3}}{4}(x-1)+\frac{1}{4}(y-\sqrt{3})\,.$$

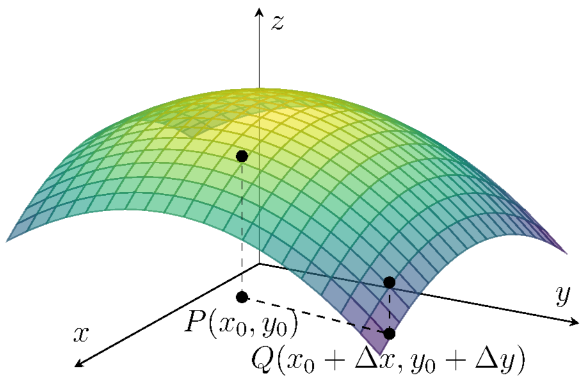
### 11.4.1 Incremental approximations (two variables $f$)

If $f(x, y)$ and its partial derivaties $f_x$ and $f_y$ are defined in an open region $R$ containing the point $P(x_0, y_0)$ and $f_x$ and $f_y$ are continuous at $P$, then

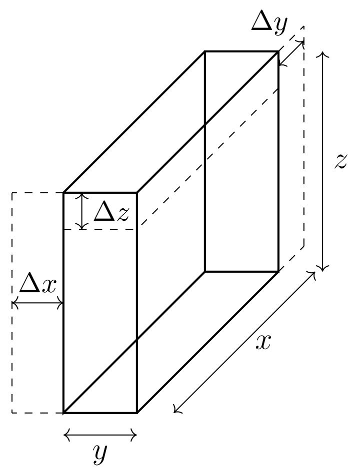
$$\begin{aligned}\Delta f &= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)\\ &\approx \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \Delta y\end{aligned}$$

so that

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \Delta y\,.$$




**Example 11.20.** An open box has length 3 Ft, width 1 Ft and height 2 Ft. Material costs $2/Ft$^2$ for the sides and $3/Ft$^2$ for the bottom. Compute the cost of constructing the box, and then use increments to estimate the change in cost if the length and the width are each increased by 3 in, and the height decreased by 4 in.




**Solution:** The surface area is given by the bottom plus four sides

$$S = xy + 2xz + 2zy\,.$$

Therefore the cost (\$3/Ft$^2$ for the bottom, \$2/Ft$^2$ for the sides) is:

$$C(x, y, z) = 3xy + 2(2xz + 2zy)\,.$$

Partial derivative:

$$C_x = 3y + 4z \qquad C_y = 3x + 4z \qquad C_z = 4x + 4y\,.$$

Change of dimensions

$$\Delta x = \frac{3\text{ in}}{12\text{ in/Ft}} = 0.25\text{ Ft} \qquad \Delta y = 0.25\text{ Ft} \qquad \Delta x = -\frac{4}{12\text{ in/Ft}} = -0.33\text{ Ft}\,.$$

Thus, the change in the total cost is approximated by

$$\begin{aligned}\Delta C &= C_x(3,1,2)\Delta x + C_y(3,1,2)\Delta y + C_z(3,1,2)\Delta z = \\ &= (3\cdot 1 + 4\cdot 2)\,0.25 + (3\cdot 3 + 4\cdot 2)\,0.25 + (4\cdot 3 + 4\cdot 1)(-0.33) \simeq 1.67\,\$\,.\end{aligned}$$

**The total differential** For function of one variable we have: $dy = f'(x)dx$. For two-variable case, the total differential of the function $f(x, y)$ is:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = f_x(x, y)\,dx + f_y(x, y)\,dy\,.$$

For three-variable functions, we have:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\,.$$

**Example 11.21.** Determine the total differential of

a. $f(x, y) = x^2 \ln(3y^2 - 2x)\,.$

b. $f(x, y) = 2x^3 + 5y^4 - 6z\,.$

**Solution:**

a. $df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy = \left[2x\ln(3y^2 - 2x) + x^2\dfrac{-2}{3y^2 - 2x}\right]dx + \left[\dfrac{6x^2y}{3y^2 - 2x}\right]\,.$

b. $df = 6x^2dx + 20y^3dy - 6dz\,.$

### 11.4.2 Differentiability

In two-dimension, the increment of $f(x)$ at a point $x_0$ is

$$\Delta f = f(x_0 + \Delta x) - f(x_0) = f'(x_0)\Delta x + \varepsilon\Delta x$$

where $\varepsilon \to 0$ as $\Delta x \to 0\,.$
For two-variable functions, the differentiability can be defined as:

**Definition 11.5.** The function $f(x, y)$ is **differentiable** at $(x_0, y_0)$ if the increment of $f$ can be expressed as

$$\Delta f = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$, respectively. In addition, $f(x, y)$ is said to be **differentiable in the region $\mathcal{R}$** if $f$ is differentiable $\forall\,(x, y) \in \mathcal{R}$.

**Theorem 11.2** (Differentiability implies continuity)**.** If $f(x, y)$ is differentiable at $(x_0, y_0)$, it is also continuous there.

**Example 11.22** (A non-differentiable function for which $f_x$ and $f_y$ exist.)**.** Let

$$f(x,y) = \begin{cases} 1 & \text{if } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

Show that the partial derivatives $f_x$ and $f_y$ exist at the origin, but $f$ is not differentiable there.

**Solution:** Since $f(0,0) = 0$ on all the $x$-axis and $y$-axis, we have

$$\begin{aligned} f_x(0,0) &= \lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x} = 0 \\ f_y(0,0) &= \lim_{\Delta y \to 0} \frac{f(0, 0+\Delta y) - f(0,0)}{\Delta y} = 0\,. \end{aligned}$$

The partial derivatives both exist at the origin. If $f(x,y)$ were differentiable at the origin, it would have to be continuous there. Thus, we can show that $f$ is **not** differentiable by showing that is **not** continuous at $(0,0)$.

$$\left.\begin{aligned} x = y: & \lim_{(x,x)\to(0,0)} f(x,x) = 1 \\ x = 0: & \lim_{(0,y)\to(0,0)} f(0,y) = 0 \end{aligned}\right\} \text{not continuous} \to \text{not differentiable.}$$

**Theorem 11.3** (Sufficient condition for differentiability)**.** If $f$ is a function and $f$, $f_x$ and $f_y$ are continuous in a disk $D$ centered in $(x_0, y_0) \to f$ is differentiable at $(x_0, y_0)$.

