Geometric Transformations and Wallpaper Groups

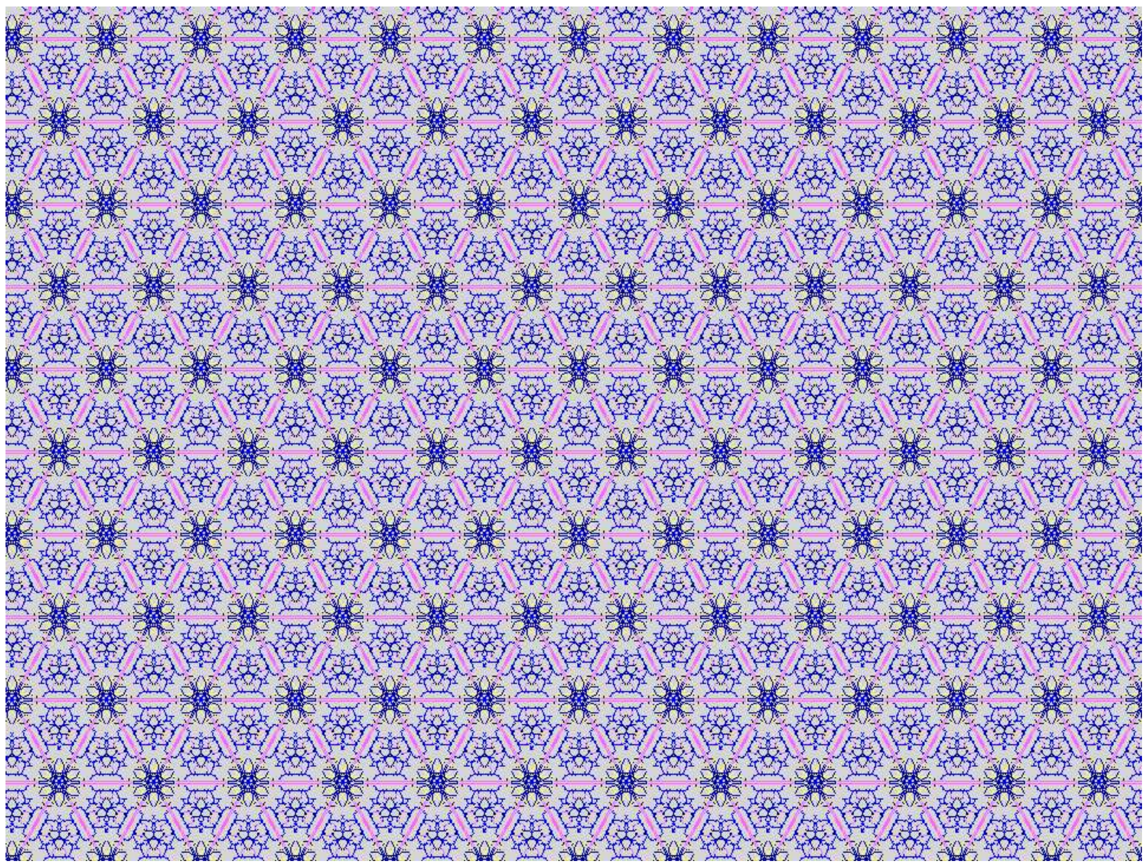
Lance Drager

Texas Tech University

Wallpaper Groups

Introduction to Wallpaper Groups

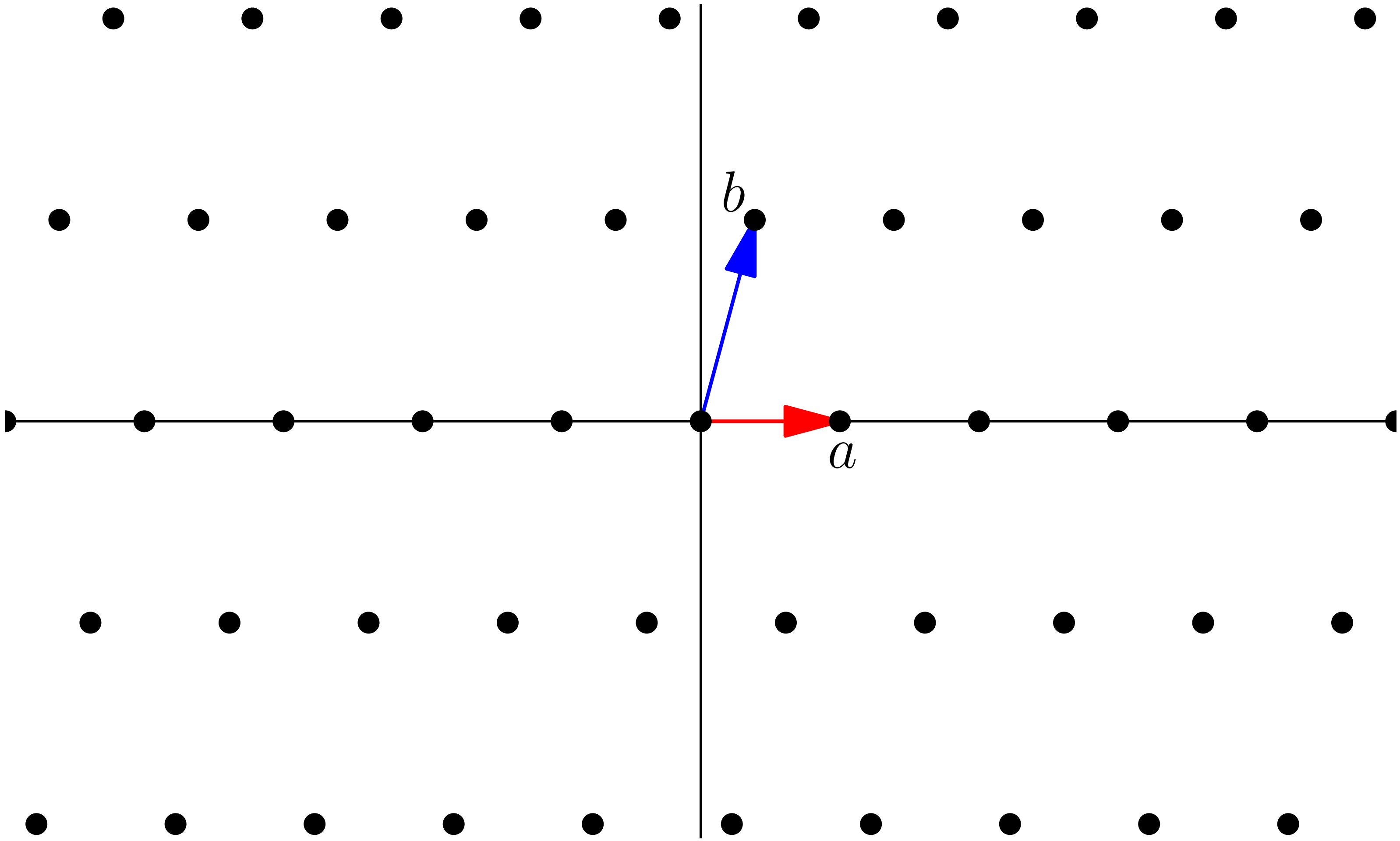
- A **Wallpaper Group** is a discrete group of isometries of the plane that contains noncollinear translations. These are the symmetries of wallpaper patterns.



Structure of Wallpaper Groups

- Let G be a wallpaper group. Choose an element $a \neq 0$ of the lattice L so that $\|a\|$ is a small as possible. We can do this because that group is discrete. By assumption, there are elements of the lattice that are skew to a. From all of these, choose b so that $\|b\|$ is as small as possible. From these choices, we have $\|a\| \leq \|b\|$.
- **Lemma**. The lattice is $L = \{ma + nb \mid m, n \in \mathbb{Z}\}$.

The Structure of Wallpaper Groups



A Lattice

The Structure of Wallpaper Groups

- Here's an **exercise** on a standard bit of linear algebra. If A is a matrix, the **trace of** A, $\mathrm{tr}(A)$, is defined as

$$\mathrm{tr}(A) = \mathrm{tr}\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = a + d.$$

  - Show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for 2×2 matrices. Use brute force.
  - Show that if P is an invertible matrix $\mathrm{tr}(PAP^{-1}) = \mathrm{tr}(A)$.

