Green Measures for (Time Changed) Markov Processes.

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The Problem

- Consider a time homogeneous Markov process $X(t)$, $t \geq 0$ starting from $x \in \mathbb{R}^d$.
- For suitable function $f : \mathbb{R}^d \to \mathbb{R}$ we consider the the potential for the function f:

$$V(f,x) := \int_0^\infty \mathbb{E}^x[f(X(t)]\,\mathrm{d}t. \tag{1}$$

- The existence of $V(f,x)$ is a difficult question regarding the class of admissible f for each process $X(t)$.
- We propose an alternative and write equality (1) as

$$V(f,x) = \int_{\mathbb{R}^d} f(y)\,\mathcal{G}(x,\mathrm{d}y),$$

where $\mathcal{G}(x,\mathrm{d}y)$ is a measure on $\mathbb{R}^d$. This measure is the fundamental solution to the equation

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Markov Semigroups

- We can start with a Markov semigroup $T(t), t \geq 0$, that is, a family of linear operators in a Banach space

$$E = B(\mathbb{R}^d) \quad \text{or} \quad E = C_b(\mathbb{R}^d) \quad \text{or} \quad E = L^p(\mathbb{R}^d),\ p \geq 1, \ldots$$

depends on each particular case.
- This family of operators satisfy the following properties:

1. $T(t) \in \mathcal{L}(E), \quad t \geq 0,$
2. $T(0) = I_E$ (identity operator on E),
3. $\lim_{t \to 0^+} T(t)f = f, \quad f \in E,$
4. $T(t+s) = T(t)T(s),$
5. $\forall f \geq 0 \quad T(t)f \geq 0.$

The semigroup is conservative if

6. $T(t)1 = 1.$

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Markov Semigroups

- The semigroup $T(t)$, $t \geq 0$ is associated with a time homogeneous Markov process $\{X(t), t \geq 0 \mid \mathbb{P}^x, x \in \mathbb{R}^d\}$ if

$$(T(t)f)(x) = \mathbb{E}^x[f(X(t))] = \int_{\mathbb{R}^d} f(y) P_t(x, \mathrm{d}y), \quad f \in E,$$

  where $P_t(x, B)$ is the probability of the transition from the point $x \in \mathbb{R}^d$ to the Borel set $B \subset \mathbb{R}^d$ in the time $t > 0$.
- The transition probabilities may be constructed from the semigroup by choosing $f = \mathbb{1}_A$, $A \in \mathcal{B}(\mathbb{R}^d)$, that is,

$$P_t(x, A) = (T(t)\mathbb{1}_A)(x).$$

- Then we have

$$\mathcal{G}(x, \mathrm{d}y) := \int_0^\infty P_t(x, \mathrm{d}y)\, \mathrm{d}t, \qquad x \in \mathbb{R}^d.$$

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Green Measure for Markov Processes

Definition 1. The Green measure for a Markov process $X(t)$, $t \geq 0$ with transition probability $P_t(x,B)$ is defined by

$$\mathcal{G}(x,B) := \int_0^\infty P_t(x,B)\,\mathrm{d}t, \quad B \in \mathcal{B}_b(\mathbb{R}^d),$$

or

$$\int_{\mathbb{R}^d} f(y)\mathcal{G}(x,\mathrm{d}y) = \int_0^\infty f(y)P_t(x,\mathrm{d}y)\,\mathrm{d}t, \quad f \in C_0(\mathbb{R}^d)$$

whenever these integrals exist.

Green Measure for Markov Processes

- From the relation between semigroup and generator we have

$$\mathbb{E}^x\left[\int_0^\infty f(X(t))\,\mathrm{d}t\right] = \int_{\mathbb{R}^d} f(y)\mathcal{G}(x,\mathrm{d}y) = -(L^{-1}f)(x) = \int_0^\infty (T(t)f)(x)\,\mathrm{d}t \tag{2}$$

for every $f \in C_0(\mathbb{R}^d)$.

- The Green measure is the fundamental solution corresponding to the generator operator L.

$$\mathcal{G}(x,\mathrm{d}y) = \mathcal{G}(x,y)\,\mathrm{d}y,$$

where $\mathcal{G}(x,\cdot) \in D'(\mathbb{R}^d)$ is a positive generalized function for all $x \in \mathbb{R}^d$.