## 11.5 Chain rules

If the two variables $x$ and $y$ of a function $f(x, y)$ can be both written as a function of a parameter $t$, then $z = f(x(t), y(t))$ and the following theorem can be stated.

**Theorem 11.4.** Let $f(x, y)$ be a differentiable function of $x$ and $y$, and let $x = x(t)$ and $y = y(t)$ be differentiable function of $t$. Then, $z = f(x, y)$ is a differentiable function of $t$, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}\,.$$

**Example 11.23.** Let $z = x^2 + y^2$, with $x = \dfrac{1}{t}$ and $y = t^2$. Compute $\dfrac{dz}{dt}$ in two ways:

1) expressing $x$ explicitly in terms of $t$;

2) using the chain rule.

**Solution:**

1) $z = \left(\dfrac{1}{t}\right)^2 + t^4 \Rightarrow \dfrac{dz}{dt} = -2t^{-3} + 4t^3 = -\dfrac{2}{t^3} + 4t^3\,.$

2) $\dfrac{dx}{dt} = \dfrac{d\left(\frac{1}{t}\right)}{dt} = -\dfrac{1}{t^2}$ $\qquad \dfrac{dy}{dt} = \dfrac{d\left(t^2\right)}{dt} = 2t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = 2x \cdot \left(-\frac{1}{t^2}\right) + 2y \cdot (2t) = 2 \cdot \frac{1}{t} \cdot \left(-\frac{1}{t^2}\right) + 2t^2 \cdot (2t^2) = -\frac{2}{t^3} + 4t^3\,.$$

**Example 11.24.** Let $z = \sqrt{x^2 + 2xy}$, where $x = \cos\theta$ and $y = \sin\theta$. Find $\dfrac{dz}{d\theta}$ in term of $x, y$ and $z$.

**Solution:**

$$\frac{\partial z}{\partial x} = \frac{1}{2}(x^2 + 2xy)^{\frac{1}{2}-1}(2x + 2y) = \frac{1}{2\sqrt{x^2 + 2xy}}(2x + 2y) = \frac{x + y}{\sqrt{x^2 + 2xy}}\,,$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(x^2 + 2xy)^{\frac{1}{2}-1}2x = \frac{x}{\sqrt{x^2 + 2xy}}\,.$$

Therefore, we have

$$\frac{dz}{d\theta} = \frac{\partial z}{\partial x}\frac{dx}{d\theta} + \frac{\partial z}{\partial y}\frac{dy}{d\theta} = \frac{x+y}{\sqrt{x^2+2xy}}(-\sin\theta) + \frac{x}{\sqrt{x^2+2xy}}(\cos\theta) = \frac{-(x+y)\sin\theta + x\cos\theta}{\sqrt{x^2+2xy}}.$$

**Theorem 11.5** (Implicit function theorem)**.** Let $F$ be defined on a disk containing $(a, b)$ as an interior point, such that $F(a, b) = 0$, and assume $F_x$ and $F_y$ both continuous on the disk, with $F_y(a, b) \neq 0$. Then, there exist an interval $I$ on the real line containing $a$ as an interior point and a unique function $y = y(x)$ defined on the interval $I$, such that $y(a) = b$ and $F(x, y(x)) = 0$, for every value $x$ on the interval $I$. Furthermore, the derivative of $y$ is given by

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

**Example 11.25** (Implicit differentiation using partial derivatives.)**.** If $y$ is a function of $x$ such that

$$\sin(x+y) + \cos(x-y) = y,$$

find $\frac{dy}{dx}$.

**Solution:** Let $F(x, y) = \sin(x + y) + \cos(x - y) - y$ so that $F(x, y) = 0$.
Then

$$F_x(x,y) = \cos(x+y) - \sin(x-y)$$
$$F_y(x,y) = \cos(x,y) + \sin(x-y) - 1.$$

By using the theorem: $\frac{dy}{dx} = -\frac{F_x}{F_y}$

$$\Rightarrow \frac{dy}{dx} = -\frac{\cos(x+y) - \sin(x-y)}{\cos(x+y) + \sin(x-y) - 1}.$$

**Extension of the chain rule** Now we consider $z = F(x, y)$, where $x = x(u, v)$ and $y = y(u, v)$ are both functions of independent parameters $u$ and $v$. Then, using the chain rule we can find $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$.

**Theorem 11.6** (Chain rule for two independent parameters)**.** Suppose $z = f(x, y)$ is differentiable at $(x, y)$ and the partial derivatives of $x$ and $y$ exist at $(u, v)$. Then, the composite function $z = f[x(u, v), y(u, v)]$ is differentiable at $(u, v)$ with

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \qquad \text{and} \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}\,.$$

**Example 11.26.** Let $z = 4x - y^2$, where $x = uv^2$ and $y = u^3v$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

**Solution:**

$$\frac{\partial z}{\partial x} = 4 \qquad \frac{\partial z}{\partial y} = -2y$$
$$\frac{\partial x}{\partial u} = v^2 \qquad \frac{\partial x}{\partial v} = 2uv$$
$$\frac{\partial y}{\partial u} = 3u^2v \qquad \frac{\partial y}{\partial v} = u^3$$

$$\Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = 4v^2 + (-2y)(3u^2v) = 4v^2 - 2(u^3v)(3u^2v) = 4v^2 - 6u^5v^2\,.$$
$$\Rightarrow \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = 4(2uv) - 2yu^3 = 8uv - 2(u^3v)u^3 = 8uv - 2u^6v\,.$$

**Example 11.27** (Implicit differentiation with chain rule)**.** Let $z = u + f(u^2v^2)$, with $f$ differentiable. Show that $u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = u$.

