CHAPTER 9

Vectors in Plane and Space

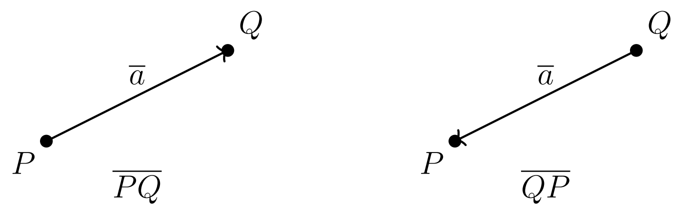
Contents

In this Chapter, we are going to introduce the following concepts:

- Vectors in $\mathbb{R}^2$
- Vectors in $\mathbb{R}^3$
- The dot product
- The cross product
- Lines in $\mathbb{R}^3$
- Planes in $\mathbb{R}^3$
- Quadric surfaces

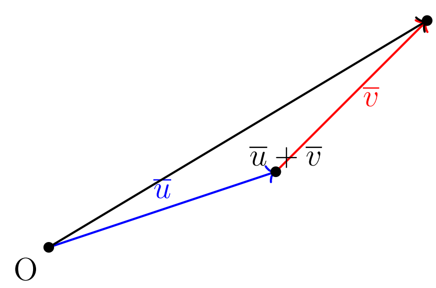
9.1 Vectors in $\mathbb{R}^2$

Vector: a directed line segment; an arrow with an initial point and a terminal point. A vector is defined uniquely by its *magnitude* and *direction*.



Vector operations.

- **Vector addition.** Given two vectors $\overline{u}$ and $\overline{v}$, their vector addition $\overline{u}+\overline{v}$ is defined by placing the tail of $\overline{v}$ at the head of $\overline{u}$. The resulting vector goes from the tail of $\overline{u}$ to the head of $\overline{v}$. This is known as the triangle rule.



- **Scalar multiplication.** The operation of scalar multiplication involves a scalar $c \in \mathbb{R}$ and a vector $\overline{u}$. The result $c\overline{u}$ is a vector with the same direction as $\overline{u}$ if $c > 0$, the opposite direction if $c < 0$, and is the zero vector if $c = 0$. Its length is $\|c\|$ times the length of $\overline{u}$.

Vector components and magnitude. Any vector $\bar{v}$ in the plane can be represented by its **components** along the x and y axes. We write this as

$$\bar{v} = \langle v_x, v_y \rangle ,$$

where v_x and v_y are the scalar projections of the vector in the horizontal and vertical directions, respectively. This means that $\bar{v}$ can be interpreted as the sum of two orthogonal vectors: one in the direction of the x-axis and one in the direction of the y-axis.

The **magnitude** (or **length**) of the vector $\bar{v}$ is defined as

$$\|\bar{v}\| = \sqrt{v_x^2 + v_y^2}\,.$$

This value represents the distance from the origin to the point (v_x, v_y) in the plane, and quantifies how long or strong the vector is, regardless of its direction.

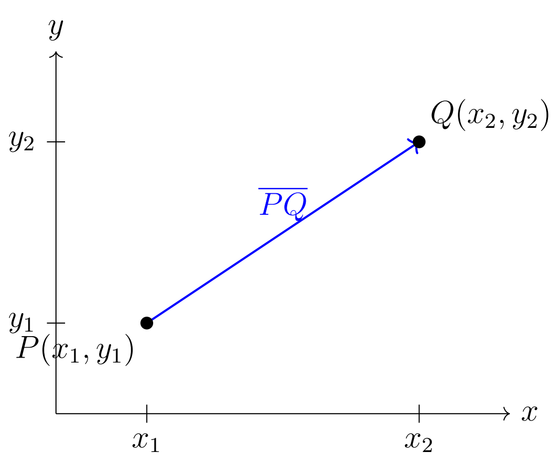
Vector between two points. Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the plane, the vector from P to Q, denoted by $\overline{PQ}$, is given by the difference of their coordinates:

$$\overline{PQ} = \langle x_2 - x_1,\ y_2 - y_1 \rangle.$$

This vector represents the displacement from point P to point Q, and can be interpreted as a directed arrow starting at P and ending at Q.

We also have:

$$\|\overline{PQ}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Remark 9.1. All the vector operations can be performed on the components.

E.g. Sum: $\langle 2, 3 \rangle + \langle 4, 5 \rangle = \langle 6, 8 \rangle$

Scalar product: $3\langle 2, 3 \rangle = \langle 6, 9 \rangle$

Properties of the vector operations. Vector operations follow well-defined algebraic properties that ensure consistency and structure. Given three vectors $\overline{u}$, $\overline{v}$, and $\overline{w}$, and scalars a and b, the following properties hold:

- **Commutativity of addition:** $\overline{u} + \overline{v} = \overline{v} + \overline{u}$.
- **Associativity of addition:** $(\overline{u} + \overline{v}) + \overline{w} = \overline{u} + (\overline{v} + \overline{w})$.
- **Scalar distributivity over vector addition:** $a(\overline{u} + \overline{v}) = a\overline{u} + a\overline{v}$.
- **Vector distributivity over scalar addition:** $(a + b)\overline{u} = a\overline{u} + b\overline{u}$.

Standard representation of vectors in the plane. In two-dimensional space, any vector can be expressed as a linear combination of two fundamental unit vectors:

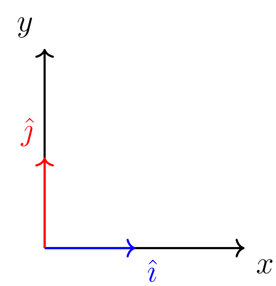
$$\hat{\imath} = \langle 1, 0 \rangle \quad \text{and} \quad \hat{\jmath} = \langle 0, 1 \rangle.$$

The vector $\hat{\imath}$ points in the direction of the positive x-axis, while $\hat{\jmath}$ points in the direction of the positive y-axis. These unit vectors form the basis of the Cartesian coordinate system, and any vector

$$\overline{v} = \langle v_x, v_y \rangle\,,$$

can be written as

$$\overline{v} = v_x\hat{\imath} + v_y\hat{\jmath}\,.$$



Example 9.1. A body is subject to two different forces: $\overline{F}_1 = -2\,\hat{\imath} + 2\,\hat{\jmath}$; the second force $\overline{F}_2$ has the direction of the vector $\overline{u} = -3\,\hat{\imath} - 4\,\hat{\jmath}$ and magnitude 10. Find the resulting force on the body.

Solution. First, compute the norm of the direction vector:

$$\|\overline{u}\| = \sqrt{(-3)^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

Then, the unit vector in the direction of $\overline{u}$ is:

$$\hat{u} = \frac{\overline{u}}{\|\overline{u}\|} = -\frac{3}{5}\,\hat{\imath} - \frac{4}{5}\,\hat{\jmath}.$$

Now we compute the force $\overline{F}_2$ of magnitude 10 in the direction of $\hat{u}$:

$$\overline{F}_2 = 10 \cdot \hat{u} = -6\,\hat{\imath} - 8\,\hat{\jmath}.$$

The resulting force is the sum of $\overline{F}_1$ and $\overline{F}_2$:

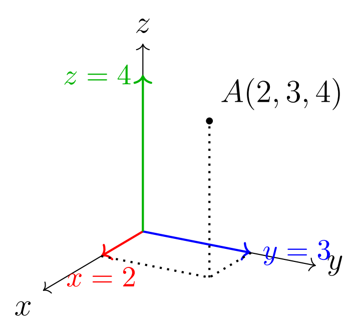
$$\overline{F}_{\text{res}} = \overline{F}_1 + \overline{F}_2 = (-2\,\hat{\imath} + 2\,\hat{\jmath}) + (-6\,\hat{\imath} - 8\,\hat{\jmath}) = -8\,\hat{\imath} - 6\,\hat{\jmath}.$$

9.2 Vectors in $\mathbb{R}^3$

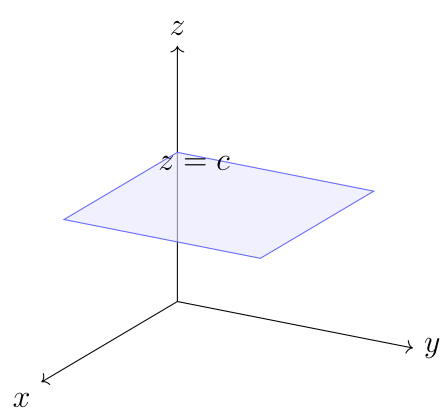
Plotting a point in $\mathbb{R}^3$. A point in space, such as $A(2,\ 3,\ 4)$, can be located by traveling along each axis:

- $x = 2$ units out of the page,
- $y = 3$ units to the right,
- and $z = 4$ units upward.

We illustrate this using components and dotted lines in the figure below.



Representing the plane $z =$ const. The plane $z = c$ consists of all points in space whose height is fixed at $z = c$. It is parallel to the xy-plane and can be visualized as a horizontal sheet:



Vectors in $\mathbb{R}^3$. All the basic vector operations and properties defined in the plane $\mathbb{R}^2$ naturally extend to three-dimensional space $\mathbb{R}^3$. In this setting, vectors are represented using the three unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, corresponding respectively to the x-, y-, and z-axes. In particular:

$$\hat{\imath} = \langle 1, 0, 0\rangle$$
$$\hat{\jmath} = \langle 0, 1, 0\rangle$$
$$\hat{k} = \langle 0, 0, 1\rangle$$

Any vector $\bar{v}$ in $\mathbb{R}^3$ can be written as

$$\bar{v} = \langle v_x, v_y, v_z\rangle = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}\,.$$

Given two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in space, the vector from P_1 to P_2 is

$$\overline{P_1P_2} = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1\rangle = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}\,. \tag{9.1}$$

This vector represents the displacement from P_1 to P_2 in three-dimensional space.

Example 9.2. Given the vectors

$$\mathbf{u} = \langle -1, 1, 2\rangle, \quad \mathbf{v} = \langle 0, 2, -3\rangle, \quad \mathbf{w} = \langle 5, -1, 0\rangle,$$

find the vector $\mathbf{q}$ such that

$$2\mathbf{u} - \mathbf{v} + 5\mathbf{q} = 3\mathbf{w}.$$

Solution: First, compute $2\mathbf{u}$:

$$2\mathbf{u} = 2 \cdot \langle -1, 1, 2\rangle = \langle -2, 2, 4\rangle.$$

Then compute $2\mathbf{u} - \mathbf{v}$:

$$\langle -2, 2, 4\rangle - \langle 0, 2, -3\rangle = \langle -2 - 0, 2 - 2, 4 - (-3)\rangle = \langle -2, 0, 7\rangle.$$

Now compute $3\mathbf{w}$:

$$3 \cdot \langle 5, -1, 0\rangle = \langle 15, -3, 0\rangle.$$

Now solve for $5\mathbf{q}$:

$$5\mathbf{q} = 3\mathbf{w} - (2\mathbf{u} - \mathbf{v}) = \langle 15, -3, 0\rangle - \langle -2, 0, 7\rangle = \langle 17, -3, -7\rangle.$$

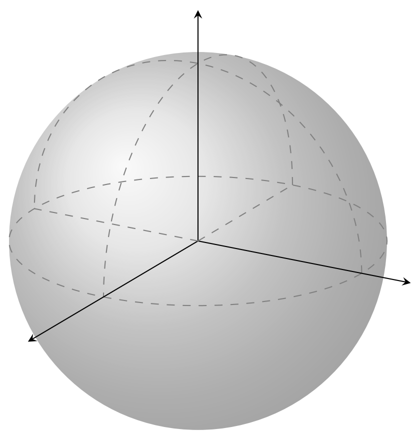
Divide by 5:

$$\mathbf{q} = \frac{1}{5} \cdot \langle 17, -3, -7\rangle = \left\langle \frac{17}{5}, -\frac{3}{5}, -\frac{7}{5} \right\rangle.$$

Sphere and Cylinder. In three-dimensional space, several important surfaces can be defined.

A **sphere** of radius r centered at a point (x_0, y_0, z_0) is the set of all points (x, y, z) such that:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2. \tag{9.2}$$



Example 9.3. Find the center and the radius of the sphere

$$x^2 + y^2 + z^2 \ = \ 3x + 3y + 3z.$$

Solution. Bring all terms to the left and complete the square in each variable:

$$x^2 - 3x + y^2 - 3y + z^2 - 3z = 0.$$

Complete the square:

$$x^2 - 3x = (x - \tfrac{3}{2})^2 - \left(\tfrac{3}{2}\right)^2,$$
$$y^2 - 3y = (y - \tfrac{3}{2})^2 - \left(\tfrac{3}{2}\right)^2,$$
$$z^2 - 3z = (z - \tfrac{3}{2})^2 - \left(\tfrac{3}{2}\right)^2.$$

Summing,

$$(x - \tfrac{3}{2})^2 + (y - \tfrac{3}{2})^2 + (z - \tfrac{3}{2})^2 \ - \ 3\left(\tfrac{3}{2}\right)^2 \ = \ 0.$$

Hence,

$$(x - \tfrac{3}{2})^2 + (y - \tfrac{3}{2})^2 + (z - \tfrac{3}{2})^2 = 3\left(\tfrac{3}{2}\right)^2 = \frac{27}{4}.$$

This is the standard sphere form (see (9.2)):

$$(x - x_0)^2 + (y - x_0)^2 + (z - z_0)^2 = r^2$$

with center

$$\boxed{C = \left(\tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}\right)}$$

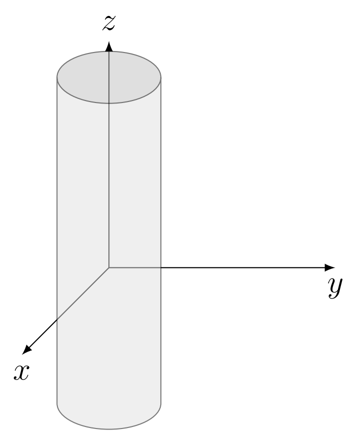
and radius

$$\boxed{r = \sqrt{\tfrac{27}{4}} = \tfrac{3\sqrt{3}}{2}}.$$

A **cylinder** of radius r aligned along the z-axis and centered at (x_0, y_0) in the xy-plane is the set of all points satisfying:

$$(x - x_0)^2 + (y - y_0)^2 = r^2,$$

independently of the z-coordinate.



9.3 The Dot Product

Definition 9.1. Given two vectors $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ and $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, their *dot product* (also called *scalar product* or *inner product*) is defined as

$$\mathbf{v}\cdot\mathbf{u} = v_1u_1 + v_2u_2 + v_3u_3.$$

The result is a scalar (a number, not a vector).