Jump Kernel

Let $a : \mathbb{R}^d \to \mathbb{R}$ be a fixed kernel such that:

- Symmetric, $a(-x) = a(x)$, for every $x \in \mathbb{R}^d$.
- Positive and bounded, $a \geq 0$, $a \in C_b(\mathbb{R}^d)$.
- Integrable: $\displaystyle\int_{\mathbb{R}^d} a(y)\,\mathrm{d}y = 1$.
- The Fourier transform $\hat{a} \in L^1(\mathbb{R}^d)$ and has finite second moment:

$$\int_{\mathbb{R}^d} |x|^2 a(x)\,\mathrm{d}x < \infty.$$

Jump Generators

Consider the generator L defined on thr Banach space E by

$$(Lf)(x) := \int_{\mathbb{R}^d} a(x-y)[f(y)-f(x)]\,\mathrm{d}y = (a*f)(x) - f(x), \quad x \in \mathbb{R}^d.$$

- In particular, $L^* = L$ in $L^2(\mathbb{R}^d)$ and L is a bdd linear operator in all $L^p(\mathbb{R}^d)$, $p \geq 1$.
- We call this operator the jump generator with jump kernel a.
- The corresponding Markov process is of a pure jump type and is known in stochastic as compound Poisson process (Skorohod [1991]).
- In terms of the Fourier image L is the multiplication operator by

$$\hat{L}(k) = \hat{a}(k) - 1 \qquad \text{(symbol of } L\text{)}.$$

- Several analytic properties of the jump generator L were studied recently, see for example (Grigor'yan et al. [2018], Kondratiev et al. [2018, 2017]).

Resolvent Kernel

- For any $\lambda \in (0,\infty)$, let $\mathcal{G}_\lambda(x,y)$, $x,y \in \mathbb{R}^d$ be the resolvent kernel of $R_\lambda(L) := (\lambda - L)^{-1}$.
    - This kernel $\mathcal{G}_\lambda(x,y)$ admits the representation:

$$\mathcal{G}_\lambda(x,y) = \frac{1}{1+\lambda}\big(\delta(x-y) + G_\lambda(x-y)\big), \quad \lambda \in (0,\infty),$$

with

$$G_\lambda(x) = \sum_{k=1}^{\infty} \frac{a_k(x)}{(1+\lambda)^k}, \tag{3}$$

$$a_k(x) = a^{*k}(x) \qquad (k\text{-times convolution of } a).$$

- The resolvent kernel $\mathcal{G}_\lambda(x,y)$ has a singular part, $\delta(x-y)$ and a regular part $G_\lambda(x-y)$.
- The Green function, as a generalized function, has the form

$$\mathcal{G}_0(x) = \delta(x) + G_0(x).$$

Main Result

Theorem 2. Under the above assumptions the Fourier representation for $G_0(x)$ is given by

$$G_0(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(k,x)} \frac{\hat{a}(k)}{1-\hat{a}(k)} \, \mathrm{d}k.$$

For $d \geq 3$ this integral exists for all $x \in \mathbb{R}^d$.

Proof. The existence of the integral follows from the integrable singularity of $(1-\hat{a}(k))^{-1}$ at $k=0$ as a consequence of the assumptions on $a(x)$. □

Particular Models: Gauss Kernels

- Assume that the jump kernels $a(x)$ has the following form:

$$a(x) = C \exp\left(-\frac{b|x|^2}{2}\right), \quad C, b > 0. \tag{4}$$

Proposition 3. If the jump kernel $a(x)$ be given by (4) and $d \geq 3$, then holds

$$G_0(x) \leq C_1 \exp\left(-\frac{b|x|^2}{4}\right).$$

Proof. By a direct calculation we find

$$a_k(x) = \frac{C}{k^{d/2}} \exp\left(-\frac{b|x|^2}{2k}\right).$$

Substituting $a_k(x)$ in $G_0(x)$ and estiamting the result follows. □

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Proposition 4. If the jump kernel $a(x)$ satisfies (5) and $d \geq 3$, then there exist $A, B > 0$ such that the bound for $G_0(x)$ holds