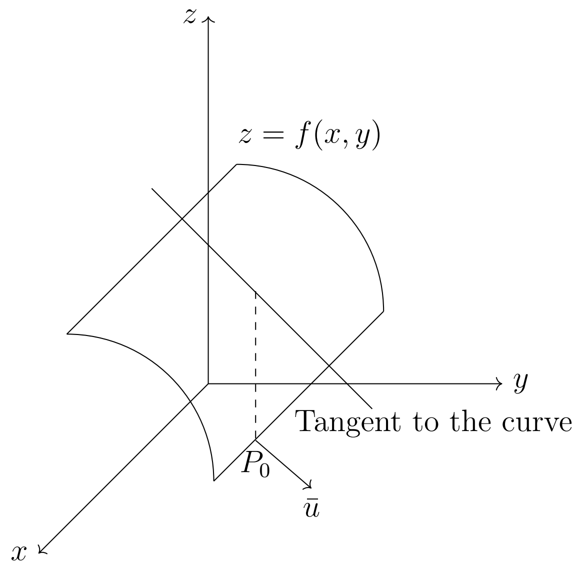
**Solution:** Let $w = u^2v^2 \Rightarrow z = u + f(w)$.

$$\frac{\partial z}{\partial u} = 1 + \frac{df}{dw}\frac{\partial w}{\partial u} = 1 + \frac{df}{dw}(2uv^2) \qquad \text{and} \qquad \frac{\partial z}{\partial v} = \frac{df}{dw}\frac{\partial w}{\partial v} = \frac{df}{dw}(2u^2v)$$

$$\Rightarrow u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = u\left[1 + \frac{df}{dw}(2uv^2)\right] - v\left[\frac{df}{dw}(2u^2v)\right] = u + \frac{df}{dw}2u^2v^2 - \frac{df}{dw}2u^2v^2 = u\,.$$

## 11.6 Directional derivatives and the gradient

To determine the slope of a line tangent to a curve at a point $P_0(x_0, y_0)$, with function $z = f(x, y)$, we need to specify the **direction** in which we wish to measure. We know that in $\hat{\imath}$-direction $\rightarrow \dfrac{\partial f}{\partial x}$; and in the $\hat{\jmath}$-direction $\rightarrow \dfrac{\partial f}{\partial y}$.
To consider the generic direction, we consider a **unit** vector $\bar{u} = u_1\hat{\imath} + u_2\hat{\jmath}$.




### 11.6.1 Directional derivative

Let $f$ be a function of two variables, and let $\bar{u} = u_1\hat{\imath} + u_2\hat{\jmath}$ be a unit vector. The **directional derivative of $f$ at $P_0(x_0, y_0)$ in the direction of $\bar{u}$ is given by**

$$D_{\bar{u}} f(x_0, y_0) = \lim_{(x,y)\to(x_0,y_0)} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

provided the limit exists.

**Theorem 11.7** (Directional derivative using partial derivatives)**.** Let $f(x, y)$ be a function that is differentiable at $P_0(x_0, y_0)$. Then, $f$ has a directional derivative in direction $\bar{u} =$

$u_1\hat{i} + u_2\hat{j}$ given by:

$$D_{\bar{u}}f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2\,.$$

**Example 11.28.** Find the directional derivative of the function $f(x, y) = 3 - 2x^2 + y^3$ at the point $P(1, 2)$ in the direction of the unit vector $\bar{u} = \frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j}$.

**Solution:**

$$f_x = -4x \qquad f_y = 3y^2$$

$$D_{\bar{u}}f(1, 2) = f_x(1, 2)\frac{1}{2} - f_y(1, 2)\frac{\sqrt{3}}{2} = -4 \cdot 1 \cdot \frac{1}{2} - 3 \cdot (2)^2 \cdot \frac{\sqrt{3}}{2} = -2 - 6\sqrt{3}\,.$$

**Note**: The value of the directional derivative can be interpreted as the slope of the line tangent to the curve with direction $\bar{u}$.

### 11.6.2 The gradient

Let $f$ be a differentiable function at $(x, y)$, and let $f(x, y)$ have partial derivatives $f_x(x, y)$ and $f_y(x, y)$. Then, the **gradient of** $f$ denoted by $\nabla f$, is a vector given by

$$\nabla f(x, y) = f_x(x, y)\hat{i} + f_y(x, y)\hat{j}\,.$$

The value of the gradient at a point $P_0(x_0, y_0)$ is denoted by

$$\nabla f_0 = f_x(x_0, y_0)\hat{i} + f_y(x_0, y_0)\hat{j}\,.$$

**Example 11.29.** Find $\nabla f(x, y)$ for $f(x, y) = x^2y + y^3$.

**Solution:**

$$f_x = 2xy \qquad f_y = x^2 + 3y^2$$

$$\nabla f(x, y) = 2xy\hat{i} + (x^2 + 3y^2)\hat{j}\,.$$

**Theorem 11.8** (The gradient formula for directional derivative)**.** If $f$ is a differentiable function of $x$ and $y$, then the directional derivative of $f$ at the point $P_0(x_0, y_0)$ in the direction of the unit vector $\bar{u}$ is

$$D_{\bar{u}}f(x_0, y_0) = \nabla f_0 \cdot \bar{u}\,.$$

**Example 11.30.** Find the directional derivative of $f(x,y)$, with $f(x,y) = \ln(x^2+y^3)$ at $P_0(1,-3)$ in the direction of $\bar{v} = 2\hat{\imath} - 3\hat{\jmath}$.

**Solution:**

$$f_x = \frac{1}{x^2+y^3}\cdot 2x \qquad f_y = \frac{1}{x^2+y^3}\cdot 3y^2$$

$$f_x(1,-3) = \frac{2\cdot 1}{1-27} = \frac{-1}{13} \qquad f_y(1,-3) = \frac{3\cdot 9}{1-27} = -\frac{27}{26}$$

$$\nabla f_0 = \nabla f(1,-3) = -\frac{1}{13}\hat{\imath} - \frac{27}{26}\hat{\jmath}$$

$$\bar{u} = \frac{\bar{v}}{||\bar{v}||} = \frac{2\hat{\imath} - 3\hat{\jmath}}{\sqrt{4+9}} = \frac{2\hat{\imath}}{\sqrt{13}} - \frac{3\hat{\jmath}}{\sqrt{13}}$$

$$D_{\bar{u}}f(x_0,y_0) = \nabla f_0 \cdot \bar{u} = -\frac{1}{26}(2\hat{\imath} + 27\hat{\jmath}) \cdot \left(\frac{2\hat{\imath}}{\sqrt{13}} - \frac{3\hat{\jmath}}{\sqrt{13}}\right) = \frac{77\sqrt{13}}{288}\,.$$

**Basic properties of the gradient**

- Constant rule: $\nabla c = \bar{0}$ (for any constant $c$)
- Linearity rule: $\nabla(af + bg) = a\nabla f + b\nabla g$ (with $a, b$ constants)
- Product rule: $\nabla(f\cdot g) = f\nabla g + g\nabla f$
- Quotient rule: $\nabla\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$, with $g \neq 0$
- Power rule: $\nabla(f^n) = nf^{n-1}\nabla f$

**Maximal property of the gradient.** Depending on the direction of the derivative at one point $P_0(x_0, y_0)$, we obtain different values of the slope of the lines tangent to the curve $f$ in $P_0$. We now aim to find the **largest** value of the directional derivative.