Example 9.4. Let $\mathbf{v} = \langle 1, 2, -1\rangle$ and $\mathbf{u} = \langle 3, 0, 5\rangle$. Then:

$$\mathbf{v}\cdot\mathbf{u} = 1\cdot 3 + 2\cdot 0 + (-1)\cdot 5 = 3 + 0 - 5 = -2.$$

Properties of the Dot Product.

- **Magnitude squared:** The square of the magnitude of a vector is equal to the dot product of the vector with itself:

$$|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v} = v_1^2 + v_2^2 + v_3^2.$$

- **Commutativity:**

$$\mathbf{v}\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{v}.$$

- **Distributivity over addition:**

$$\mathbf{v}\cdot(\mathbf{u}+\mathbf{w}) = \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{w}.$$

- **Scalar multiplication:**

$$(a\mathbf{v})\cdot\mathbf{u} = a(\mathbf{v}\cdot\mathbf{u}).$$

  More generally:

$$(a\mathbf{v} + b\mathbf{w})\cdot\mathbf{u} = a(\mathbf{v}\cdot\mathbf{u}) + b(\mathbf{w}\cdot\mathbf{u}).$$

Theorem 9.1. Let θ be the angle between the nonzero vectors $\overline{v}$ and $\overline{w}$, then:

$$\cos\theta = \frac{\overline{v}\cdot\overline{w}}{\|\overline{v}\|\,\|\overline{w}\|}$$

Remark 9.2. Note that it is easy to show that

$$\overline{v} \cdot \overline{w} = \|\overline{v}\| \, \|\overline{w}\| \cos\theta \,,$$

that is also called *geometrical formula for the dot product*.

Example 9.5 (Finding the angle between two sides of a triangle in 3D)**.** Given the triangle with vertices

$$A(1,1,8), \quad B(4,-3,4), \quad C(-3,1,5),$$

find the angle θ between the sides $\overline{BA}$ and $\overline{BC}$.

Solution. Considering equation (9.1), we compute the vectors:

$$\overline{BA} = A - B = \begin{bmatrix} 1-4 \\ 1-(-3) \\ 8-4 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 4 \end{bmatrix}, \quad \overline{BC} = C - B = \begin{bmatrix} -3-4 \\ 1-(-3) \\ 5-4 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix}.$$

The dot product is:

$$\overline{BA} \cdot \overline{BC} = (-3)(-7) + (4)(4) + (4)(1) = 21 + 16 + 4 = 41.$$

The magnitudes are:

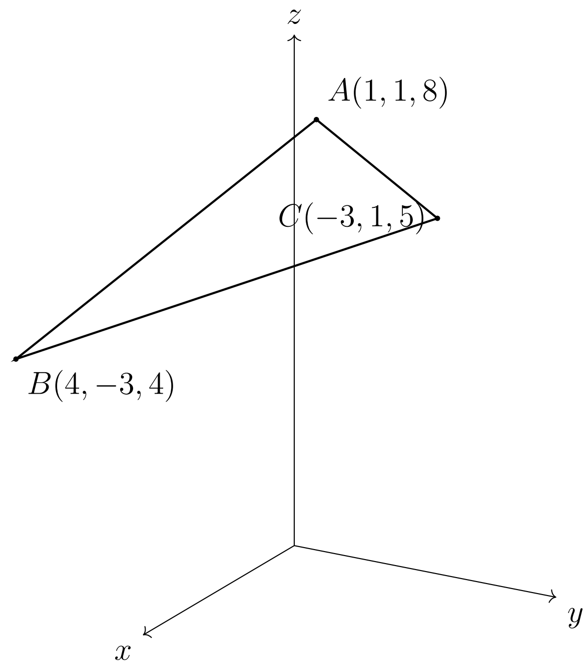
$$|\overline{BA}| = \sqrt{(-3)^2 + 4^2 + 4^2} = \sqrt{9 + 16 + 16} = \sqrt{41},$$
$$|\overline{BC}| = \sqrt{(-7)^2 + 4^2 + 1^2} = \sqrt{49 + 16 + 1} = \sqrt{66}.$$

Then,

$$\cos\theta = \frac{\overline{BA} \cdot \overline{BC}}{|\overline{BA}||\overline{BC}|} = \frac{41}{\sqrt{41}\sqrt{66}} = \frac{41}{\sqrt{2706}}.$$

Finally,

$$\theta = \cos^{-1}\left(\frac{41}{\sqrt{2706}}\right) \approx \cos^{-1}(0.7885) \approx 37.8^\circ.$$



Theorem 9.2. Nonzero vectors $\overline{u}$ and $\overline{v}$ are orthogonal iif (if and only if) $\overline{u} \cdot \overline{v} = 0$.

Proof. Since the statement presents an "iif", we need to prove both the implications of the theorem (direct and inverse):

1. If $\overline{u}$ and $\overline{v}$ are orthogonal $\Rightarrow \theta = \frac{\pi}{2} \Rightarrow \overline{u} \cdot \overline{v} = \|\overline{u}\| \, \|\overline{v}\| \cos\theta = 0$;

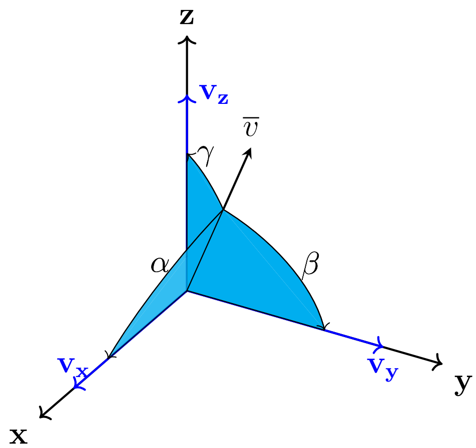
2. If $\overline{u} \cdot \overline{v} = 0 \Rightarrow \|\overline{u}\| \, \|\overline{v}\| \cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$.

□

Direction Cosines. Given a vector $\overline{v} = \langle v_x, v_y, v_z \rangle \in \mathbb{R}^3$, the **direction cosines** of $\overline{v}$ are the cosines of the angles that $\overline{v}$ makes with the positive x-, y-, and z-axes. Denoting these angles by α, β, γ respectively, we define:

$$\cos\alpha = \frac{v_x}{\|\overline{v}\|}, \qquad \cos\beta = \frac{v_y}{\|\overline{v}\|}, \qquad \cos\gamma = \frac{v_z}{\|\overline{v}\|}, \tag{9.3}$$

where $\|\overline{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$ is the Euclidean norm of the vector.



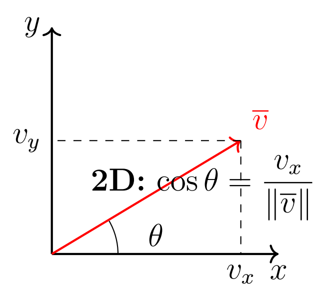
These three direction cosines fully describe the orientation of $\overline{v}$ in space, and they satisfy the identity:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Remark 9.3. In two-dimensional geometry, a vector $\overline{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$ forms an angle θ with the x-axis such that

$$\tan\theta = \frac{v_y}{v_x},$$

which is the slope of the line in Cartesian coordinates. The direction cosines generalize this concept to three dimensions, where slope is no longer a single number but rather a triple of cosines measuring the vector's alignment with each axis.



Example 9.6. Find the direction cosines and the angles that the vector $\overline{v} = (3, 4, 12)$ makes with the coordinate axes.

Solution. We first compute the magnitude of $\overline{v}$.

$$\|\overline{v}\| = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{9 + 16 + 144} = \sqrt{169} = 13$$

Then, following (9.3), we have

$$\cos\alpha = \frac{v_x}{\|\overline{v}\|} = \frac{3}{13}$$
$$\cos\beta = \frac{v_y}{\|\overline{v}\|} = \frac{4}{13}$$
$$\cos\gamma = \frac{v_z}{\|\overline{v}\|} = \frac{12}{13}$$

Note that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{9}{169} + \frac{16}{169} + \frac{144}{169} = 1$$

Lastly, the angles are:

$$\alpha = \cos^{-1}\left(\frac{3}{13}\right) \approx 76.6^\circ, \quad \beta = \cos^{-1}\left(\frac{4}{13}\right) \approx 72.5^\circ, \quad \gamma = \cos^{-1}\left(\frac{12}{13}\right) \approx 21.8^\circ$$

Projection of a Vector in the Direction of Another. Given two nonzero vectors $\overline{v}$ and $\overline{w}$ in $\mathbb{R}^n$, we define the **scalar projection** of $\overline{v}$ in the direction of $\overline{w}$ as

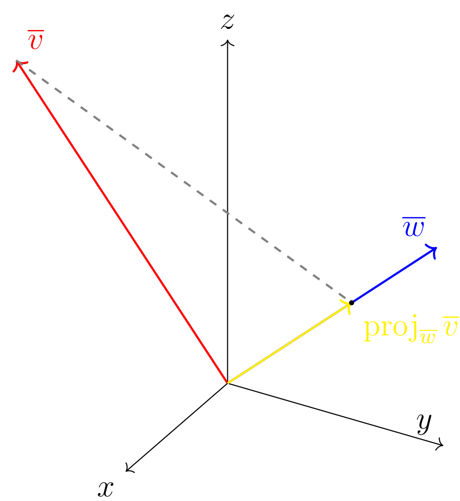
$$\text{comp}_{\overline{w}}\,\overline{v} = \frac{\overline{v}\cdot\overline{w}}{\|\overline{w}\|},$$

which represents the signed length of the projection of $\overline{v}$ onto the line defined by $\overline{w}$.

The **vector projection** of $\overline{v}$ in the direction of $\overline{w}$ is the vector

$$\text{proj}_{\overline{w}}\,\overline{v} = \left(\frac{\overline{v}\cdot\overline{w}}{\|\overline{w}\|^2}\right)\overline{w},$$

which lies on the line spanned by $\overline{w}$ and has magnitude equal to the scalar projection. If the angle between $\overline{v}$ and $\overline{w}$ is acute, the scalar projection is positive; if the angle is obtuse, the scalar projection is negative.



Example 9.7. Let $\overline{v} = (2, -1, 4)$ and $\overline{w} = (1, 2, 2)$. Find $\text{comp}_{\overline{w}}$ and $\text{proj}_{\overline{w}}\,\overline{v}$

We first compute the dot product:

$$\overline{v} \cdot \overline{w} = 2 \cdot 1 + (-1) \cdot 2 + 4 \cdot 2 = 2 - 2 + 8 = 8.$$

Then, compute the magnitude of $\overline{w}$:

$$\|\overline{w}\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3.$$

Compute the scalar projection:

$$\text{comp}_{\overline{w}}\,\overline{v} = \frac{8}{3}.$$

Lastly, we can compute the vector projection:

$$\text{proj}_{\overline{w}}\,\overline{v} = \left(\frac{8}{9}\right)\langle 1, 2, 2\rangle = \left\langle \frac{8}{9}, \frac{16}{9}, \frac{16}{9} \right\rangle.$$

Thus, the scalar projection is $\frac{8}{3}$ and the vector projection is $\left\langle \frac{8}{9}, \frac{16}{9}, \frac{16}{9} \right\rangle$.

9.4 The Cross Product

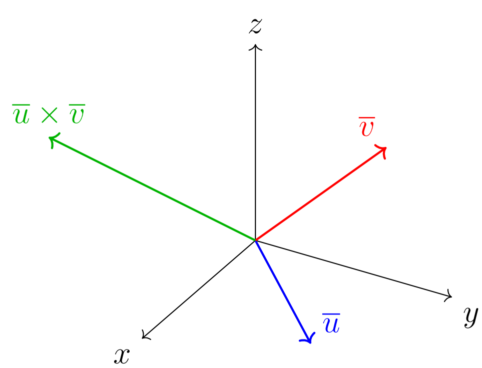
Definition 9.2 (Cross Product)**.** Given two vectors $\overline{u} = \langle u_1, u_2, u_3 \rangle$ and $\overline{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$, their *cross product* $\overline{u} \times \overline{v}$ is defined using the determinant of a 3×3 matrix:

$$\begin{aligned} \overline{u} \times \overline{v} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (-1)^{1+1}\hat{\imath} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} + (-1)^{1+2}\hat{\jmath} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + (-1)^{1+3}\hat{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\ &= \hat{\imath} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \end{aligned}$$

Thus, we have:

$$\overline{u} \times \overline{v} = (u_2 v_3 - u_3 v_2)\,\hat{\imath} - (u_1 v_3 - u_3 v_1)\,\hat{\jmath} + (u_1 v_2 - u_2 v_1)\,\hat{k}$$

Geometric Interpretation of the Cross Product Let $\overline{u}, \overline{v} \in \mathbb{R}^3$ be two non-parallel vectors. The cross product $\overline{u} \times \overline{v}$ is a vector in $\mathbb{R}^3$ defined to be orthogonal to both $\overline{u}$ and $\overline{v}$. Its orientation is given by the right-hand rule: if the fingers of the right hand curl from $\overline{u}$ toward $\overline{v}$, then the thumb points in the direction of $\overline{u} \times \overline{v}$. Geometrically, this vector represents the normal to the plane spanned by $\overline{u}$ and $\overline{v}$, and its magnitude equals the area of the parallelogram constructed on $\overline{u}$ and $\overline{v}$ as adjacent sides.

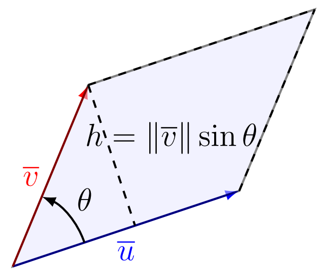


Definition 9.3 (Magnitude of the Cross Product)**.** Let $\theta \in [0, \pi]$ be the angle between two vectors $\overline{u}$ and $\overline{v}$ in $\mathbb{R}^3$. The magnitude of their cross product is given by the formula:

$$\|\overline{u} \times \overline{v}\| = \|\overline{u}\|\,\|\overline{v}\|\,\sin\theta.$$

This expression confirms that the magnitude is maximal when $\overline{u}$ and $\overline{v}$ are orthogonal ($\theta = \frac{\pi}{2}$), and vanishes when they are parallel ($\theta = 0$ or π). Consequently, the cross product also serves as a test of orthogonality and coplanarity.

Remark 9.4. The magnitude $\|\overline{u} \times \overline{v}\|$ also represents the **area of the parallelogram** determined by $\overline{u}$ and $\overline{v}$ when placed tail to tail in space.