The Structure of Wallpaper groups

- **The Crystallographic Restriction.** The possible orders for a rotation in a wallpaper group are 2, 3, 4 and 6.
- Suppose that $(R \,|\, a)$ is a rotation in our wallpaper group G. We know that $RL \subseteq L$. This means that Ra must be a lattice point, so $Ra = ma + nb$ for some integers m and n. Similarly, $Rb \in L$, so $Rb = pa + qb$ for some integers p and q. The equations

$$Ra = ma + nb, \qquad Rb = pa + qb$$

can be written in matrix form as

$$R[a \mid b] = [Ra \mid Rb] = [a \mid b] \begin{bmatrix} m & p \\ n & q \end{bmatrix}.$$

The Structure of Wallpaper Groups

- To continue with the crystallographic restriction, let $P = [a \mid b]$. Then our matrix equation can be written as $RP = PM$ where $M = \begin{bmatrix} m & p \\ n & q \end{bmatrix}$. The matrix P must be invertible, because the vectors a and b are not collinear. Thus, we have $R = PMP^{-1}$. From the previous exercise, we have $\mathrm{tr}(R) = \mathrm{tr}(PMP^{-1}) = \mathrm{tr}(M) = m + q$. Since m and q are integers, we conclude that $\mathrm{tr}(R)$ is an integer. Let θ be the angle $0 < \theta < 360°$ so that $R = R(\theta)$. Then

$$R = R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \implies \mathrm{tr}(R) = 2\cos(\theta).$$