**Theorem 11.9.** Suppose $f$ is differentiable at the point $P_0$, and $\nabla f|_{P_0} = \nabla f_0 \neq 0$. Then

1) The largest value of the directional derivative $D_{\bar{u}} f$ at $P_0$ is $||\nabla f_0||$ and occurs when the unit vector $\bar{u}$ points in the direction of $\nabla f_0$.

2) The smallest value of $D_{\bar{u}} f$ at $P_0$ is $-||\nabla f_0||$ and occurs when $\bar{u}$ points in the direction of $-\nabla f_0$.

**Note**: This theorem means that the function $f$ increases most rapidly in the direction of the gradient $\nabla f_0$ and decreases most rapidly in the opposite direction.

**Example 11.31.** Given $f(x, y) = xe^{2y-x}$, find the direction where $f$ shows the most rapid increase in $P_0(2, 1)$.

**Solution:** We begin with the gradient: $\nabla f = f_x \hat{i} + f_y \hat{j}$.

$$f_x = e^{2y-x} + x \cdot e^{2y-x} \cdot (-1) = e^{2y-x}(1-x)\,, \qquad f_y = x \cdot e^{2y-x} \cdot 2 = 2xe^{2y-x}\,.$$

$$\nabla f = \left[e^{2y-x}(1-x)\right]\hat{i} + \left[2xe^{2y-x}\right]\hat{j}$$
$$\nabla f_0 = \nabla f(2,1) = e^{2-2}(1-2)\hat{i} + \left[2 \cdot 2 \cdot e^{2-2}\right]\hat{j} = -\hat{i} + 4\hat{j}$$

The increase is $||\nabla f_0|| = \sqrt{1+16} = \sqrt{17}\,.$
The maximum decrease is $-\nabla f_0 = \hat{i} - 4\hat{j}\,.$

**Three-variable gradient**

| | |
|---|---|
| $\nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$ | Gradient |
| $D_{\bar{u}} f = \nabla f_0 \cdot \bar{u}$ | Directional derivative |

**Example 11.32.** Consider the function $f(x, y, z) = xy\sin(xz)$ and the point $P_0(1, -2, \pi)$. Find:

1) $\nabla f_0$ (the gradient in $P_0$).

2) The directional derivative in $P_0$ in the direction $\nabla = -2\hat{i} + 3\hat{j} - 5\hat{k}\,.$

**Solution:**

1) $\nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$

$$f_x = y\sin(xz) + xy\cos(xz)\cdot z = y\sin(xz) + xyz\cos(xz)$$
$$f_y = x\sin(xz)$$
$$f_z = xy\cos(xz)\cdot x = x^2y\cos(xz)$$

$\nabla f = [y\sin(xz) + xyz\cos(xz)]\,\hat{i} + [x\sin(xz)]\,\hat{j} + \left[x^2y\cos(xz)\right]\hat{k}$

$\nabla f_0 = \nabla f|_{(1,-2,\pi)} = [-2\cdot\sin(\pi) + (-2\pi)\cdot\cos(\pi)]\,\hat{i} + [\sin(\pi)]\,\hat{j} + [-2\cos(\pi)] + \hat{k} = 2\pi\hat{i} + 2\hat{k}\,.$

2) To find $D_{\bar{u}}f$ we need $\bar{u}$:

$$\bar{u} = \frac{\bar{v}}{||\bar{v}||} = \frac{-2\hat{i}+3\hat{j}-5\hat{k}}{\sqrt{4+9+25}} = \frac{1}{\sqrt{38}}\cdot(-2\hat{i}+3\hat{j}-5\hat{k})\,.$$

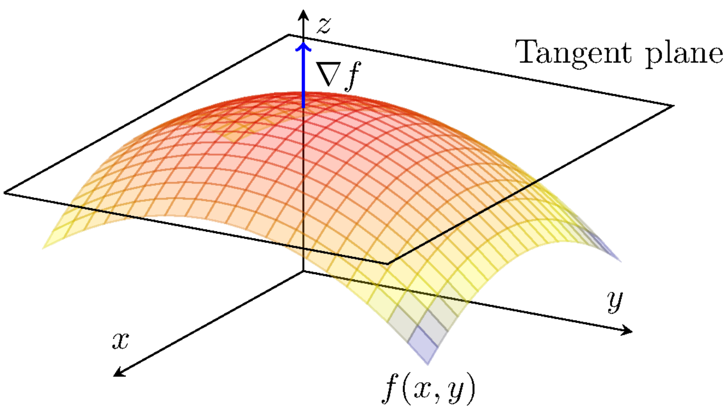
Finally we have

$$D_{\bar{u}}f(1,-2,\pi) = \nabla f_0\cdot\bar{u} = \frac{1}{\sqrt{38}}(-2\cdot 2\pi - 10) = \frac{-4\pi-10}{\sqrt{38}}\,.$$

**Normal property of the gradient**

**Theorem 11.10.** Suppose that the function $f$ is differentiable at the point $P_0$, and that the gradient at $P_0$ satisfies $\nabla f_0 \neq 0$. Then, $\nabla f_0$ is **orthogonal** to the **level surface** of $f$ through $P_0$.

**Note**: The gradient is orthogonal to the tangent vectors at one point.



**Example 11.33** (Finding a vector normal to a surface)**.** Find a vector that is normal to the surface $f(x,y,z) = x^2 + 2xy - yz + 3z^2 = 7$ at the point $P_0(1,1,-1)$.

**Solution:** Note that $f(x,y,z)$ represents a level surface. Computing the gradient we have

$$\begin{aligned}\nabla f &= (2x+2y)\hat{\imath} + (2x-z)\hat{\jmath} + (-y+6z)\hat{k}\\ \nabla f_0 &= \nabla f(1,1,-1) = 4\hat{\imath} + 3\hat{\jmath} - 7\hat{k} \to \text{required normal.}\end{aligned}$$

**Example 11.34** (Finding a vector normal to a level curve)**.** Find the level curve for $c = 1$ of the curve $f(x,y) = x^2 - y^2$ and find a normal vector at the point $P_0(2,\sqrt{3})$.