Properties of the Cross Product. Let $\overline{u}, \overline{v}, \overline{w} \in \mathbb{R}^3$ and let $a, b \in \mathbb{R}$ be scalars. Then:

- **Scalar distributivity:** $(a\overline{u}) \times \overline{v} = a(\overline{u} \times \overline{v}) = \overline{u} \times (a\overline{v})$.
- **Vector distributivity:** $\overline{u} \times (\overline{v} + \overline{w}) = \overline{u} \times \overline{v} + \overline{u} \times \overline{w}$.
- **Anticommutativity:** $\overline{u} \times \overline{v} = -(\overline{v} \times \overline{u})$.
- **Lagrange's identity:** $\|\overline{u} \times \overline{v}\|^2 = \|\overline{u}\|^2 \|\overline{v}\|^2 - (\overline{u} \cdot \overline{v})^2$.
- **BAC–CAB formula:** For the triple cross product, we have:

$$\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a} \cdot \overline{c})\overline{b} - (\overline{a} \cdot \overline{b})\overline{c}.$$

Example 9.8. Find a nonzero vector orthogonal to $\overline{v} = -2\hat{\imath} + 3\hat{\jmath} - 7\hat{k}$ and $\overline{w} = 5\hat{\imath} + 9\hat{k}$. Use the cross product.

Solution. We can find a vector orthogonal to both $\overline{v}$ and $\overline{w}$ by computing their cross product:

$$\begin{aligned}\overline{v} \times \overline{w} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 3 & -7 \\ 5 & 0 & 9 \end{vmatrix} \\ &= \hat{\imath} \begin{vmatrix} 3 & -7 \\ 0 & 9 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} -2 & -7 \\ 5 & 9 \end{vmatrix} + \hat{k} \begin{vmatrix} -2 & 3 \\ 5 & 0 \end{vmatrix} \\ &= \hat{\imath}(3 \cdot 9 - (-7) \cdot 0) - \hat{\jmath}((-2) \cdot 9 - (-7) \cdot 5) + \hat{k}((-2) \cdot 0 - 3 \cdot 5) \\ &= \hat{\imath}(27) - \hat{\jmath}(-18 + 35) + \hat{k}(-15) = 27\hat{\imath} - 17\hat{\jmath} - 15\hat{k}\end{aligned}$$

Therefore, a nonzero vector orthogonal to both $\overline{v}$ and $\overline{w}$ is

$$\overline{v} \times \overline{w} = 27\hat{\imath} - 17\hat{\jmath} - 15\hat{k}.$$

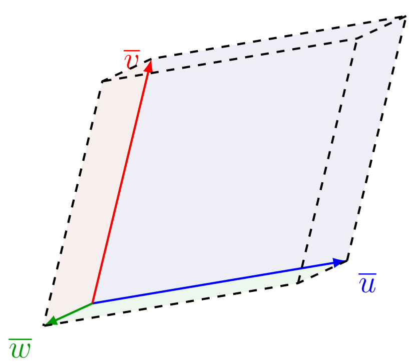
Definition 9.4 (Scalar Triple Product)**.** Given three vectors $\overline{u}, \overline{v}, \overline{w} \in \mathbb{R}^3$, the *scalar triple product* is defined as:

$$\overline{u} \cdot (\overline{v} \times \overline{w}).$$

Algebraically, it is equal to the determinant of the 3×3 matrix whose rows (or columns) are the components of the vectors:

$$\overline{u} \cdot (\overline{v} \times \overline{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Remark 9.5. Geometrically, the absolute value of the scalar triple product gives the **volume of the parallelepiped** determined by the three vectors. If the scalar triple product is zero, then the three vectors lie in the same plane, and the volume is zero.



Example 9.9. Find the volume of the parallelepiped determined by the vectors

$$\overline{u} = (2,0,1), \quad \overline{v} = (0,3,0), \quad \overline{w} = (1,0,4).$$

Solution. The volume is given by the absolute value of the scalar triple product:

$$V = \left|\overline{u} \cdot (\overline{v} \times \overline{w})\right|.$$

We first compute the cross product:

$$\overline{v} \times \overline{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 0 \\ 1 & 0 & 4 \end{vmatrix} = \mathbf{i}(3 \cdot 4 - 0 \cdot 0) - \mathbf{j}(0 \cdot 4 - 0 \cdot 1) + \mathbf{k}(0 \cdot 0 - 3 \cdot 1) = 12\,\mathbf{i} + 0\,\mathbf{j} - 3\,\mathbf{k}.$$

So,

$$\overline{v} \times \overline{w} = (12, 0, -3).$$

Now take the dot product with $\overline{u}$:

$$\overline{u} \cdot (\overline{v} \times \overline{w}) = (2,0,1) \cdot (12,0,-3) = 2 \cdot 12 + 0 \cdot 0 + 1 \cdot (-3) = 24 - 3 = 21.$$

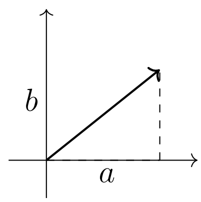
$$\boxed{V = 21}.$$

9.5 Lines in $\mathbb{R}^3$

In $\mathbb{R}^{\nvDash}$, a line is represented using the slope m, defined as the change in y-component divided by the change in x-component: $m = \frac{\Delta y}{\Delta x}$.

We reformulate the definition of m, and we say that a line has the same direction of the vector

$$\overline{v} = a\hat{\imath} + b\hat{\jmath}$$



In 3D, we say that a line has the same direction of

$$\overline{v} = A\hat{\imath} + B\hat{\jmath} + C\hat{k}$$

where A, B, C are called **direction numbers**.

Definition 9.5 (Parametric Form of a Line in $\mathbb{R}^3$)**.** A line L in three-dimensional space can be described by:

- a **point** $Q = (x_0, y_0, z_0)$ through which the line passes, and
- a **direction vector** $\overline{v} = A\hat{\imath} + B\hat{\jmath} + C\hat{k}$.

A generic point $P = (x, y, z)$ lies on the line L if and only if the vector $\overline{PQ}$ is parallel to $\overline{v}$, that is:

$$\overline{PQ} = t\,\overline{v}, \quad \text{for some } t \in \mathbb{R}.$$

Since, considering the equation of the vector between two points (9.1):

$$\overline{PQ} = (x - x_0)\hat{\imath} + (y - y_0)\hat{\jmath} + (z - z_0)\hat{k},$$

then, we have:

$$\overline{PQ} = t\,\overline{v}$$
$$(x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k} = t(A\hat{i} + B\hat{j} + C\hat{k})$$

we put each component (along $\hat{i}, \hat{j}, \hat{k}$) equal to each other, obtaining:

$$\begin{cases} x - x_0 = tA \\ y - y_0 = tB \\ z - z_0 = tC \end{cases} \quad \text{where } t \in \mathbb{R}.$$

and we obtain the **parametric equations** of the line:

$$\begin{cases} x = x_0 + tA \\ y = y_0 + tB \\ z = z_0 + tC \end{cases} \quad \text{where } t \in \mathbb{R}. \tag{9.4}$$

Example 9.10 (Parametric Equation of a Line in $\mathbb{R}^3$)**.** Find the parametric equations of the line that passes through the point $P(3, 1, 4)$ and has direction vector $\overline{v} = \langle -1, 1, -2 \rangle$. Also, find the point where this line intersects the xy-plane.

Solution. The parametric form of a line in $\mathbb{R}^3$, passing through a point (x_0, y_0, z_0) and with direction vector $\overline{v} = \langle A, B, C \rangle$, has been introduced in (9.4). Substituting $(x_0, y_0, z_0) = (3, 1, 4)$ and $\langle A, B, C \rangle = \langle -1, 1, -2 \rangle$, we get:

$$\begin{cases} x = 3 - t \\ y = 1 + t \\ z = 4 - 2t \end{cases} \quad t \in \mathbb{R}$$

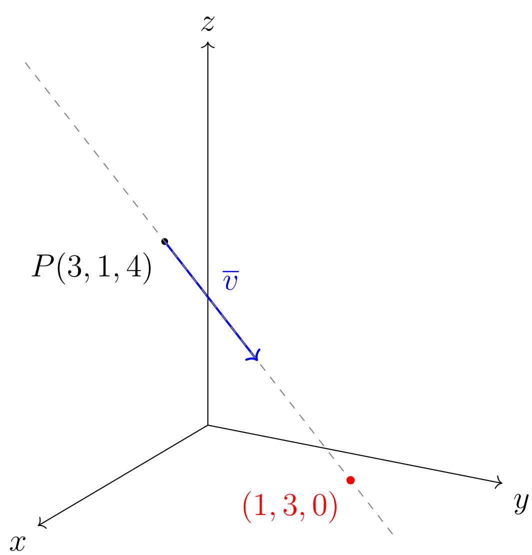
Intersection with the xy-Plane.
The xy-plane corresponds to $z = 0$. We solve:

$$4 - 2t = 0 \quad \Rightarrow \quad t = 2$$

Substitute $t = 2$ into the equations for x and y:

$$x = 3 - 2 = 1$$
$$y = 1 + 2 = 3$$

Therefore, the intersection point is $(1, 3, 0)$.



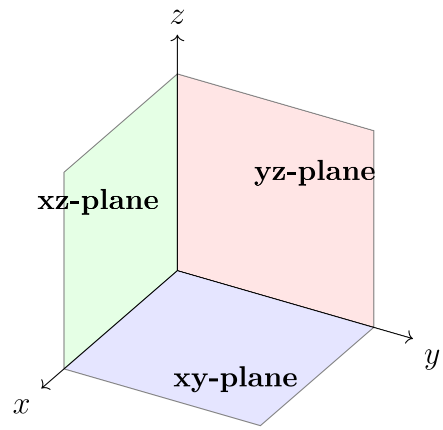
Remark 9.6. In the three-dimensional Cartesian coordinate system, the coordinate planes are defined as follows:

- The **xy-plane** is the plane where the z-coordinate is zero. It contains all points of the form $(x, y, 0)$.

- The **yz-plane** is the plane where the x-coordinate is zero. It contains all points of the form $(0, y, z)$.

- The **xz-plane** is the plane where the y-coordinate is zero. It contains all points of the form $(x, 0, z)$.

Each coordinate plane divides the space into two half-spaces and serves as a reference for measuring coordinates in the 3D space.



Symmetric form of a line in $\mathbb{R}^3$.

Definition 9.6. Consider the parametric equations of a line as in (9.4):

$$x = x_0 + At, \quad y = y_0 + Bt, \quad z = z_0 + Ct,$$

where $t \in \mathbb{R}$ is a parameter, (x_0, y_0, z_0) is a fixed point on the line, and $\mathbf{v} = (A, B, C)$ is the direction vector. Solving each equation for t, we get

$$t = \frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C},$$

which is the **symmetric form** of the line (assuming $A, B, C \neq 0$).

Equivalently, the line can be expressed in vector form as

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v},$$

where $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$. Note that sometimes Webwork will ask the final answer in this form.

Example 9.11. Find the parametric and symmetric equations of the line passing through the points

$$P_1 = (1, 2, 3) \quad \text{and} \quad P_2 = (4, 0, -1).$$

Solution: The direction vector is

$$\mathbf{v} = P_2 - P_1 = (4 - 1,\ 0 - 2,\ -1 - 3) = (3, -2, -4).$$

Using $P_1 = (1, 2, 3)$ as a point on the line, the parametric equations are:

$$x = 1 + 3t, \quad y = 2 - 2t, \quad z = 3 - 4t, \quad t \in \mathbb{R}.$$

To write the symmetric form, solve each for t:

$$t = \frac{x-1}{3} = \frac{y-2}{-2} = \frac{z-3}{-4}.$$

Therefore, the symmetric equations of the line are

$$\frac{x-1}{3} = \frac{y-2}{-2} = \frac{z-3}{-4}.$$

Example 9.12. Problem: Consider the following two lines in $\mathbb{R}^3$:

Line 1:

$$x = 1 + t, \quad y = 2, \quad z = -1 + 2t$$

Line 2:

$$x = 3, \quad y = s, \quad z = 5 - s$$

Determine whether the lines are parallel, intersecting, or skew.

Solution: We start by identifying direction vectors for the lines:

- Line 1 has direction vector $\mathbf{v}_1 = \langle 1, 0, 2 \rangle$
- Line 2 has direction vector $\mathbf{v}_2 = \langle 0, 1, -1 \rangle$

Since the direction vectors are not scalar multiples of each other (same direction vector), the lines are *not parallel*. In particular, to verify if two vectors are one scalar multiple of the other, we can find, if it exists, a number k such that

$$\mathbf{v}_1 = k\mathbf{v}_2 \quad \Rightarrow \quad \langle 1, 0, 2 \rangle = k\langle 0, 1, -1 \rangle \quad \Rightarrow \quad \langle 1, 0, 2 \rangle = \langle 0, k, -k \rangle$$

That means, find k such that

$$\begin{cases} 1 = 0 \\ 0 = k \\ 2 = -k\,, \end{cases}$$

that is clearly not possible.

Now, to check if they intersect, we try to find values of t and s such that the coordinates of the two lines match.

Line 1:

$$x = 1 + t, \quad y = 2, \quad z = -1 + 2t$$

Line 2:

$$x = 3, \quad y = s, \quad z = 5 - s$$

So we solve the system generated by putting each component of each line equal to each other (if they are all equal, that has to be an intersection point; if the system is not possible, then the two lines cannot intersect):

$$\begin{cases} 1 + t = 3 \\ 2 = s \\ 1 - 2t = 5 - s \end{cases}$$

The system has a unique solution $t = 2$, $s = 2$. Important: this system is a system of three equation in two unknowns. Make sure to verify that the solution fits all the three equations!

Thus, both lines yield the point:

$$(x, y, z) = (3, 2, 3)$$

You can obtain this point by plugging t in the first line and s in the second line. So the lines intersect at the point $(3, 2, 3)$.

9.5.1 Parametric Equations

Let $f_1(t), f_2(t), f_3(t)$ be continuous functions of t on an interval I. Then, the equations

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t)$$

are called **parametric equations** with parameter t. As t varies over the parameter set I, the point

$$(x, y, z) = (f_1(t), f_2(t), f_3(t))$$

traces a parametric curve in $\mathbb{R}^3$.

Remark 9.7. If $z = f_3(t) = 0$, the parametric curve lies in $\mathbb{R}^2$. Also, the parameter t is not necessarily time.

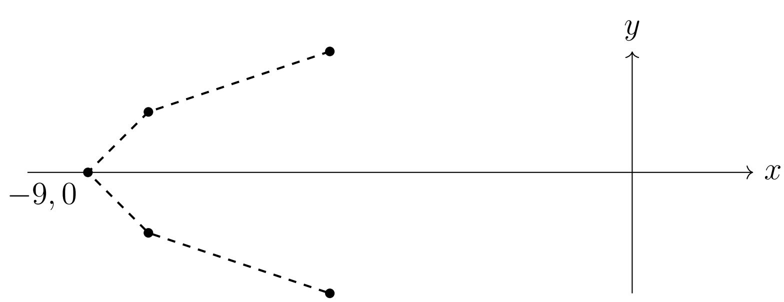
Remark 9.8. The equation of a line written in parametric form is a specific example of a parametric equation. More generally, the concept of a parametric equation is quite broad, allowing one to define arbitrary parametrizations for x, y, and z, not necessarily representing a line.