$$G_0(x) \leq A\exp(-B|x|).$$

Proof. It was shown in Kondratiev et al. [2018] that

$$a_n(x) \leq Cn^{-d/2}\exp(-c\min(|x|, |x|^2/n)).$$

Hence, the following bound for $a_n(x)$ holds

$$a_n(x) \leq Cn^{-d/2}\big(\exp(-c|x|) + \exp(-c|x|^2/n)\big).$$

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Brownian Motion

- Let $B(t)$, $t \geq 0$ be the Brownian motion in $\mathbb{R}^d$ whose generator is the Laplace operator Δ considered in a proper Banach space E.
- We are interested in studying the expectation of the random variable

$$Y(f) = \int_0^\infty f(B(t))\,\mathrm{d}t$$

for certain class of functions $f : \mathbb{R}^d \to \mathbb{R}$.

- Define the following class of functions

$$CL(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{R} \mid f \text{ is continuous, bdd and } f \in L_1(\mathbb{R}^d)\}.$$

It is a Banach space with the norm $\|f\|_{CL} := \|f\|_\infty + \|f\|_1$.

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Brownian Motion

Proposition 5. Let $d \geq 3$ be given. The Green measure of Brownian motion is

$$\mathcal{G}(x, \mathrm{d}y) = G_0(x-y)\,\mathrm{d}y = \frac{C(d)}{|x-y|^{d-2}}\,\mathrm{d}y.$$

Proof. We have

$$\mathbb{E}^x[Y(f)] = -\Delta^{-1} f(x) = \int_{\mathbb{R}^d} C(d) \frac{f(y)}{|x-y|^{d-2}}\,\mathrm{d}y.$$

Then

$$\left|\int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2}}\,\mathrm{d}y\right| \leq \left|\int_{|x-y|\leq 1} \frac{f(y)}{|x-y|^{d-2}}\,\mathrm{d}y\right| + \left|\int_{|x-y|>1} \frac{f(y)}{|x-y|^{d-2}}\mathrm{d}y\right|$$
$$\leq C_1 \|f\|_\infty + C_2 \|f\|_1$$
$$\leq C \|f\|_{CL}.$$

□

Markov Processes in Random Time

- Let $X = \{X(t),\ t \geq 0\}$ be a Markov process in $\mathbb{R}^d$ s.t. $X(0) = x \in \mathbb{R}^d$ a.s.
- Define the function $u(t,x)$ by (for suitable $f : \mathbb{R}^d \to \mathbb{R}$.)

$$u(t,x) := \mathbb{E}[f(X(t))],\ t > 0,\ x \in \mathbb{R}^d$$

This is the solution of the Kolmogorov equation

$$\frac{\partial}{\partial t} u(t,x) = Lu(t,x), \quad u(0,x) = f(x), \tag{6}$$

where L is the generator of the process $X(t)$.

Markov Processes in Random Time

- Let $Y = \{Y(t),\ t \geq 0\}$ be the random time change in X

$$Y(t) := X(E(t)), \quad t \geq 0,$$

where $E(t)$, $t \geq 0$ is the inverse of a subordinator $S \perp\!\!\!\perp X$.

- Let us define a similar function for $Y(t)$:

$$v(t,x) = \mathbb{E}[f(Y(t))].$$

Then $v(t,x)$ satisfies the following fractional evolution equation:

$$D_t^{(k)} v(t,x) = Lv(t,x). \tag{7}$$

The function $k \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is given in terms of the characteristics of S and $D_t^{(k)}$ is a differential-convolution operator defined by

$$(\mathbb{D}_t^{(k)} u)(t) := \frac{d}{dt}\int_0^t k(t-\tau)u(\tau)\,\mathrm{d}\tau - k(t)u(0),\ t > 0. \tag{8}$$

Subordination Formula

- It follows from the subordination formula (subordination principle):

$$v(t,x) = \int_0^\infty u(\tau,x) G_t(\tau)\, \mathrm{d}\tau, \tag{9}$$

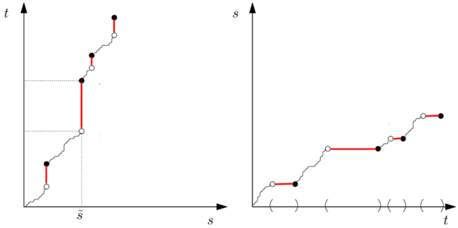
where $G_t(\tau)$ is the density of the inverse subordinator $E(t)$.