**Solution:** Level curve: $x^2 - y^2 = 1$.

$$\begin{aligned}\nabla f &= 2x\hat{\imath} - 2y\hat{\jmath}\\ \nabla f_0 &= \nabla f(2,\sqrt{3}) = 4\hat{\imath} - 2\sqrt{3}\hat{\jmath} \to \text{required normal.}\end{aligned}$$

### 11.6.3 Tangent planes and normal line to a surface

Suppose $S$ is a surface with the equation $F(x,y,z) = C$, and let $P_0(x_0,y_0,z_0)$ be a point on $S$ where $F$ is differentiable with $\nabla f \neq 0$. Then, the equation of the **tangent plane** to $S$ at $P_0$ is

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

and the **normal line** to $S$ at $P_0$ is:

$$\begin{cases} x = x_0 + F_x(x_0,y_0,z_0)t\\ y = y_0 + F_y(x_0,y_0,z_0)t\\ z = z_0 + F_z(x_0,y_0,z_0)t\,. \end{cases}$$

**Example 11.35.** Find the tangent plane and the tangent normal line at the point $P_0(1,-1,2)$ on the surface $S$ given by $x^2y + y^2z + z^2x = 5$.

**Solution:** We consider $F(x,y,z) = x^2y + y^2z + z^2x$. The problem is to consider the level surface $F(x,y,z) = 5$. We first find the gradient

$$\nabla F(x,y,z) = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k} = (2xy + z^2)\hat{\imath} + (x^2 + 2yz)\hat{\jmath} + (y^2 + 2xz)\hat{k}\,.$$

$$\begin{aligned}\nabla F_0 = \nabla F_0(1,-1,2) \qquad &= (-2+4)\hat{\imath} + (1-4)\hat{\jmath} + (1+4)\hat{k} = \\ &= 2\hat{\imath} - 3\hat{\jmath} + 5\hat{k} \rightarrow \text{normal to } F \Rightarrow \text{normal to the plane.}\end{aligned}$$

The tangent plane is:

$$\begin{aligned}&2(x-x_0) - 3(y-y_0) + 5(z-z_0) = 0\\ &2(x-1) - 3(y+1) + 5(z-2) = 0\\ &2x - 3y + 5z = 15\,.\end{aligned}$$

The normal line is:

$$\begin{cases} x - 1 = 2t \\ y + 1 = -3t \\ z - 2 = 5t\,. \end{cases}$$

## 11.7 Extrema of functions of two variables

**Absolute extrema.** The function $f(x, y)$ is said to have an absolute maximum at $(x_0, y_0)$ if $f(x_0, y_0) \geq f(x, y)$, $\forall (x, y)$ in the domain $D$ of $f$. Similarly, $f$ has an absolute minimum at $(x_0, y_0)$ if $f(x_0, y_0) \leq f(x, y)$ for all $(x, y)$ in $D$. The absolute maxima and minima are called absolute extrema.

**Relative extrema.** Let $f$ be a function defined on a region containing $(x_0, y_0)$. Then,

- $f(x_0, y_0)$ is a relative maximum if $f(x, y) \leq f(x_0, y_0)$ $\forall$ $(x, y) \in$ an open disk containing $(x_0, y_0)$;
- $f(x_0, y_0)$ is a relative minimum if $f(x, y) \geq f(x_0, y_0)$ $\forall$ $(x, y) \in$ an open disk containing $(x_0, y_0)$.

**Theorem 11.11** (Partial derivatives criteria for relative extrema)**.** If $f$ has a relative extremum (maximum or minimum) and partial derivatives $f_x$ and $f_y$ both exist at $(x_0, y_0)$, then

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0\,.$$

### 11.7.1 Critical points

**Definition 11.6.** A critical point of a function $f$ defined on an open set $D$ is a point $(x_0, y_0)$ in $D$ where either one of the following is true

1) $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$;
2) At least one of $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

**Example 11.36** (Distinguish critical point)**.** Discuss the nature of the critical point $(0, 0)$ for the quadric surfaces

$$\text{a) } z = x^2 + y^2 \qquad \text{b) } z + x^2 + y^2 = 1 \qquad \text{c) } z = y^2 - x^2\,.$$

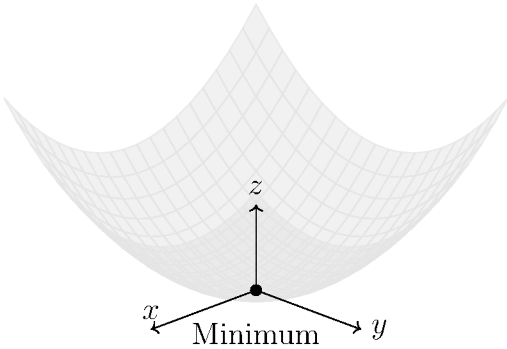
**Solution:**

a) $z = k \to x^2 + y^2 = k$

Ellipse if $k \geq 0$

$(0, 0)$ is a minimum for $f(x, y) = z = x^2+y^2$

$f_x = 2x$
$f_y = 2y$

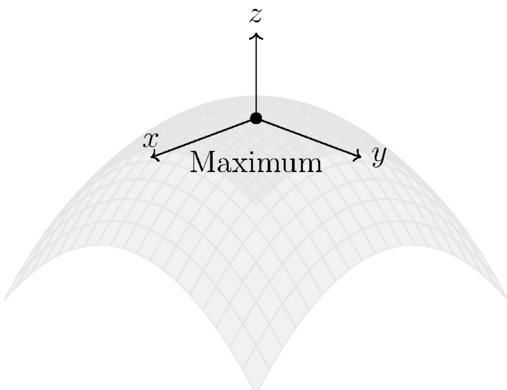



b) $z = k \to x^2 + y^2 = 1 - k$

Ellipse if $1 - k \geq 0 \to k \leq 1$

$(0, 0)$ is a maximum
for $z = f(x, y) = 1 - x^2 - y^2$

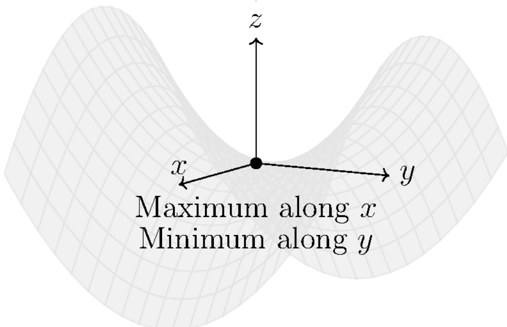
$f_x = -2x$
$f_y = -2y$




c) $z = k \to y^2 - x^2 = k \to$ Hyperbole

$$\left.\begin{aligned} f_x &= -2x \\ f_y &= 2y \end{aligned}\right\} = 0 \text{ in } (0, 0)$$

$\to (0, 0)$ is a critical point




**Saddle point:** a critical point where every open disk contains both points such that $f(x_0, y_0) \leq f(x, y)$ and $f(x_0, y_0) \geq f(x, y)$.