Example 9.13 (Sketching a parametric curve)**.** Sketch the path of the curve defined by

$$x = t^2 - 9, \quad y = \frac{1}{3}t, \quad \text{for } -3 \le t \le 2.$$

Solution. We consider different values of t and we find the corresponding points of the curve:
Table of values

t	$x = t^2 - 9$	$y = \frac{1}{3}t$
-3	0	-1
-2	-5	-$\frac{2}{3}$
-1	-8	-$\frac{1}{3}$
0	-9	0
1	-8	$\frac{1}{3}$
2	-5	$\frac{2}{3}$



Example 9.14. Eliminate the parameter from the parametric equations:

$$x = t^2 - 9, \quad y = \frac{1}{3}t$$

Solution. Solve $y = \frac{1}{3}t \Rightarrow t = 3y$, then substitute into x:

$$x = (3y)^2 - 9 = 9y^2 - 9$$

So the Cartesian equation of the curve is:

$$x = 9y^2 - 9$$

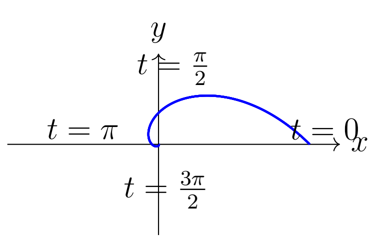
This is a parabola opening to the right, traced as t increases from -3 to 2.

Example 9.15 (Describing a spiraling path)**.** Describe the path of the curve described by the parametric equations:

$$x = e^{-t}\cos t, \quad y = e^{-t}\sin t, \quad \text{for } t \geq 0$$

Solution. We cannot find a simple relation between x and y without the parameter t, but we can still understand the behavior by plotting points for different values of t:

t	x	y
0	1.00	0.00
$\frac{\pi}{4}$	0.32	0.32
$\frac{\pi}{2}$	0.00	0.21
$\frac{3\pi}{4}$	-0.31	0.00
π	-0.31	0.00
$\frac{3\pi}{2}$	0.00	-0.05
2π	0.00	0.00



The equation describes a *spiral* that converges to the origin as $t \to \infty$. We can verify:

$$x^2 + y^2 = e^{-2t}(\cos^2 t + \sin^2 t) = e^{-2t}$$

As $t \to \infty$, we have $x^2 + y^2 \to 0$, so the distance from the origin decreases.

Curve Parametrization Given a function $y = g(x)$, we can construct a parametric representation of the curve.

Example 9.16. Let $y = 9x$. One possible parametrization is:

$$x = t, \quad y = 9t$$

Alternatively, we could write $x = \frac{t}{3}$, which implies $y = 3t$.

In general, any function $y = f(x)$ can be written parametrically as:

$$x = t, \quad y = f(t)$$

Parametrization in Polar Coordinates We can also describe curves using polar coordinates. Recall:

$$x = r\cos\theta, \quad y = r\sin\theta$$

Example 9.17. Let $x^2 + y^2 = 9$, a circle of radius 3. Then a natural parametrization is:

$$x = 3\cos\theta, \quad y = 3\sin\theta, \quad \theta \in [0, 2\pi]$$

This is a parametrization using the polar angle θ.

Remark 9.9. A single curve can have infinitely many parametrizations. Choosing one often depends on the application: animation, physics, geometry, etc.

Example 9.18. Find an explicit relation between x and y by eliminating the parameter t:

$$x(t) = 2\sin(t) + 12, \qquad y(t) = \cos^2(t)$$

Solution We begin by recalling the Pythagorean identity:

$$\sin^2(t) + \cos^2(t) = 1 \quad \Rightarrow \quad \sin^2(t) = 1 - \cos^2(t)$$

From the second equation, we write:

$$y = \cos^2(t) \quad \Rightarrow \quad \sin^2(t) = 1 - y$$

Now consider the first equation:

$$x = 2\sin(t) + 12 \quad \Rightarrow \quad \sin(t) = \frac{x - 12}{2}$$

We square both sides:

$$\sin^2(t) = \left(\frac{x - 12}{2}\right)^2 = \frac{(x - 12)^2}{4}$$

Equating with the previous expression for $\sin^2(t)$:

$$1 - y = \frac{(x - 12)^2}{4}$$

Solving for y, we obtain the relation:

$$\boxed{y = 1 - \frac{(x - 12)^2}{4}}$$

This is the explicit relation between x and y, describing a downward-opening parabola.

9.6 Planes in $\mathbb{R}^3$

In three-dimensional space, a **plane** is a flat, two-dimensional surface that extends infinitely. Geometrically, a plane can be uniquely defined by a **point** lying on the plane and a **direction perpendicular to the plane**, i.e., a **normal vector**.

Let $\overline{v} = \langle v_x, v_y, v_z \rangle$ be a vector perpendicular to the plane, and let $P(x_0, y_0, z_0)$ be a point on the plane. Then, for any point $Q(x, y, z)$ lying on the plane, the vector $\overline{PQ} = \langle x - x_0, y - y_0, z - z_0 \rangle$ must be perpendicular to the normal vector $\overline{v}$ (by definition of normal vector itself). Considering Theorem 9.2, this implies that their dot product is zero.

Definition 9.7. The equation of a plane in $\mathbb{R}^3$ that passes through a point $P(x_0, y_0, z_0)$ and is perpendicular to the vector $\overline{v} = \langle v_x, v_y, v_z \rangle$ is given by the condition:

$$\overline{v} \cdot \overline{PQ} = 0$$

which expands to:

$$v_x(x - x_0) + v_y(y - y_0) + v_z(z - z_0) = 0$$

This is the general form of the equation of a plane in three-dimensional space.

Starting from the general equation of a plane:

$$v_x(x - x_0) + v_y(y - y_0) + v_z(z - z_0) = 0$$

we can distribute each term:

$$v_x x - v_x x_0 + v_y y - v_y y_0 + v_z z - v_z z_0 = 0$$

Grouping the variable terms and the constants:

$$v_x x + v_y y + v_z z = v_x x_0 + v_y y_0 + v_z z_0$$

Let us denote the constant on the right-hand side as:

$$D = v_x x_0 + v_y y_0 + v_z z_0$$

Thus, the **standard form** of the plane is:

$$v_x x + v_y y + v_z z = D$$

Example 9.19. Given the equation of the plane

$$2x - 3y + 6z = 5,$$

find:

(a) A unit vector perpendicular to the plane.

(b) The equation of the line that passes through the point $P(1, -2, 0)$ and is perpendicular to the plane.

(c) The intersection point between the line and the plane.

Solution.
(a) The normal vector to the plane is given directly by the coefficients of x, y, and z in the plane equation:

$$\overline{v} = \langle 2, -3, 6 \rangle.$$

To obtain the unit vector, we normalize it:

$$\|\overline{v}\| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7,$$

so the unit vector perpendicular to the plane is:

$$\boxed{\overline{u} = \left\langle \frac{2}{7}, -\frac{3}{7}, \frac{6}{7} \right\rangle}.$$

(b) We now want the equation of the line that:

- passes through the point $P(1, -2, 0)$,
- and has direction vector $\overline{v} = \langle 2, -3, 6 \rangle$.

Using the parametric form of a line:

$$\boxed{\begin{cases} x(t) = 1 + 2t \\ y(t) = -2 - 3t \\ z(t) = 0 + 6t \end{cases}}$$

(c) To find the intersection point, substitute the parametric equations of the line into the equation of the plane:

$$2x - 3y + 6z = 5 \quad \Rightarrow \quad 2(1 + 2t) - 3(-2 - 3t) + 6(6t) = 5.$$

Simplifying:

$$2 + 4t + 6 + 9t + 36t = 5 \quad \Rightarrow \quad 8 + 49t = 5 \quad \Rightarrow \quad 49t = -3 \quad \Rightarrow \quad t = -\frac{3}{49}.$$

Now substitute back to find the intersection point:

$$x = 1 + 2t = 1 - \frac{6}{49} = \frac{43}{49}, \quad y = -2 - 3t = -2 + \frac{9}{49} = -\frac{89}{49}, \quad z = 6t = -\frac{18}{49}.$$

So the point of intersection is:

$$\boxed{\left(\frac{43}{49}, -\frac{89}{49}, -\frac{18}{49}\right)}$$

Example 9.20. Find the equation of the plane that passes through the three points:

$$A(1, 2, 3), \quad B(4, 0, -1), \quad C(-2, 1, 5).$$

Solution. We start by finding two vectors that lie on the plane (any couple of vectors between the given points are good):

$$\overline{AB} = \langle 4 - 1,\ 0 - 2,\ -1 - 3\rangle = \langle 3, -2, -4\rangle,$$
$$\overline{AC} = \langle -2 - 1,\ 1 - 2,\ 5 - 3\rangle = \langle -3, -1, 2\rangle.$$

The normal vector to the plane is given by the following cross product:

$$\overline{n} = \overline{AB} \times \overline{AC}.$$

Using the determinant formula:

$$\overline{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & -4 \\ -3 & -1 & 2 \end{vmatrix} = \hat{i}((-2)(2) - (-4)(-1)) - \hat{j}((3)(2) - (-4)(-3)) + \hat{k}((3)(-1) - (-2)(-3))$$

$$= \hat{\imath}(-4-4) - \hat{\jmath}(6-12) + \hat{k}(-3-6) = \hat{\imath}(-8) - \hat{\jmath}(-6) + \hat{k}(-9) = \langle -8, 6, -9 \rangle.$$

So the normal vector is:

$$\overline{n} = \langle -8, 6, -9 \rangle.$$

Now we use point $A(1,2,3)$ and the normal vector to write the plane equation:

$$-8(x-1) + 6(y-2) - 9(z-3) = 0.$$

Expanding:

$$-8x + 8 + 6y - 12 - 9z + 27 = 0 \quad \Rightarrow \quad -8x + 6y - 9z + 23 = 0.$$

Multiplying by -1 for standard convention:

$$\boxed{8x - 6y + 9z = 23}.$$

This is the equation of the plane through points A, B, and C.

Example 9.21. Find the equation of the line that passes through the point $P(-1,2,3)$ and is **parallel** to the line of intersection of the planes

$$\Pi_1 : \ 3x - 2y + z = 4 \qquad \text{and} \qquad \Pi_2 : \ x + 2y + 3z = 5.$$

Solution. To find the direction of the line of intersection of two planes, we compute the cross product of their normal vectors.
The normal vector to Π_1 is $\overline{n}_1 = \langle 3, -2, 1 \rangle$, and to Π_2 is $\overline{n}_2 = \langle 1, 2, 3 \rangle$.
Their cross product is:

$$\overline{d} = \overline{n}_1 \times \overline{n}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \hat{\imath}((-2)(3) - (1)(2)) - \hat{\jmath}((3)(3) - (1)(1)) + \hat{k}((3)(2) - (-2)(1))$$

$$= \hat{\imath}(-6-2) - \hat{\jmath}(9-1) + \hat{k}(6+2) = \langle -8, -8, 8 \rangle.$$

We can simplify this direction vector diving by 8 (the rescaled vector will maintain the direction):

$$\overline{d} = \langle -1, -1, 1 \rangle.$$

Now we use the point $P(-1,2,3)$ and this direction to write the parametric equations of the line:

$$\begin{cases} x = -1 - t \\ y = 2 - t \\ z = 3 + t \end{cases} \qquad \text{or in vector form:} \quad \overline{r}(t) = \langle -1, 2, 3\rangle + t\langle -1, -1, 1\rangle$$

Example 9.22. Find the equation of the plane that contains the following two lines, given in symmetric form:

$$\ell_1 : \quad \frac{x-1}{2} = \frac{y+3}{-1} = \frac{z}{4}, \qquad \ell_2 : \quad \frac{x+2}{1} = \frac{y-1}{2} = \frac{z+3}{-2}$$

Solution. We first extract a point and a direction vector from each line.

- From ℓ_1:

$$\text{Point } P_1 = (1, -3, 0), \qquad \text{Direction } \overline{v}_1 = \langle 2, -1, 4\rangle$$

- From ℓ_2:

$$\text{Point } P_2 = (-2, 1, -3), \qquad \text{Direction } \overline{v}_2 = \langle 1, 2, -2\rangle$$

The plane must contain both P_1 and P_2, and be parallel to both direction vectors $\overline{v}_1$ and $\overline{v}_2$. To find the normal vector to the plane, we take the cross product:

$$\overline{n} = \overline{v}_1 \times \overline{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 4 \\ 1 & 2 & -2 \end{vmatrix} = \hat{i}((-1)(-2) - (4)(2)) - \hat{j}((2)(-2) - (4)(1)) + \hat{k}((2)(2) - (-1)(1))$$

$$= \hat{i}(2-8) - \hat{j}(-4-4) + \hat{k}(4+1) = \langle -6, 8, 5\rangle$$

So the normal vector is $\overline{n} = \langle -6, 8, 5\rangle$.
We now use the point $P_1 = (1, -3, 0)$ to write the equation of the plane:

$$-6(x-1) + 8(y+3) + 5(z-0) = 0$$

Expanding:

$$-6x + 6 + 8y + 24 + 5z = 0 \quad \Rightarrow \quad -6x + 8y + 5z + 30 = 0$$

$$\boxed{-6x + 8y + 5z + 30 = 0}$$

This is the equation of the plane containing both lines ℓ_1 and ℓ_2.

9.6.1 Distances in $\mathbb{R}^3$

Distance from a Point to a Plane

Definition 9.8. Given a plane in space with equation

$$ax + by + cz + d = 0,$$

and a point $P(x_0, y_0, z_0)$ not lying on the plane, the *distance* from the point to the plane is defined as the length of the perpendicular segment from the point to the plane. This distance can be computed using the following formula:

$$\text{Distance} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}. \tag{9.5}$$

The distance formula is derived by projecting the vector from any point on the plane to the point P onto the normal vector $\overline{n} = \langle a, b, c\rangle$ of the plane. The numerator gives the signed magnitude of this projection, and the denominator normalizes it by the length of the normal vector.

Example 9.23. Find the equation of the sphere with center $C(-3, 1, 5)$ and tangent to the plane $6x - 2y + 3z = 9$.

Solution. Since the sphere is tangent to the plane, the **distance** from the center of the sphere to the plane equals the radius r of the sphere.

Recall the formula (9.5) for the distance from a point (x_0, y_0, z_0) to the plane $ax + by + cz + d = 0$:

$$\text{Distance} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

First, rewrite the plane equation in standard form:

$$6x - 2y + 3z - 9 = 0,$$

so $a = 6,\ b = -2,\ c = 3,\ d = -9$, and $(x_0, y_0, z_0) = (-3, 1, 5)$.