- If μ_t^x and ν_t^x denote the marginal distrib. of $X(t)$ and $Y(t)$, resp., then the subordination relations implies

$$\nu_t^x = \int_0^\infty \mu_\tau^x G_t(\tau)\, \mathrm{d}\tau. \tag{10}$$

Trapping Effect

- For every jump of the subordinator S there is a corresponding flat period of its inverse E.



- These flat periods represent trapping events in which the test particle gets immobilized in a trap.
- Trapping slows down the overall dynamics of the initial MP X.
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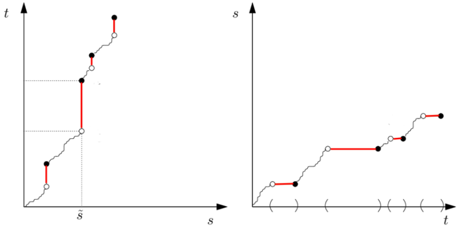

- These flat periods represent trapping events in which the test particle gets immobilized in a trap.
- Trapping slows down the overall dynamics of the initial MP X.
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Assumptions

- Let $S = \{S(t),\ t \geq 0\}$ be a subordinator without drift starting at zero with Laplace transform
$$\mathbb{E}(e^{-\lambda S(t)}) = e^{-t\Phi(\lambda)}, \quad \lambda \geq 0$$
and
$$\Phi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda\tau})\, \mathrm{d}\sigma(\tau), \qquad \sigma((0,\infty)) = \infty.$$
- Define the kernel k as follows $k(t) := \sigma\big((t,\infty)\big)$, $t > 0$ and $\mathcal{K}(\lambda) := (\mathcal{L}k)(\lambda)$.

(H) The Lévy measure σ has a completely monotone density $\rho(t)$ w.r.t. the Lebesgue measure (i.e., $(-1)^n \rho^{(n)}(t) \geq 0$ for all $t > 0$, $n = 0, 1, 2, \ldots$) and the functions $\mathcal{K}$, Φ satisfy
$$\mathcal{K}(\lambda) \to \infty,\ \text{as } \lambda \to 0; \quad \mathcal{K}(\lambda) \to 0,\ \text{as } \lambda \to \infty; \tag{11}$$
$$\Phi(\lambda) \to 0,\ \text{as } \lambda \to 0; \quad \Phi(\lambda) \to \infty,\ \text{as } \lambda \to \infty. \tag{12}$$

Green Measure does not exists!

- The Green measure for the time change process $Y(t)$ is defined by

$$\mathcal{G}(x, \mathrm{d}y) := \int_0^\infty v_t^x(\mathrm{d}y)\,\mathrm{d}t.$$

Lemma 6. Under the assumptions formulated for any dimension d the Green measure for $Y(t)$ does not exists.

Proof. Using the subordination formula (10) we obtain

$$\int_0^\infty \nu_t^x\,\mathrm{d}t = \int_0^\infty \int_0^\infty \mu_\tau^x G_t(\tau)\,\mathrm{d}\tau\,\mathrm{d}t.$$

But we know that for each τ we have (Kochubei et al. [2020]),

$$\int_0^\infty G_t(\tau)\,\mathrm{d}t = \mathcal{K}(0) = +\infty.$$

□

Renormalized Green Measures

- As the Green measure $\mathcal{G}(x, \mathrm{d}y)$ does not exists for a general subordinated process Y, we have to consider instead a renormalized Green measure

$$\mathcal{G}_r(x, \mathrm{d}y) := \lim_{T\to\infty} \frac{1}{N(T)} \int_0^T \nu_t^x(\mathrm{d}y)\,\mathrm{d}t.$$

Theorem 7. Assume that the Markov process $X(t)$ in $\mathbb{R}^d$, $d \geq 3$ has a Green measure $\mathcal{G}(x, \mathrm{d}y)$ and define

$$N(T) := \int_0^T k(s)\,\mathrm{d}s, \quad T \geq 0. \tag{13}$$

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Proof. The idea of the proof is based on the subordination formula and the result from Kochubei et al. [2020]

$$\lim_{t\to\infty}\left(\int_0^t G_s(\tau)\,ds\right)\left(\int_0^t k(s)\,ds\right)^{-1}=1. \qquad \square$$

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Thank You