**Theorem 11.12** (Second partial test)**.** Let $f(x, y)$ have a critical point at $P_0(x_0, y_0)$ and assume $f$ has continuous second-order partial derivatives in a disk centered at $(x_0, y_0)$. The **discriminant** of $f$ is the expression

$$D = f_{xx}f_{yy} - f_{xy}^2 .$$

Then:

1) A **relative maximum** occurs at $P_0$ if $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$ (or, equivalently, $D(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) < 0$).

2) A **relative minimum** occurs at $P_0$ if $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$ (or $f_{yy}(x_0, y_0) > 0$).

3) A **saddle point** occurs at $P_0$ if $D(x_0, y_0) < 0$.

4) If $D(x_0, y_0) = 0$, the test is **inconclusive**. Further analysis needed.

**Note:** The discriminant formula can be remembered by considering the determinant of the matrix

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} .$$

**Example 11.37.** Find all relative extrema and saddle points of: $f(x, y) = 2x^2 + 2xy + y^2 - 2x - 2y + 5$.

**Solution:**

$$f_x = 4x + 2y - 2 \qquad\qquad f_y = 2x + 2y - 2\,.$$

We want $f_x = f_y = 0$.

$$\begin{cases} 4x + 2y - 2 = 0 \\ 2x + 2y - 2 = 0\,. \end{cases}$$

Subtracting the second equation to the first we obtain

$$\begin{cases} 2x = 0 \\ 2x + 2y - 2 = 0 \end{cases} \quad \to \quad \begin{cases} x = 0 \\ 2y = 2 \end{cases} \quad \to \quad \begin{cases} x = 0 \\ y = 1\,. \end{cases}$$

The critical point is $P(0, 1)$. Since we have $f_{xx} = 4$, $f_{yy} = 2$ and $f_x y = 2$, we compute

$$D = f_{xx} f_{yy} - f_{xy}^2 = 4 \cdot 2 - 2^2 = 4 > 0\,.$$

Since $f_{xx} > 0$ we conclude that we have a minimum at $P(0, 1)$.

**Example 11.38.** Find the critical points of $f(x,y) = 8x^3 - 24xy + y^3$ and classify each point.

**Solution:**

$$f_x = 24x^2 - 24y \qquad \Rightarrow \qquad \begin{cases} f_x = 24x^2 - 24y = 0 \\ f_y = -24x + 3y^2 = 0 \end{cases}$$
$$f_y = -24x + 3y^2$$

$$\begin{cases} -24y = -24x^2 \\ -24x + 3y^2 = 0 \end{cases} \qquad \Rightarrow \qquad \begin{cases} y = x^2 \\ -24x + 3y^2 = 0\,. \end{cases}$$

Substituting $y$ in the second equation we have

$$-24x + 3x^4 = 0 \quad \rightarrow \quad 3x(-8 + x^3) = 0$$

$$x = 0 \quad \vee \quad x^3 - 8 = 0$$
$$x^3 = 8$$
$$x = 2\,.$$

Therefore, we have $P_1(0,0)$ and $P_2(2,4)$. Computing the second derivatives we have

$$f_{xx} = 48x \qquad f_{yy} = 6y \qquad f_{xy} = -24$$

$$\Rightarrow D = 48x \cdot 6y - (-24)^2 = 288xy - 576$$

$P_1(0,0) \rightarrow \quad D(0,0) = -576 < 0 \rightarrow$ Saddle point at $(0,0)$.

$P_2(2,4) \rightarrow \quad D(2,4) = 288 \cdot 8 - 576 = 1728 > 0$

$f_{xx}(2,4) = 48 \cdot 2 = 96 > 0 \quad \rightarrow$ Relative minimum at $(2,4)$.

### 11.7.2 Absolute extrema of continuous functions

**Theorem 11.13** (Extreme value for a function of two variables)**.** A function of two variables $f(x,y)$ attains both an absolute maximum and an absolute minimum on any closed, bounded set $S$ where it is continuous.

**Procedure to find the absolute extrema:** Given a continuous functions $f(x,y)$ on a closed, bounded set $S$:

Step 1: Find all the critical points of $f$ in $S$.

Step 2: Find all the points on the boundary of $S$ where an absolute extrema can occur (e.g. boundary points, critical points, endpoints).

Step 3: Compute the value of $f(x_0, y_0)$ for each of the points $(x_0, y_0)$ found in Step 1 and Step 2.

Step 4: The absolute maximum of $f$ on $S$ is the largest of the values computed in Step 3, and the absolute minimum is the smallest of the computed values.

**Example 11.39.** Find the absolute extrema of the function $f(x, y) = e^{x^2-y^2}$ over the disk $x^2 + y^2 \leq 1$.

**Solution:**

Step 1:

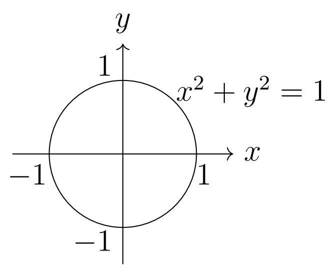
$$f_x = e^{x^2-y^2} \cdot 2x \qquad \begin{cases} e^{x^2-y^2} \cdot 2x = 0 \\ e^{x^2-y^2} \cdot (-2y) = 0 \end{cases} \qquad \begin{cases} x = 0 \\ y = 0 \end{cases} \quad \Rightarrow P(0,0) \text{ is a critical point}$$
$$f_y = e^{x^2-y^2} \cdot (-2y)$$

Step 2:
Examine the values of the function on the boundary: $x^2 + y^2 = 1 \quad \Rightarrow \quad y^2 = 1 - x^2$

$$f(x, y) = e^{x^2-y^2} = e^{x^2-(1-x^2)} = e^{2x^2-1}\,.$$

Consider now the single-valued function $F(x)$ :

$$F(x) = e^{2x^2-1}, \quad F'(x) = e^{2x^2-1}4x\,.$$




We considered the interval: $-1 \leq x \leq 1$.