Compute the numerator:

$$|6(-3) - 2(1) + 3(5) - 9| = |-18 - 2 + 15 - 9| = |-14| = 14.$$

Compute the denominator:

$$\sqrt{6^2 + (-2)^2 + 3^2} = \sqrt{36 + 4 + 9} = \sqrt{49} = 7.$$

So the radius is:

$$r = \frac{14}{7} = 2.$$

Now write the equation of the sphere centered at $(-3, 1, 5)$ with radius 2:

$$(x + 3)^2 + (y - 1)^2 + (z - 5)^2 = 4.$$

Distance from a Point to a Line Given a point $P_0 = (x_0, y_0, z_0)$ and a line $\mathscr{L}$ in space, we often want to compute the shortest distance from the point to the line. Geometrically, this corresponds to the length of the segment that goes from P_0 and is orthogonal to the line $\mathscr{L}$. This is summarized in the following definition:

Definition 9.9. Let $\mathscr{L}$ be a line passing through a point $P_1 = (x_1, y_1, z_1)$ and directed by a vector $\overline{v} = \langle v_x, v_y, v_z \rangle$. The distance d from a point $P_0 = (x_0, y_0, z_0)$ to the line $\mathscr{L}$ is given by:

$$d = \frac{\left\|\overline{v} \times \overline{P_0P_1}\right\|}{\|\overline{v}\|},$$

where $\overline{P_0P_1}$ is the vector from P_1 to P_0, and $\times$ denotes the cross product.

To compute this, one follows these steps:

- Construct the vector $\overline{P_0P_1} = \langle x_0 - x_1,\ y_0 - y_1,\ z_0 - z_1 \rangle$ (note: P_1 can be *any* point of the line).
- Compute the cross product $\overline{v} \times \overline{P_0P_1}$.
- Take the norm (length) of that cross product vector.
- Divide by the norm of $\overline{v}$.

This formula measures how "far off" the point is from the direction of the line by projecting it orthogonally.

Example 9.24. Find the distance from the point $P_0 = (2, -1, 4)$ to the line $\mathscr{L}$ given by the parametric equations:

$$x = 1 + 2t, \quad y = t, \quad z = -1 + 2t.$$

Solution. The line $\mathscr{L}$ has direction vector $\overline{v} = \langle 2, 1, 2 \rangle$. A point on the line can be found by taking $t = 0$, which gives:

$$P_1 = (1, 0, -1).$$

We construct the vector from P_1 to P_0:

$$\overline{P_1P_0} = \langle 2 - 1,\ -1 - 0,\ 4 - (-1) \rangle = \langle 1, -1, 5 \rangle.$$

Compute the cross product:

$$\overline{v} \times \overline{P_1P_0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 1 & -1 & 5 \end{vmatrix} = \mathbf{i}(1 \cdot 5 - 2 \cdot (-1)) - \mathbf{j}(2 \cdot 5 - 2 \cdot 1) + \mathbf{k}(2 \cdot (-1) - 1 \cdot 1)$$

$$= \mathbf{i}(5 + 2) - \mathbf{j}(10 - 2) + \mathbf{k}(-2 - 1) = \langle 7, -8, -3 \rangle.$$

Now take the norm of the cross product:

$$\left\| \overline{v} \times \overline{P_1P_0} \right\| = \sqrt{7^2 + (-8)^2 + (-3)^2} = \sqrt{49 + 64 + 9} = \sqrt{122}.$$

And the norm of $\overline{v}$:

$$\|\overline{v}\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3.$$

Thus, the distance from the point to the line is:

$$d = \frac{\sqrt{122}}{3}.$$

9.7 Quadric Surfaces

Catalog of Quadric Surfaces

In general, a **quadric surface** has an equation of the form:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

where $A, B, C, D, E, F, G, H, I, J$ are constants.

Using rotations and translations, all these equations can be rewritten in simplified form as either:

$$z = Mx^2 + Ny^2 \qquad \text{or} \qquad Pz + Qy + Rz = S$$

where M, N, P, Q, R, S are constants.

We obtain the following classic **quadric surfaces**:

Elliptic Cone.

$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Traces: to visualize a 3D quadric surface, we often examine its **traces**, which are the curves formed by intersecting the surface with planes like $z =$const. These cross-sectional views help reconstruct the full 3D shape.

- In the plane $z = 0$:

  $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$$

  Since both $\frac{x^2}{a^2} \geq 0$ and $\frac{y^2}{b^2} \geq 0$, the only solution is:

  $$x = 0,\ y = 0 \quad \Rightarrow \quad \text{a point (the vertex)}$$

- For $z = k \neq 0$:

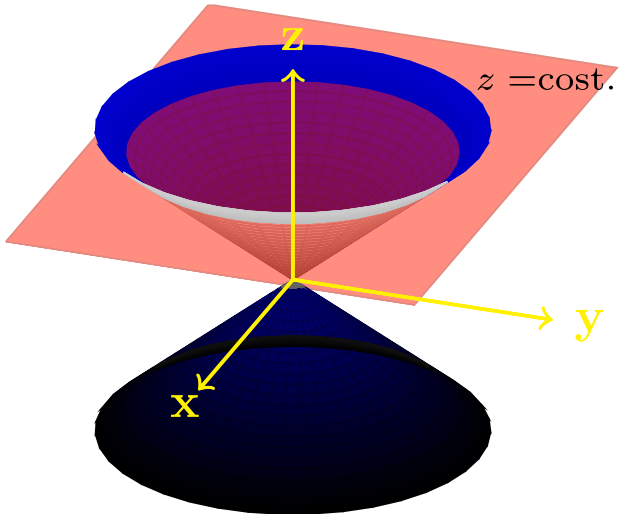
  $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = k^2 \quad \Rightarrow \quad \text{Ellipses for any } k \text{ value.}$$

Thus, horizontal cuts ($z = k$) yield ellipses, while the intersection at $z = 0$ is just a single point.

If $a = b$:

$$z^2 = \frac{x^2 + y^2}{a^2} \quad \Rightarrow \quad \frac{x^2 + y^2}{z^2} = a^2$$

the quadric is called **circular cone**.



Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

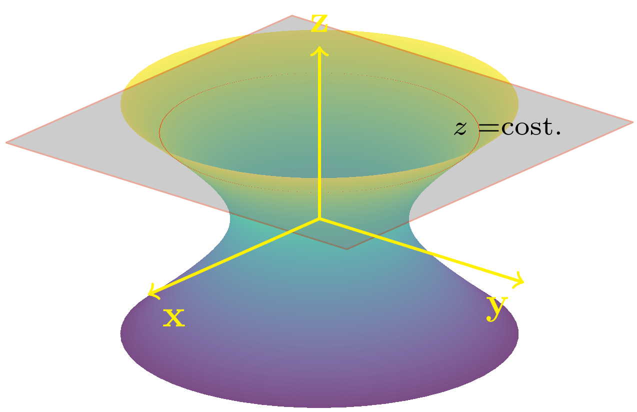
Traces:

- In the plane $z = 0$:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \text{Ellipse}$$

- For $z = k$:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2} \quad \Rightarrow \quad \text{Ellipse for any } k \text{ value, since } 1 + \frac{k^2}{c^2} \text{ is always positive.}$$



Hyperboloid of Two Sheets

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

Traces:

- In the plane $z = 0$:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1 \quad \Rightarrow \quad \text{No points (empty)}$$

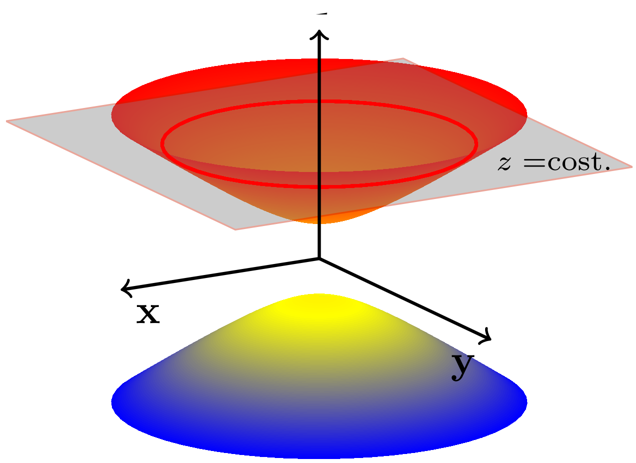
- For $z = k$:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2} - 1 > 0 \quad \Rightarrow \quad \text{Ellipses}$$

  In this case, it is an ellipse iif

$$\frac{k^2}{c^2} - 1 > 0 \quad \Rightarrow \quad k^2 > c^2 \quad \Rightarrow \quad \text{Ellipse if } |k| > c$$

  .



Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Traces.

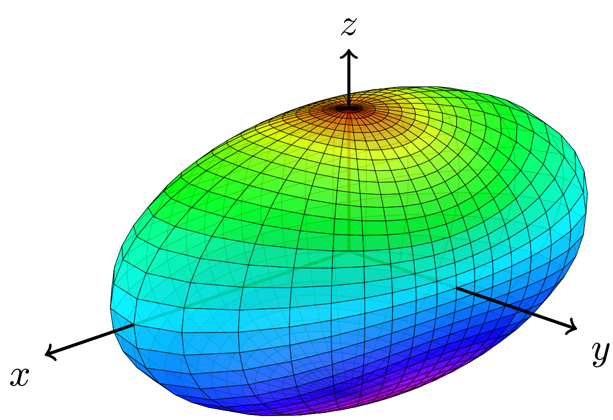
- In the plane $z = 0$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \text{Ellipse}$$

- For $z = k$:
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}$$
In this case, it is an ellipse iif
$$1 - \frac{z^2}{c^2} > 0 \quad \Rightarrow \quad z^2 < c^2 \quad \Rightarrow \quad \text{Ellipse if } |k| < c$$
.

If $a = b = c$, then the ellipsoid is a **sphere**:

$$x^2 + y^2 + z^2 = r^2$$

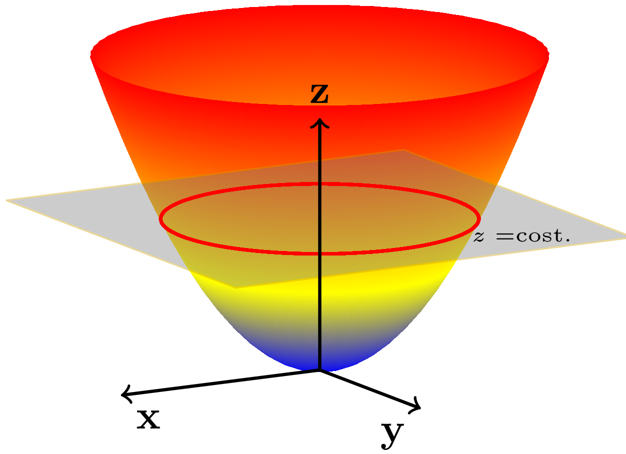


Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Traces:

- For $z = c > 0$:
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = c \quad \Rightarrow \quad \text{Ellipses for} c > 0$$
- For $y = k$:
$$z = \frac{x^2}{a^2} + \frac{k^2}{b^2} \quad \Rightarrow \quad \text{Parabolas}$$
- For $x = h$:
$$z = \frac{h^2}{a^2} + \frac{y^2}{b^2} \quad \Rightarrow \quad \text{Parabolas}$$



Hyperbolic Paraboloid

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

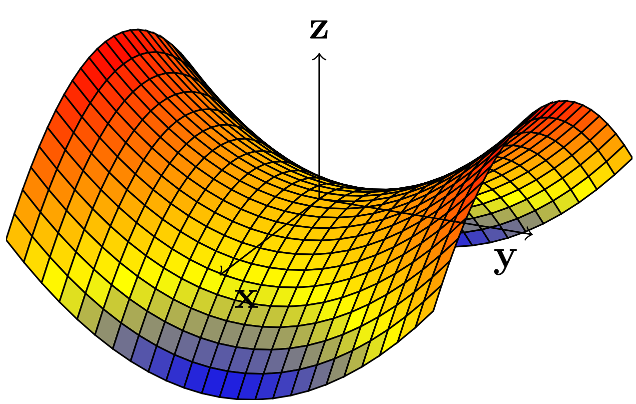
Traces:

- For $z = 0$:

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 0 \quad \Rightarrow \quad \text{Two lines } y = \pm x$$

- For $z = k$:

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = k \quad \Rightarrow \quad \text{Hyperbolas}$$



Example 9.25. Recognize the following quadric surface using the trace technique:

$$9x^2 + 4y^2 - z^2 = 36$$

Find the type of surface and sketch the shape of the traces in the three coordinate planes.

Solution. First, we put the equation in the forms that we have seen previously, by dividing by 36 (in order to normalize the right-hand-side):

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{36} = 1$$

Then, we analyze the traces for different values of z:

- Trace in the xy-plane ($z = 0$):

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \quad \Rightarrow \quad \text{Ellipse.}$$

- Trace in the for a constant z plane ($z = k$):

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{k^2}{36} = 1 \quad \Rightarrow \quad \frac{x^2}{4} + \frac{y^2}{9} = \frac{k^2}{36} + 1$$
$$\text{Ellipse if } \frac{k^2}{36} + 1 > 0 \quad \Rightarrow \quad \text{Ellipse } \forall k \in \mathbb{R}\,.$$

Optionally, we can also consider the intersection of the quadric with the other coordinate planes:

- Trace in the yz-plane ($x = 0$):

$$\frac{y^2}{9} - \frac{z^2}{36} = 1$$

  This is a **hyperbola** opening along the y-axis.

- Trace in the xz-plane ($y = 0$):

$$\frac{x^2}{4} - \frac{z^2}{36} = 1$$

  This is a **hyperbola** opening along the x-axis.

By sketching the given curves, it is easy to see that the quadric is a Hyperboloid of One Sheet.

∎

Example 9.26. Recognize the following quadric surface using the trace technique:

$$z = 2x^2 + 2y^2 + 4$$

Find the type of surface and sketch the shape of the traces in the three coordinate planes.

Solution. The given curve has z as a symmetry axis. Thus, we consider the traces of the surface when cut by $z =$ constant planes.

- $z = 0$ gives:
$$2x^2 + 2y^2 = -4$$
which is impossible.

- $z = k$ gives:
$$k = 2x^2 + 2y^2 + 4 \quad \Rightarrow \quad 2x^2 + 2y^2 = k - 4$$
which is an ellipse (or better, a circle), when
$$k - 4 > 0 \quad \Rightarrow \quad k > 4\,.$$

The figure represented is a paraboloid. In fact, if we consider the traces for x and $y =$ constant, we obtain:

- $x = k$: $z = 2y^2 + \underbrace{4 + 2k^2}_{\text{const.}}$ it is a parabola under certain conditions for k.
- $y = k$: $z = 2x^2 + \underbrace{4 + 2k^2}_{\text{const.}}$ it is a parabola under certain conditions for k.

Remark 9.10. In the examples presented, the quadric surfaces are expressed with the z-axis as the axis of symmetry. However, the same classification and interpretation of traces apply to analogous surfaces with symmetry about the x-axis or y-axis. In those cases, the role of the variables in the equation changes accordingly: typically, the position of the minus sign (or the variable associated with it) shifts to reflect the new orientation. This adjustment does not alter the overall nature of the surface. An exception is the ellipsoid, which remains unchanged under cyclic permutations of x, y, and z, due to its full rotational symmetry.

Example 9.27 (Changing the Axis of Symmetry)**.** Let us illustrate how the axis of symmetry can be changed by permuting the variables in the standard form of a quadric surface.

- **Paraboloid along the y-axis:**

$$y = \frac{x^2}{a^2} + \frac{z^2}{b^2}$$

  This is a *paraboloid* that opens along the y-axis. The cross-sections for $y = k$ are ellipses in the xz-plane, and the traces in planes parallel to yz or xy are parabolas.

- **Hyperboloid of one sheet along the x-axis:**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

  This is a *hyperboloid of one sheet* with its central axis along the x-axis. The cross-sections perpendicular to x (i.e., in the yz-plane) are hyperbolas, while traces for $x = k$ are ellipses.

These examples confirm that quadric surfaces can be oriented along any axis by relabeling the variables in their standard forms.

CHAPTER 10

Vector Valued Functions

Contents

This chapter introduces the main concepts related to vector functions. It is organized as follows:

- **Introduction to vector functions**: definition and basic properties.
- **Differentiation and integration of vector functions**: fundamental rules and applications.
- **Unit tangent, principal unit normal, and curvature**: geometric tools for analyzing the behavior of vector functions.

10.1 Introduction to Vector Functions

A curve in $\mathbb{R}^3$ is the set of all ordered triples

$$(x(t), y(t), z(t)) = (f_1(t), f_2(t), f_3(t))$$

depending on the real variable t, satisfying the parametric equations:

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t).$$

Definition 10.1 (Vector Valued Function)**.** A *vector function* $\overline{F}$ of a real variable t on a domain D assigns to each $t \in D$ a unique vector $\overline{F}(t)$. The set of all vectors of the form $\overline{r} = \overline{F}(t)$, for $t \in D$, is the graph of $\overline{F}$.

$$\overline{F}(t) = \langle f_1(t), f_2(t), f_3(t)\rangle$$

$$\overline{F}(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$$

where $f_1(t), f_2(t), f_3(t)$ are real-valued functions of the real variable t, defined on the domain D. These are called the *components* of $\overline{F}$.

So, we can write:

$$\overline{F}(t) = \langle f_1(t), f_2(t), f_3(t)\rangle$$

Example 10.1. Consider

$$\overline{F}(t) = \langle 3 - t,\, 2t,\, 3t - 4\rangle \quad \forall t$$

This represents the equation of a line defined in $\mathbb{R}^{3}$‘:

$$x = 3 - t \qquad y = 2t \qquad z = 3t - 4$$

That represents the line (in parametric form):

$$\begin{cases} x - 3 = -t \\ y = 2t \\ z + 4 = 3t \end{cases} \qquad \text{Thus: } \overline{v} =< -1, 2, 3 > \qquad P_0(3, 0, -4)$$

Example 10.2. Sketch the graph of the function

$$\overline{F}(t) = (2\sin t)\hat{\imath} - (2\cos t)\hat{\jmath} + (3t)\hat{k}$$

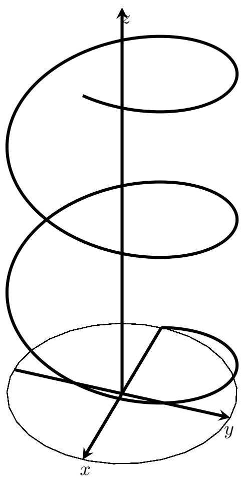
Solution.

$$\begin{cases} x = 2\sin(t) \\ y = -2\cos(t) \\ z = 3t \end{cases}$$

If we consider the first two components squared:

$$(\underbrace{2\sin(t)}_{x})^2 + (\underbrace{-2\cos(t)}_{y})^2 = 4(\sin^2(t) + \cos^{(} t)) = 4 \quad \Rightarrow \quad x^2 + y^2 = 4$$

In the x-y plane the graph of the vector function represents a circle. In the z-direction, it grows linearly ($z = 3t$).



Example 10.3. Find the vector function $\overline{F}(t)$ whose graph is the intersection of the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and the parabolic cylinder $y = x^2$

Solution. Put $x = t$, then the equation of the parabolic cylinder gives $y = t^2$. Plugging everything in the equation of the hemisphere we obtain

$$z = \sqrt{4 - t^2 - (t^2)^2} = \sqrt{4 - t^2 - t^4}\,.$$

Thus,

$$\begin{cases} x = t \\ y = t^2 \\ z = \sqrt{4 - t^2 - t^4} \end{cases}$$

$$\overline{F}(t) = < t, t^2, \sqrt{4 - t^2 - t^4} >$$

Operations of Vector Functions. Given the vector functions $F(t), G(t)$, and the scalar function $f(t)$, we can define the following operations between vector functions:

$$(\overline{F} + \overline{G})(t) = \overline{F}(t) + \overline{G}(t)$$

$$(\overline{F} - \overline{G})(t) = \overline{F}(t) - \overline{G}(t)$$

$$(f \cdot \overline{F})(t) = f(t)\,\overline{F}(t)$$

$$(\overline{F} \times \overline{G})(t) = \overline{F}(t) \times \overline{G}(t)$$

$$(\overline{F} \cdot \overline{G})(t) = \overline{F}(t) \cdot \overline{G}(t)$$

Note: the result of the last operation is a scalar function.

Remark 10.1. If the domain of $\overline{F}$ is D_1 and of $\overline{G}$ is D_2, these operations are defined on $D_1 \cap D_2$.

Example 10.4. Let

$$\overline{F}(t) = t\hat{\imath} + t^2\hat{\jmath} - (t^3 + 1)\hat{k}, \quad \overline{G}(t) = 6t\hat{\imath} + t^3\hat{\jmath} - 5t\hat{k}.$$

Find:

1. $e^t\overline{F}(t) + t\overline{G}(t)$
2. $(\overline{F} \times \overline{G})(t)$
3. $(\overline{F} \cdot \overline{G})(t)$

Solution:

1.
$$e^t\overline{F}(t) + t\overline{G}(t) = (e^t t + 6t^2)\hat{\imath} + (e^t t^2 + t^4)\hat{\jmath} + (-e^t(t^3 + 1) - 5t^2)\hat{k}$$

2.

$$\overline{F}(t) \times \overline{G}(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ t & t^2 & -(t^3+1) \\ 6t & t^3 & -5t \end{vmatrix}.$$

Expanding:

$$= \hat{\imath}\,(t^2(-5t) - (-(t^3+1))t^3) - \hat{\jmath}\,(t(-5t) - (-(t^3+1))(6t)) + \hat{k}\,(t \cdot t^3 - t^2(6t)).$$

$$= \hat{\imath}\,(-5t^3 + (t^3+1)t^3) - \hat{\jmath}\,(-5t^2 + 6t(t^3+1)) + \hat{k}\,(t^4 - 6t^3).$$

$$= \hat{\imath}(t^6 - 4t^3) - \hat{\jmath}(t^2 + 6t) + \hat{k}(t^4 - 6t^3).$$

3.

$$\overline{F}(t) \cdot \overline{G}(t) = (t)(6t) + (t^2)(t^3) + (-(t^3+1))(-5t).$$

$$= 6t^2 + t^5 + 5t^4 + 5t.$$

Limits of a vector function. Given a vector function

$$\overline{F}(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k},$$

we say that the limit $\lim_{t\to t_0} \overline{F}(t)$ exists if and only if each of the scalar limits $\lim_{t\to t_0} f_1(t)$, $\lim_{t\to t_0} f_2(t)$ and $\lim_{t\to t_0} f_3(t)$ exist. In such a case,

$$\lim_{t\to t_0} \overline{F}(t) = \left(\lim_{t\to t_0} f_1(t)\right)\mathbf{i} + \left(\lim_{t\to t_0} f_2(t)\right)\mathbf{j} + \left(\lim_{t\to t_0} f_3(t)\right)\mathbf{k}.$$

Example 10.5. Consider $\overline{F}(t) = (2t)\,\mathbf{i} + (t^2)\,\mathbf{j} + (3)\,\mathbf{k}$. Then

$$\lim_{t\to 1} \overline{F}(t) = (2)\,\mathbf{i} + (1)\,\mathbf{j} + (3)\,\mathbf{k}.$$

Example 10.6. Consider $\overline{F}(t) = \frac{\sin t}{t}\,\mathbf{i} + (t+1)\,\mathbf{j} + (t^2)\,\mathbf{k}$. Then

$$\lim_{t\to 0} \overline{F}(t) = (1)\,\mathbf{i} + (1)\,\mathbf{j} + (0)\,\mathbf{k},$$

since $\lim_{t\to 0} \frac{\sin t}{t} = 1$.

Properties. The operations on limits of vector functions follow the same rules as scalar limits. In particular, if $\lim_{t\to t_0}\overline{F}(t)$ and $\lim_{t\to t_0}\overline{G}(t)$ both exist, then:

$$\lim_{t\to t_0}\left(\overline{F}(t)+\overline{G}(t)\right)=\lim_{t\to t_0}\overline{F}(t)+\lim_{t\to t_0}\overline{G}(t),$$

$$\lim_{t\to t_0}\left(c\,\overline{F}(t)\right)=c\cdot\lim_{t\to t_0}\overline{F}(t),$$

for any scalar $c\in\mathbb{R}$. Moreover, the following hold:

$$\lim_{t\to t_0}\left(\overline{F}(t)\cdot\overline{G}(t)\right)=\left(\lim_{t\to t_0}\overline{F}(t)\right)\cdot\left(\lim_{t\to t_0}\overline{G}(t)\right),$$

$$\lim_{t\to t_0}\left(\overline{F}(t)\times\overline{G}(t)\right)=\left(\lim_{t\to t_0}\overline{F}(t)\right)\times\left(\lim_{t\to t_0}\overline{G}(t)\right).$$

Example 10.7. Let $\overline{F}(t)=(t)\,\mathbf{i}+(1)\,\mathbf{j}+(0)\,\mathbf{k}$ and $\overline{G}(t)=(2)\,\mathbf{i}+(t)\,\mathbf{j}+(1)\,\mathbf{k}$. Then

$$\overline{F}(t)\cdot\overline{G}(t)=2t+t+0=3t,$$

so

$$\lim_{t\to 1}\left(\overline{F}(t)\cdot\overline{G}(t)\right)=3.$$

On the other hand,

$$\lim_{t\to 1}\overline{F}(t)=(1)\,\mathbf{i}+(1)\,\mathbf{j}+(0)\,\mathbf{k},\quad \lim_{t\to 1}\overline{G}(t)=(2)\,\mathbf{i}+(1)\,\mathbf{j}+(1)\,\mathbf{k},$$

and their dot product is

$$(1)(2)+(1)(1)+(0)(1)=3.$$

Thus we confirm

$$\lim_{t\to 1}\left(\overline{F}(t)\cdot\overline{G}(t)\right)=\left(\lim_{t\to 1}\overline{F}(t)\right)\cdot\left(\lim_{t\to 1}\overline{G}(t)\right).$$

Example 10.8. Let $\overline{F}(t)=(t)\,\mathbf{i}+(0)\,\mathbf{j}+(1)\,\mathbf{k}$ and $\overline{G}(t)=(1)\,\mathbf{i}+(t)\,\mathbf{j}+(0)\,\mathbf{k}$. Then

$$\overline{F}(t)\times\overline{G}(t)=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ t&0&1\\ 1&t&0\end{vmatrix}=(-t)\,\mathbf{i}+(1)\,\mathbf{j}+(t^2)\,\mathbf{k}.$$

Hence

$$\lim_{t\to 1}\left(\overline{F}(t)\times\overline{G}(t)\right)=(-1)\,\mathbf{i}+(1)\,\mathbf{j}+(1)\,\mathbf{k}.$$

On the other hand,

$$\lim_{t\to 1} \overline{F}(t) = (1)\,\mathbf{i} + (0)\,\mathbf{j} + (1)\,\mathbf{k}, \quad \lim_{t\to 1} \overline{G}(t) = (1)\,\mathbf{i} + (1)\,\mathbf{j} + (0)\,\mathbf{k},$$

and their cross product is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = (-1)\,\mathbf{i} + (1)\,\mathbf{j} + (1)\,\mathbf{k}.$$

Thus we confirm

$$\lim_{t\to 1} \left(\overline{F}(t) \times \overline{G}(t)\right) = \left(\lim_{t\to 1} \overline{F}(t)\right) \times \left(\lim_{t\to 1} \overline{G}(t)\right).$$

Continuity of a vector function.

Definition 10.2 (Continuity of a vector function)**.** A vector function

$$\overline{F}(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$$

is said to be **continuous at** t_0 if

$$\lim_{t\to t_0} \overline{F}(t) = \overline{F}(t_0).$$

Equivalently, $\overline{F}(t)$ is continuous at t_0 if and only if each component function $f_1(t)$, $f_2(t)$, and $f_3(t)$ is continuous at t_0.

Example 10.9. Given

$$\overline{F}(t) = \sin t\,\mathbf{i} + \frac{1}{1-t}\,\mathbf{j} + \ln(t)\,\mathbf{k},$$

determine where $\overline{F}$ is continuous.

Solution. A vector function is continuous where all its components are continuous.

- $\sin t$ is continuous for all $t \in \mathbb{R}$.
- $\dfrac{1}{1-t}$ is continuous for $t \neq 1$.
- $\ln(t)$ is continuous for $t > 0$.

Intersecting these domains gives $t > 0$ and $t \neq 1$. Hence

$$\overline{F}(t) \text{ is continuous on } (0,1) \cup (1,\infty).$$

10.2 Differentiation and Integration of Vector Functions

10.2.1 Vector Derivatives.

Definition 10.3. Derivative of a Vector Function The *difference quotient* of a vector function $\overline{F}$ is given by

$$\Delta\overline{F} = \overline{F}(t+\Delta t) - \overline{F}(t), \qquad \frac{\Delta\overline{F}}{\Delta t}.$$

The *derivative* of $\overline{F}$ at t is defined as

$$\overline{F}'(t) = \lim_{\Delta t\to 0}\frac{\Delta\overline{F}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\overline{F}(t+\Delta t) - \overline{F}(t)}{\Delta t},$$

whenever this limit exists.

In short notation, we write

$$\overline{F}'(t) = \frac{d}{dt}\overline{F}(t).$$

We say that $\overline{F}$ is differentiable at t_0 if $\overline{F}'(t)$ exists at t_0.