$F'(x) = 0 \quad \to \quad e^{2x^2-1}4x = 0 \quad \Rightarrow \quad x = 0\,.$
$y^2 = 1 - x^2 \quad \to \quad y^2 = 1 \quad \Rightarrow \quad y = \pm 1\,.$

Endpoint of the interval $\Rightarrow x = \pm 1 \quad \Rightarrow \quad y = 0\,.$

Points: $(-1, 0)$, $(1, 0)$, $(0, -1)$, $(0, 1)$.

| Points to check | $f(x_0, y_0) = e^{x_0^2 - y_0^2}$ | |
|---|---|---|
| $(0, 0)$ | $e^0 = 1$ | |
| $(0, 1)$ | $e^{-1}$ | Minimum |
| $(0, -1)$ | $e^{-1}$ | Minimum |
| $(1, 0)$ | $e^1$ | Maximum |
| $(-1, 0)$ | $e^1$ | Maximum. |

## 11.8 Lagrange multipliers

Many applied problems of two variables have to be optimized subject to a restriction to a constraint on the variables.

**Theorem 11.14** (Lagrange's theorem)**.** Assume that $f$ and $g$ have first partial derivatives and that $f$ has an extremum at $P_0(x_0, y_0)$ when restricted to the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq 0$, there is a number $l$ such that

$$\nabla f(x_0, y_0) = l\, \nabla g(x_0, y_0)\,.$$

**Constrained optimization problems (method of Lagrange multipliers)** Suppose $f$ and $g$ satisfy the hypotheses of Lagrange's theorem, and that $f(x, y)$ has an extremum subject to the constraint $g(x, y) = c$. To find the extreme value, proceed as follows:

Step 1: Simultaneously solve the following three equations for $x$, $y$ and $l$

$$f_x(x, y) = l\, g_x(x, y)\,, \qquad f_y(x, y) = l\, g_y(x, y)\,, \qquad g(x, y) = c\,.$$

Step 2: Evaluate $f$ at all points found in Step 1 and all the points of the boundary of the constraint. The extremum we seek must be among these values.

**Example 11.40** (Optimization with Lagrange multipliers)**.** Given that the smallest and the largest values of $f(x, y) = 1 - x^2 - y^2$ subject to the constraint $x + y = 1$ with $x \geq 0$, $y \geq 0$ exists, use the method of the Lagrange multipliers to find these extrema.

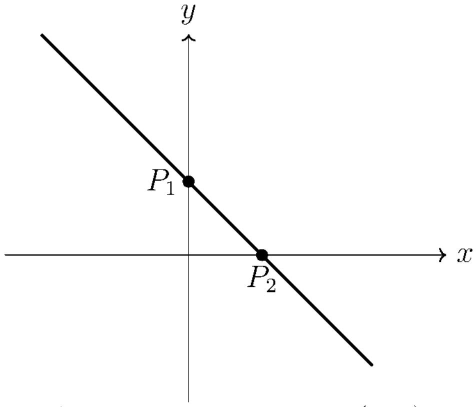
**Solution:** Constraint: $x + y = 1 \quad \Rightarrow \quad g(x, y) = x + y \quad (c = 1)\,.$

$$f_x = -2x \qquad f_y = -2y \qquad g_x(x, y) = 1 \qquad g_y(x, y) = 1\,.$$

Then, the system is:

$$\begin{cases} -2x = l \cdot (1) \\ -2y = l \cdot (1) \\ x + y = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} l = -2x \\ -2y = -2x \\ x + y = 1 \end{cases} \quad \begin{cases} l = -2x \\ x = y \\ 2x = 1\,. \end{cases}$$

$$\begin{cases} x = \dfrac{1}{2} \\ y = \dfrac{1}{2} \\ l = -1 \end{cases} \qquad \Rightarrow \qquad f\left(\frac{1}{2}, \frac{1}{2}\right) = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}.$$



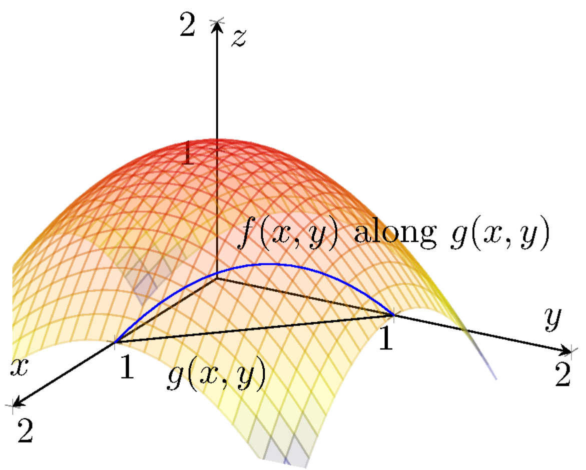

Consider now the equation:

$$g(x) = x + y - 1 \quad \rightarrow \quad y = -x + 1 .$$

Endpoints ($x \geq 0$; $y \geq 0$):

$P_1(0, 1) \quad \rightarrow \quad f(0, 1) = 0 .$
$P_2(1, 0) \quad \rightarrow \quad f(1, 0) = 0 .$

We have a minimum at $P_1(0, 1)$ and $P_2(1, 0) \rightarrow f(P_1) = f(P_2) = 0$.
We have a maximum at $P\left(\frac{1}{2}, \frac{1}{2}\right) \rightarrow f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$.



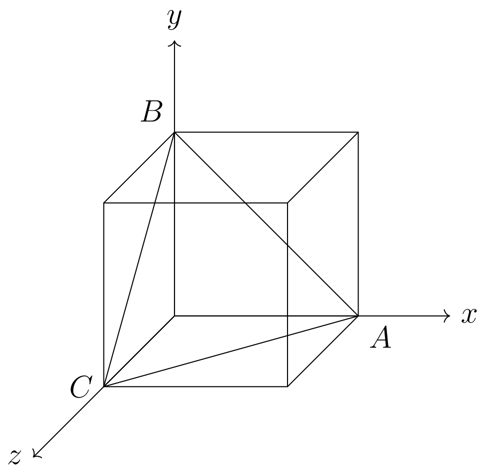

**Example 11.41** (Hottest and coldest points on a plate)**.** A container in $\mathbb{R}^3$ has the shape of the cube given by $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$. A plate is placed in the cube in such a way that it occupies that portion of the plane $x + y + z = 1$ that lies inside the

cubical container. The container is heated so that the temperature is

$$T(x, y, z) = 4 - 2x^2 - y^2 - z^2 .$$

Find the hottest and the coldest points of the plane.