Theorem 10.1 (Derivative of a Vector Function)**.** The vector function

$$\overline{F}(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$$

is differentiable whenever the component functions f_1, f_2, f_3 are each differentiable, and in this case

$$\overline{F}'(t) = f_1'(t)\,\mathbf{i} + f_2'(t)\,\mathbf{j} + f_3'(t)\,\mathbf{k}.$$

Proof.

$$\begin{aligned}\overline{F}'(t) &= \lim_{\Delta t\to 0}\frac{\overline{F}(t+\Delta t) - \overline{F}(t)}{\Delta t}.\\ &= \lim_{\Delta t\to 0}\frac{\big(f_1(t+\Delta t) - f_1(t)\big)\mathbf{i} + \big(f_2(t+\Delta t) - f_2(t)\big)\mathbf{j} + \big(f_3(t+\Delta t) - f_3(t)\big)\mathbf{k}}{\Delta t}.\end{aligned}$$

Which gives

$$\overline{F}'(t) = \Bigg(\underbrace{\lim_{\Delta t \to 0} \frac{f_1(t+\Delta t) - f_1(t)}{\Delta t}}_{f_1'(t)}\Bigg)\,\mathbf{i} + \Bigg(\underbrace{\lim_{\Delta t \to 0} \frac{f_2(t+\Delta t) - f_2(t)}{\Delta t}}_{f_2'(t)}\Bigg)\,\mathbf{j} + $$
$$+ \Bigg(\underbrace{\lim_{\Delta t \to 0} \frac{f_3(t+\Delta t) - f_3(t)}{\Delta t}}_{f_3'(t)}\Bigg)\,\mathbf{k} = f_1'(t)\,\mathbf{i} + f_2'(t)\,\mathbf{j} + f_3'(t)\,\mathbf{k}.$$

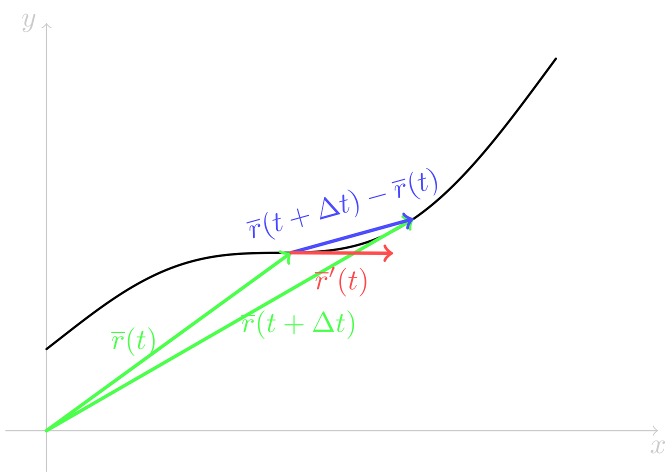
□

Geometric interpretation of the vector derivative. Let $\overline{r}(t)$ denote the position vector of a point moving along a space curve C. The derivative

$$\overline{r}'(t) = \lim_{\Delta t \to 0} \frac{\overline{r}(t+\Delta t) - \overline{r}(t)}{\Delta t}$$

is a *tangent vector* to C at the point $\overline{r}(t)$.

For a general vector function $\overline{F}(t)$ the derivative gives the instantaneous rate of change of each component, and when $\overline{F}$ is a position vector it becomes the geometric tangent described above.



Definition 10.4. Suppose that $\overline{F}(t)$ is differentiable at t_0, and that $\overline{F}'(t_0) \neq 0$. Then, $\overline{F}'(t_0)$ is a tangent vector to the graph of $\overline{F}(t)$ at $t = t_0$, and points in the direction determined by increasing t.

Example 10.10. Given the vector function

$$\overline{F}(t) = e^{2t}\,\overline{i} + (t^2 - t)\,\overline{j} + \ln(t)\,\overline{k},$$

we want to find a tangent vector to the curve at $t = 1$, and the tangent line at that point.
Solution.

1. *Compute the derivative.* The derivative $\overline{F}'(t)$ gives a tangent vector to the curve:

$$\overline{F}'(t) = \frac{d}{dt}\left(e^{2t}\right)\overline{i} + \frac{d}{dt}(t^2 - t)\,\overline{j} + \frac{d}{dt}(\ln(t))\,\overline{k}.$$

   Compute each component:

$$\frac{d}{dt}e^{2t} = 2e^{2t}, \quad \frac{d}{dt}(t^2 - t) = 2t - 1, \quad \frac{d}{dt}\ln(t) = \frac{1}{t}.$$

   Thus,

$$\overline{F}'(t) = 2e^{2t}\,\overline{i} + (2t - 1)\,\overline{j} + \frac{1}{t}\,\overline{k}.$$

2. *Evaluate at $t = 1$*

$$\overline{F}'(1) = 2e^2\,\overline{i} + 1\,\overline{j} + 1\,\overline{k}.$$

   This is the tangent vector at $t = 1$.

To find now the tangent line, we need to find the line with direction vector

$$\overline{F}'(1) = 2e^2\,\overline{i} + 1\,\overline{j} + 1\,\overline{k} = < 2e^2, 1, 1 > .$$

and that passes through the point P pointed by $\overline{F}(1)$.

$$\overline{F}(1) = e^2\,\overline{i} + 0\,\overline{j} + 0\,\overline{k} = e^2\,\overline{i} \qquad \Rightarrow \qquad P(e^2, 0, 0)\,.$$

Thus, the tangent line is (using parameter r, since t has been used for $\overline{F}$)

$$\begin{cases} x - e^2 = 2e^2 r \\ y = t \\ z = t \end{cases}$$

Definition 10.5. A *smooth curve* in $\mathbb{R}^n$ is a vector function

$$\overline{F}(t) = \big(F_1(t), F_2(t), \ldots, F_n(t)\big),$$

defined on an interval $I \subset \mathbb{R}$, such that:

1. Each component function $F_i(t)$ is *continuously differentiable* on I.
2. The derivative $\overline{F}'(t) \neq \overline{0}$ for all $t \in I$. That is, the curve has *no stationary points.*

Intuitively, a smooth curve has a well-defined *tangent vector* at every point, and no "corners" or cusps.

Higher Order Derivatives.

Definition 10.6 (Higer Order Derivatives)**.** Let

$$\overline{F}(t) = \langle f_1(t), f_2(t), f_3(t)\rangle$$

be a vector function with component functions that are sufficiently differentiable. The *higher-order derivatives* of $\overline{F}(t)$ are defined as:

$$\overline{F}''(t) = \frac{d^2\overline{F}}{dt^2} = \langle f_1''(t), f_2''(t), f_3''(t)\rangle, \quad \overline{F}'''(t) = \frac{d^3\overline{F}}{dt^3} = \langle f_1'''(t), f_2'''(t), f_3'''(t)\rangle,$$

and in general,

$$\overline{F}^{(n)}(t) = \frac{d^n\overline{F}}{dt^n} = \langle f_1^{(n)}(t), f_2^{(n)}(t), f_3^{(n)}(t)\rangle.$$

These vectors represent the rate of change of the previous derivative. For example, $\overline{F}''(t)$ is the *acceleration vector* if $\overline{F}'(t)$ is the velocity vector of a particle moving along the curve.

Example 10.11. Consider the vector function

$$\overline{F}(t) = \langle t^3, \sin(t), e^t \rangle.$$

1. Compute the first derivative (velocity)

$$\overline{F}'(t) = \langle 3t^2, \cos(t), e^t \rangle.$$

2. Compute the second derivative (acceleration)

$$\overline{F}''(t) = \langle 6t, -\sin(t), e^t \rangle.$$

3. Compute the third derivative

$$\overline{F}'''(t) = \langle 6, -\cos(t), e^t \rangle.$$

Properties of Vector Function Differentiation.

Theorem 10.2 (Rules for differentiating vector functions)**.** Let the vector functions $\overline{F}$ and $\overline{G}$ be differentiable at t, and let the scalar function h be differentiable at t. Then $\alpha\overline{F} + \beta\overline{G}$, $h\overline{F}$, $\overline{F} \cdot \overline{G}$, and $\overline{F} \times \overline{G}$ are also differentiable at t, and the following rules hold:

1. *Linearity rule:* $(\alpha\overline{F} + \beta\overline{G})'(t) = \alpha\overline{F}'(t) + \beta\overline{G}'(t)$
2. *Scalar multiple rule:* $(h\overline{F})'(t) = h'(t)\,\overline{F}(t) + h(t)\,\overline{F}'(t)$
3. *Dot product rule:* $(\overline{F} \cdot \overline{G})'(t) = \overline{F}'(t) \cdot \overline{G}(t) + \overline{F}(t) \cdot \overline{G}'(t)$
4. *Cross product rule:* $(\overline{F} \times \overline{G})'(t) = \overline{F}'(t) \times \overline{G}(t) + \overline{F}(t) \times \overline{G}'(t)$

Proof of the linearity rule. Consider the difference quotient of the linear combination $\alpha\overline{F} + \beta\overline{G}$:

$$\frac{\Delta(\alpha\overline{F} + \beta\overline{G})}{\Delta t} = \frac{\alpha\Delta\overline{F} + \beta\Delta\overline{G}}{\Delta t} = \alpha\frac{\Delta\overline{F}}{\Delta t} + \beta\frac{\Delta\overline{G}}{\Delta t}.$$

Taking the limit as $\Delta t \to 0$, we get

$$(\alpha\overline{F} + \beta\overline{G})'(t) = \alpha \lim_{\Delta t \to 0} \frac{\Delta\overline{F}}{\Delta t} + \beta \lim_{\Delta t \to 0} \frac{\Delta\overline{G}}{\Delta t} = \alpha\overline{F}'(t) + \beta\overline{G}'(t).$$

□

Example 10.12. Let

$$\overline{F}(t) = \langle t, t^2, t^3 \rangle, \quad \overline{G}(t) = \langle \sin t, \cos t, e^t \rangle, \quad h(t) = t^2.$$

Compute the derivatives using the rules from Theorem.

1. *Sum of vectors:*

$$(\overline{F}+\overline{G})'(t) = \overline{F}'(t)+\overline{G}'(t) = \langle 1, 2t, 3t^2 \rangle+\langle \cos t, -\sin t, e^t \rangle = \langle 1+\cos t, 2t-\sin t, 3t^2+e^t \rangle$$

2. *Scalar multiple:*

$$(h\overline{F})'(t) = h'(t)\,\overline{F}(t) + h(t)\,\overline{F}'(t) = 2t\langle t, t^2, t^3 \rangle + t^2\langle 1, 2t, 3t^2 \rangle = \langle 3t^2, 4t^3, 5t^4 \rangle$$

3. *Dot product:*

$$(\overline{F}\cdot\overline{G})'(t) = \overline{F}'\cdot\overline{G}+\overline{F}\cdot\overline{G}' = \langle 1, 2t, 3t^2 \rangle \cdot \langle \sin t, \cos t, e^t \rangle + \langle t, t^2, t^3 \rangle \cdot \langle \cos t, -\sin t, e^t \rangle$$

$$= \sin t + 2t\cos t + 3t^2 e^t + t\cos t - t^2 \sin t + t^3 e^t = (\sin t - t^2 \sin t) + 3t\cos t + 4t^2 e^t + t^3 e^t$$

Theorem 10.3 (Orthogonality of a function of constant length and its derivative)**.** Let $\overline{F}(t)$ be a differentiable vector function of constant length. Then $\overline{F}(t)$ is orthogonal to its derivative $\overline{F}'(t)$.

Proof. Since $\overline{F}(t)$ has constant length, we have

$$\|\overline{F}(t)\| = \text{constant} \quad \Rightarrow \quad \|\overline{F}(t)\|^2 = \overline{F}(t)\cdot\overline{F}(t) = \text{constant}^2.$$

Differentiating both sides with respect to t gives

$$\frac{d}{dt}\big[\overline{F}(t)\cdot\overline{F}(t)\big] = \frac{d}{dt}\big(\text{constant}^2\big) = 0.$$

Using the product rule for the dot product, we get

$$\overline{F}'(t)\cdot\overline{F}(t) + \overline{F}(t)\cdot\overline{F}'(t) = 2\,\overline{F}(t)\cdot\overline{F}'(t) = 0 \quad \Rightarrow \quad \overline{F}(t) \text{ and } \overline{F}'(t) \text{ are orthogonal.}$$

(To prove this theorem, we used the important Theorem 9.2). □

Example 10.13 (Spatial Application.)**.** Suppose

$$\overline{R}(t) = \langle x(t), y(t), z(t) \rangle$$

is a 3D vector function whose length lies entirely on the sphere

$$x^2 + y^2 + z^2 = r^2 \quad \text{(constant length)}.$$

Let P_0 correspond to $t = t_0$. Then $\overline{R}(t_0)$ is the radius connecting the center to P_0, and thus $\overline{R}'(t_0)$ is tangent to the sphere at P_0.

10.2.2 Integrals of Vector Functions

Let $\overline{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$ be a vector function defined on an interval $I \subset \mathbb{R}$. The *definite integral* of $\overline{F}(t)$ from $t = a$ to $t = b$ is defined component-wise as

$$\int_a^b \overline{F}(t)\, dt = \left\langle \int_a^b f_1(t)\, dt,\ \int_a^b f_2(t)\, dt,\ \int_a^b f_3(t)\, dt \right\rangle.$$

Similarly, the *indefinite integral* (antiderivative) is

$$\int \overline{F}(t)\, dt = \left\langle \int f_1(t)\, dt,\ \int f_2(t)\, dt,\ \int f_3(t)\, dt \right\rangle + \overline{C},$$

where $\overline{C}$ is a constant vector of integration.

Properties Some useful properties of vector function integrals include:

- **Linearity:** For scalar constants α, β and vector functions $\overline{F}, \overline{G}$,

$$\int \big(\alpha\overline{F}(t) + \beta\overline{G}(t)\big)\, dt = \alpha \int \overline{F}(t)\, dt + \beta \int \overline{G}(t)\, dt.$$

- **Fundamental Theorem of Calculus:** If $\overline{F}(t)$ is continuous on $[a, b]$ and $\overline{R}(t)$ is an antiderivative of $\overline{F}(t)$, then

$$\int_a^b \overline{F}(t)\, dt = \overline{R}(b) - \overline{R}(a).$$

Example 10.14. Consider the vector function

$$\overline{F}(t) = \langle 2t,\, \cos t,\, e^t \rangle.$$

Indefinite integral. The indefinite integral of $\overline{F}(t)$ is computed component-wise:

$$\int \overline{F}(t)\, dt = \left\langle \int 2t\, dt,\ \int \cos t\, dt,\ \int e^t\, dt \right\rangle + \overline{C},$$

where $\overline{C} = \langle C_1, C_2, C_3 \rangle$ is a constant vector. Evaluating each component, we get

$$\int \overline{F}(t)\, dt = \langle t^2,\, \sin t,\, e^t \rangle + \overline{C}.$$

Definite integral. To compute the definite integral from $t = 0$ to $t = 1$:

$$\int_0^1 \overline{F}(t)\, dt = \left\langle \int_0^1 2t\, dt,\ \int_0^1 \cos t\, dt,\ \int_0^1 e^t\, dt \right\rangle = \langle 1,\, \sin 1,\, e - 1 \rangle.$$

10.2.3 Modeling the motion of an object in $\mathbb{R}^3$

Let an object move in three-dimensional space and let its *position vector* at time t be

$$\overline{R}(t) = x(t)\, \overline{i} + y(t)\, \overline{j} + z(t)\, \overline{k},$$

where $x(t)$, $y(t)$, and $z(t)$ are the coordinates of the object in the standard Cartesian axes.