**Solution:**




$$g(x, y, z) = x + y + z$$

$$\begin{aligned} T_x &= -4x & g_x &= 1 \\ T_y &= -2y & g_y &= 1 \\ T_z &= -2z & g_z &= 1 \end{aligned}$$

$$\begin{cases} -4x = l \cdot (1) \\ -2y = l \cdot (1) \\ -2z = l \cdot (1) \\ x + y + z = 1 \end{cases} \quad \begin{cases} x = -\dfrac{l}{4} \\ y = -\dfrac{l}{2} \\ z = -\dfrac{l}{2} \\ -\dfrac{l}{4} - \dfrac{l}{2} - \dfrac{l}{2} = 1 \end{cases} \quad \begin{cases} l = -\dfrac{4}{5} \\ x = \dfrac{1}{5} \\ y = \dfrac{2}{5} \\ z = \dfrac{2}{5} \end{cases} \to P\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right) \to f\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right) = \frac{9}{5} .$$

Edge $AC$: $x + z = 1$, $y = 0$.

$T(1 - z, 0, z) = 4 - 2(1 - z)^2 - z^2 = 4 - 2 - 2z^2 + 4z - z^2 = 2 - 3z^2 + 4z, \qquad 0 \le z \le 1 .$

$T_z = -6z + 4 = 0 \quad \Rightarrow \quad z = \dfrac{2}{3} \quad \Rightarrow \quad$ Point $P_1\left(\dfrac{1}{3}, 0, \dfrac{2}{3}\right)$.

Do the same for $AB$, $BC$. Then evaluate $P$, $P_1$, $P_2$, $P_3$ and $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.

**Theorem 11.15** (Rate of change of the extreme value)**.** Suppose $E$ is an extreme value (maximum or minimum) of $f$ subject to the constraint $g(x, y) = c$. Then, the Lagrange multiplier $l$ is the rate of change of $E$ with respect to $c$. That is, $l = dE/dC$.

**Example 11.42.** If $x$ thousand dollars is spent on labor, and $y$ thousand dollars is spent on equipment, it is estimated that the output of a certain factory will be

$$Q(x, y) = 50x^{2/5}y^{3/5} \quad \text{units.}$$

If \$ 150000 is available, how should this capital be allocated between labor and equipment to generate the largest possible output? How does the maximum output change if the money available is increased by \$ 1000?

**Solution:** $x, y$ are in thousand $\quad\rightarrow\quad x + y = 150$, that is the constraint.

$$\Rightarrow \quad g(x, y) = x + y\,.$$

$$\begin{aligned} Q_x &= 20x^{-3/5}y^{3/5} & g_x &= 1 \\ Q_y &= 30x^{2/5}y^{-2/5} & g_y &= 1\,. \end{aligned}$$

Lagrange multipliers:

$$\begin{cases} 20x^{-3/5}y^{3/5} = l \cdot (1) \\ 30x^{2/5}y^{-2/5} = l \cdot (1) \\ x + y = 150 \end{cases} \quad \rightarrow \quad l = 20x^{-3/5}y^{3/5}$$

$$\begin{gathered} 30x^{2/5}y^{-2/5} = 20x^{-3/5}y^{3/5} \\ \frac{30}{20}xy^{-1} = 1 \quad \Rightarrow \quad x = \frac{20}{30}y \\ x + y = 150 \quad \Rightarrow \quad \frac{20}{30}y + y = 150 \\ \frac{5}{3}y = 150 \quad \Rightarrow \quad y = 150 \cdot \frac{3}{5} = 90 \\ x = 150 - y = 60\,. \end{gathered}$$

Maximum output $Q(60, 90) = 50 \cdot (60)^{2/5} \cdot (90)^{3/5} = \cdots =$

The maximum output is obtained when \$ 60000 is allocated in labor and \$ 90000 is allocated in equipment. Moreover $l = 20 \cdot x^{-3/5} \cdot y^{3/5} \approx 25.51$ units $\Rightarrow$ rate of change of the maximum with the respect of $c = 150000\,\$$.

**Lagrange multipliers with two parameters.** The Lagrange Multipliers theorem can also be applied in situations where more than one constraint equations are applied.

Constraints:

$$g(x_0, y_0, z_0) = c_1\,,$$
$$h(x_0, y_0, z_0) = c_2\,,$$

where $(x_0, y_0, z_0)$ is the desired extremum. Then

$$\nabla f(x_0, y_0, z_0) = l\,\nabla g(x_0, y_0, z_0) + \mu\, h(x_0, y_0, z_0)\,.$$

Now, we aim to find $l, \mu, x_0, y_0, z_0$ (see the last example).

**Example 11.43** (Ex 7, HW 11.7-11.8)**.** Find the point of the plane $z = 4x + 4y + 3$ closest to the origin.

**Solution:** Distance point $(x, y, z)$ origin $=\sqrt{x^2 + y^2 + z^2}$.
Distance squared $= x^2 + y^2 + z^2$.
Constraint: $z = 4x + 4y + 3 \quad \Rightarrow \quad 4x + 4y + z = 3$.

$$f = x^2 + y^2 + z^2\,, \qquad g = 4x + 4y - z\,.$$

$$f_x = 2x \qquad\qquad g_x = 4$$
$$f_y = 2y \qquad\qquad g_y = 4$$
$$f_z = 2z \qquad\qquad g_z = -1$$

$$\begin{cases} 2x = 4l \\ 2y = 4l \\ 2z = -l \\ 4x + 4y - z = 3 \end{cases} \qquad \begin{aligned} x &= 2l \\ y &= 2l \\ z &= -\frac{1}{2}l \\ 4 \cdot 2l + 4 \cdot 2l &+ \frac{1}{2}l = -3 \end{aligned}$$

$$16l + \frac{1}{2}l = -3 \quad \to \quad \frac{33}{2}l = -3 \quad \to \quad l = -\frac{2}{11}\,.$$

$$x = -\frac{4}{11}\,, \qquad y = -\frac{4}{11}\,, \qquad z = \frac{1}{11}\,.$$