Velocity. The *velocity vector* $\overline{V}(t)$ of the object is defined as the derivative of the position vector with respect to time:

$$\overline{V}(t) = \frac{d\overline{R}}{dt} = \frac{dx}{dt}\, \overline{i} + \frac{dy}{dt}\, \overline{j} + \frac{dz}{dt}\, \overline{k}.$$

Acceleration. The *acceleration vector* $\overline{A}(t)$ is the derivative of the velocity vector with respect to time:

$$\overline{A}(t) = \frac{d\overline{V}}{dt} = \frac{d^2x}{dt^2}\, \overline{i} + \frac{d^2y}{dt^2}\, \overline{j} + \frac{d^2z}{dt^2}\, \overline{k}.$$

Speed. The *speed* $v(t)$ of the object is the magnitude of the velocity vector:

$$v(t) = \|\overline{V}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}.$$

Trajectory. The *trajectory* of the object is the curve in $\mathbb{R}^3$ traced by the tip of the position vector $\overline{R}(t)$ as t varies. In other words, it is the set

$$\left\{(x(t), y(t), z(t)) \in \mathbb{R}^3 \mid t \in I\right\},$$

where I is the time interval during which the motion occurs.

Example 10.15. Consider a particle whose position vector is

$$\overline{R}(t) = \langle \cos t,\, \sin t,\, t^3 \rangle.$$

Find the velocity, acceleration, and speed of the object. Also, find the direction of the motion for $t = 2s$.

Solution. The velocity vector is the derivative of the position:

$$\overline{V}(t) = \frac{d\overline{R}}{dt} = \langle -\sin t,\, \cos t,\, 3t^2 \rangle.$$

The acceleration vector is the derivative of the velocity:

$$\overline{A}(t) = \frac{d\overline{V}}{dt} = \langle -\cos t,\, -\sin t,\, 6t \rangle.$$

The speed is the magnitude of the velocity:

$$v(t) = \|\overline{V}(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (3t^2)^2} = \sqrt{1 + 9t^4}.$$

Direction of motion at $t = 2$ s: The direction of motion is given by the *velocity vector* at $t = 2$.

$$\overline{V}(2) = \langle -\sin 2,\, \cos 2,\, 12 \rangle,$$

Example 10.16 (Position of a particle given its velocity)**.** The velocity of a particle is

$$\overline{V}(t) = e^t\,\overline{i} + t^2\,\overline{j} + \cos(2t)\,\overline{k}, \quad \text{with} \quad \overline{R}(0) = 2\,\overline{i} + 1\,\overline{j} - 1\,\overline{k}.$$

Find the position of the particle.

Solution. Since $\overline{V}(t) = \frac{d\overline{R}(t)}{dt}$, then

$$\overline{R}(t) = \int \overline{V}(t)\, dt + \overline{C}$$

where $\overline{C} = \langle C_1, C_2, C_3 \rangle$. So we have:

$$\int \overline{V}(t)\, dt = \left\langle \int e^t dt,\ \int t^2 dt,\ \int \cos(2t) dt \right\rangle = \left\langle e^t,\ \frac{t^3}{3},\ \tfrac{1}{2}\sin(2t) \right\rangle + \overline{C}\,. \qquad (10.1)$$

Determining the constants with the initial condition. Imposing $\overline{R}(0) = \langle 2, 1, -1 \rangle$ in (10.1):

$$\overline{R}(0) = \left\langle e^0 + C_1,\ 0 + C_2,\ \tfrac{1}{2}\sin 0 + C_3 \right\rangle = \langle 1 + C_1,\ C_2,\ C_3 \rangle = \langle 2, 1, -1 \rangle.$$

Thus,

$$\begin{cases} 1 + C_1 = 2 \\ C_2 = 1 \\ C_3 = -1 \end{cases} \qquad \Rightarrow \qquad C_1 = 1,\ C_2 = 1,\ C_3 = -1$$

Concluding, the position of the particle is

$$\overline{R}(t) = \left\langle e^t + 1,\ \frac{t^3}{3} + 1,\ \frac{1}{2}\sin(2t) - 1 \right\rangle$$

10.4 Unit Tangent, unit principal normal and curvature

In the study of curves defined by a vector function of a real parameter, the notions of *unit tangent*, *principal unit normal*, and *curvature* are fundamental in understanding the geometry of the motion. These vectors describe how the trajectory evolves and how it bends at each point.

10.4.1 Unit Tangent and unit principal normal

Definition 10.7. Given a differentiable vector function $\overline{R}(t)$ that describes a motion in space, the *unit tangent vector* is defined as

$$\overline{T}(t) = \frac{\overline{R}'(t)}{\|\overline{R}'(t)\|},$$

which points in the direction of motion. The *principal unit normal vector* is then defined as

$$\overline{N}(t) = \frac{\overline{T}'(t)}{\|\overline{T}'(t)\|},$$

which points in the direction of the instantaneous change of the tangent.

Example 10.17. Consider the curve defined by

$$\overline{R}(t) = \langle 3\sin(t),\, 4t,\, 3\cos(t)\rangle.$$

Find its unit tangent and principal unit normal vectors.
Solution. First, we compute the derivative

$$\overline{R}'(t) = \langle 3\cos(t),\, 4,\, -3\sin(t)\rangle,$$

with norm

$$\|\overline{R}'(t)\| = \sqrt{(3\cos(t))^2 + 4^2 + (-3\sin(t))^2} = \sqrt{9\cos^2(t) + 16 + 9\sin^2(t)} = 5.$$

Hence, the unit tangent is

$$\overline{T}(t) = \left\langle \tfrac{3}{5}\cos(t),\, \tfrac{4}{5},\, -\tfrac{3}{5}\sin(t) \right\rangle.$$

Next, we compute

$$\overline{T}'(t) = \left\langle -\tfrac{3}{5}\sin(t),\, 0,\, -\tfrac{3}{5}\cos(t) \right\rangle,$$

with norm

$$\|\overline{T}'(t)\| = \frac{3}{5}\sqrt{\sin^2(t) + \cos^2(t)} = \tfrac{3}{5}.$$

Thus, the unit normal vector is

$$\overline{N}(t) = \langle -\sin(t),\, 0,\, -\cos(t) \rangle.$$

10.4.2 Arc length as a parameter.

When a curve is described by a parameter t, the *distance* traveled along the curve between two points can be computed using the arc length. This provides a natural reparametrization of the curve in terms of the distance traveled rather than the parameter t.

Definition 10.8. In the plane, consider a curve given by the parametric representation

$$\overline{R}(t) = x(t)\,\overline{i} + y(t)\,\overline{j}, \qquad a \le t \le b.$$

The **arc length** of the curve from $t = a$ to $t = b$ is defined as

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

This definition extends naturally to three dimensions.

Definition 10.9. For a space curve represented by

$$\overline{R}(t) = x(t)\,\overline{i} + y(t)\,\overline{j} + z(t)\,\overline{k}, \qquad a \le t \le b,$$

the **arc length** of the curve from $t = a$ to $t = b$ is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Equivalently, using the derivative of the vector function, we can write

$$s = \int_a^b \left\| \overline{R}'(t) \right\| \, dt. \tag{10.2}$$

Definition 10.10. The **arc length function** associated with a vector function $\overline{R}(t)$ is defined as

$$s(t) = \int_{t_0}^{t} \left\| \overline{R}'(u) \right\| \, du,$$

where t_0 is the initial parameter value. This function measures the distance traveled along the curve from t_0 up to t.

Example 10.18 (Arc length between two given values of t)**.** Consider the curve described by

$$\overline{R}(t) = \langle 2t,\, 3\sin(t),\, 5 - 3\cos(t) \rangle, \quad t \in [0, 2].$$

Solution. The derivative is

$$\overline{R}'(t) = \langle 2, 3\cos(t), 3\sin(t) \rangle,$$

so that

$$\left\| \overline{R}'(t) \right\| = \sqrt{4 + 9\cos^2(t) + 9\sin^2(t)} = \sqrt{4+9} = \sqrt{13}.$$

The arc length from $t = 0$ to $t = 2$ is therefore

$$s = \int_0^2 \sqrt{13}\, dt = 2\sqrt{13}\,.$$

Example 10.19 (Arc length for a generic t)**.** Let

$$\overline{R}(t) = \langle \cos(t),\, \sin(t),\, t \rangle.$$

Find the arc length function for $t_0 = 0$.
Solution. We have

$$\overline{R}'(t) = \langle -\sin(t),\, \cos(t),\, 1 \rangle,$$

so that

$$\left\| \overline{R}'(t) \right\| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2} = \sqrt{2}.$$

Hence, the arc length from $t = 0$ to a generic time t is

$$s(t) = \int_0^t \sqrt{2}\, du = \sqrt{2}\, t.$$

Theorem 10.4 (Speed as derivative of the arc length)**.** Let $\overline{R}(t)$ be a smooth vector function that describes a motion in space, and let $s(t)$ denote the arc length defined in (10.2). Then, the derivative of $s(t)$ with respect to t equals the speed of the motion:

$$\frac{ds}{dt} = \left\|\overline{R}'(t)\right\|.$$

Theorem 10.5 (Formulas for $\overline{T}$ and $\overline{N}$ in terms of the arc length s)**.** Let $\overline{R}(s)$ be a smooth curve parametrized by the arc length s. Then:

- The *unit tangent vector* is given by

$$\overline{T}(s) = \frac{d\overline{R}}{ds}.$$

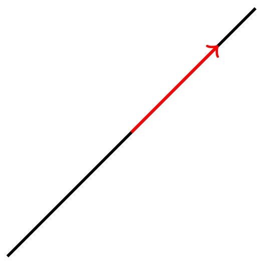
- The *principal unit normal vector* is obtained by differentiating the unit tangent and normalizing:

$$\overline{N}(s) = \frac{d\overline{T}/ds}{\left\|d\overline{T}/ds\right\|} = \frac{d\overline{T}/ds}{k}.$$

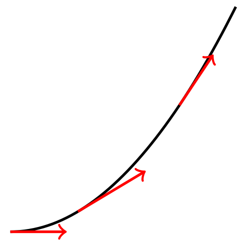
  where $k = \left\|d\overline{T}/ds\right\|$.

10.4.3 The curvature

The *curvature* is used to represent how the tangent vector changes when the arc length changes.



No change: always the same tangent



The tangent vector changes as a function of s

Definition 10.11 (Curvature)**.** Suppose the smooth curve C is the graph of the vector function $\overline{R}(s)$ parametrized in terms of the arc length s. Then the *curvature* of C is the function

$$\kappa(s) = \left\| \frac{d\overline{T}}{ds} \right\|,$$

where $\overline{T}(s)$ is the unit tangent vector.

Remark 10.2. In terms of the variable t, the curvature can be expressed as

$$\kappa(t) = \frac{\left\| \overline{T}'(t) \right\|}{\left\| \overline{R}'(t) \right\|}.$$

Example 10.20. Show that the circle with radius r,

$$\overline{R}(t) = \langle r\cos t,\ r\sin t \rangle, \quad r > 0,$$

has curvature $\kappa = \frac{1}{r}$.

Solution.

$$\overline{R}'(t) = \langle -r\sin t,\ r\cos t \rangle,$$
$$\overline{R}''(t) = \langle -r\cos t,\ -r\sin t \rangle.$$

The unit tangent vector is

$$\overline{T}(t) = \frac{\overline{R}'(t)}{\|\overline{R}'(t)\|} = \frac{\langle -r\sin t,\ r\cos t \rangle}{\sqrt{r^2\sin^2 t + r^2\cos^2 t}} = \langle -\cos t,\ \sin t \rangle.$$

Thus,

$$\overline{T}'(t) = \langle \sin t,\ \cos t \rangle, \quad \|\overline{T}'(t)\| = 1,$$

and

$$\|\overline{R}'(t)\| = r.$$

Therefore,

$$\kappa = \frac{\|\overline{T}'(t)\|}{\|\overline{R}'(t)\|} = \frac{1}{r}.$$

Theorem 10.6 (The Cross Product Derivative Formula for Curvature)**.** Suppose the curve C is the graph of the vector function $\overline{R}(t)$. Then the curvature is given by

$$\kappa = \frac{\|\overline{R}'(t) \times \overline{R}''(t)\|}{\|\overline{R}'(t)\|^3}.$$

Example 10.21. Compute the curvature of the helix

$$\overline{R}(t) = \langle a\cos t,\ a\sin t,\ bt\rangle.$$

Solution.

$$\overline{R}'(t) = \langle -a\sin t,\ a\cos t,\ b\rangle, \qquad \overline{R}''(t) = \langle -a\cos t,\ -a\sin t,\ 0\rangle.$$

Cross product:

$$\overline{R}'(t) \times \overline{R}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = \langle ab\sin t,\ -ab\cos t,\ a^2\rangle.$$

Magnitudes:

$$\left\|\overline{R}'(t) \times \overline{R}''(t)\right\| = \sqrt{a^2b^2(\sin^2 t + \cos^2 t) + a^4} = a\sqrt{a^2+b^2},$$

$$\left\|\overline{R}'(t)\right\| = \sqrt{a^2(\sin^2 t + \cos^2 t) + b^2} = \sqrt{a^2+b^2}.$$

Therefore, by the cross–product formula for curvature,

$$\kappa(t) = \frac{\left\|\overline{R}'(t) \times \overline{R}''(t)\right\|}{\left\|\overline{R}'(t)\right\|^3} = \frac{a\sqrt{a^2+b^2}}{(a^2+b^2)^{3/2}} = \frac{a}{a^2+b^2}$$

which is constant (independent of t).

Theorem 10.7 (Curvature of a Planar Curve)**.** Let the curve be given in Cartesian form by

$$y = f(x),$$

where f has continuous second derivative. The *curvature* κ of the curve at a point $(x, f(x))$ is

$$\kappa(x) = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}}.$$