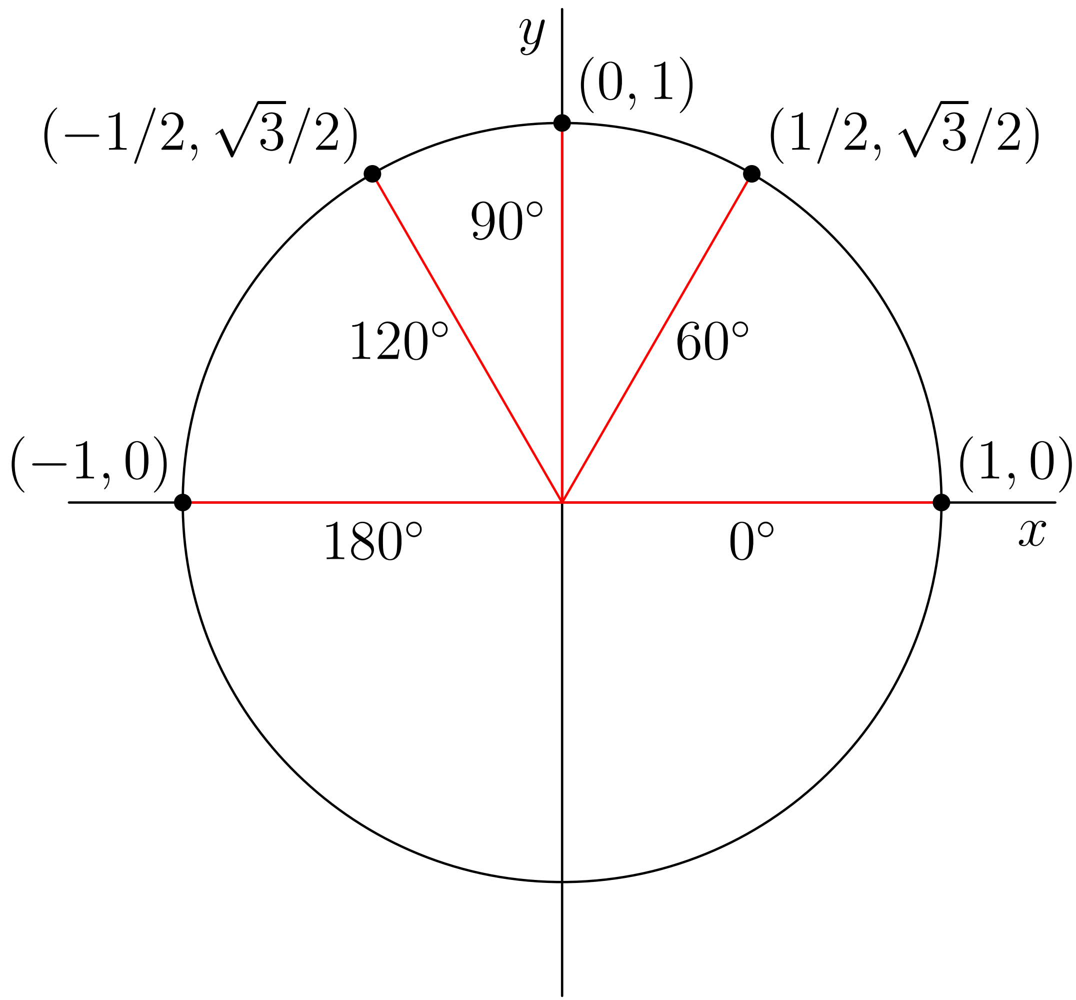
The Structure of Wallpaper Groups

- We conclude that $2\cos(\theta)$ is integer, so $\cos(\theta)$ is a half integer. Since $-1 \leq \cos(\theta) \leq 1$ the possibilities are $\cos(\theta) = -1, -1/2, 0, 1/2, 1$.

$$
\begin{array}{lclcl}
\cos(\theta) = -1 & \implies & \theta = 180^\circ & \implies & o(R) = 2, \\
\cos(\theta) = -1/2 & \implies & \theta = 120^\circ & \implies & o(R) = 3, \\
\cos(\theta) = 0 & \implies & \theta = 90^\circ & \implies & o(R) = 4, \\
\cos(\theta) = 1/2 & \implies & \theta = 60^\circ & \implies & o(R) = 6, \\
\cos(\theta) = 1 & \implies & \theta = 0 & \implies & R = I.
\end{array}
$$

The Structure of Wallpaper Groups

- Here's the picture of the angles.



Classification of Lattices

- The lattices of wallpaper groups can be divided into 5 classes. We have chosen a and b above. Since $a+b$ and $a-b$ are skew to a, we must have $\|b\| \leq \|a-b\|$ and $\|b\| \leq \|a+b\|$. We can arrange that $\|a-b\| \leq \|a+b\|$ by replacing b by $-b$ if necessary. We then have

$$\|a\| \leq \|b\| \leq \|a-b\| \leq \|a+b\|.$$

  We can then investigate when we have $=$ or $<$ for each of the $\leq$'s above. This gives 8 cases. We can then investigate what the lattice looks like in each case.

Classification of Lattices

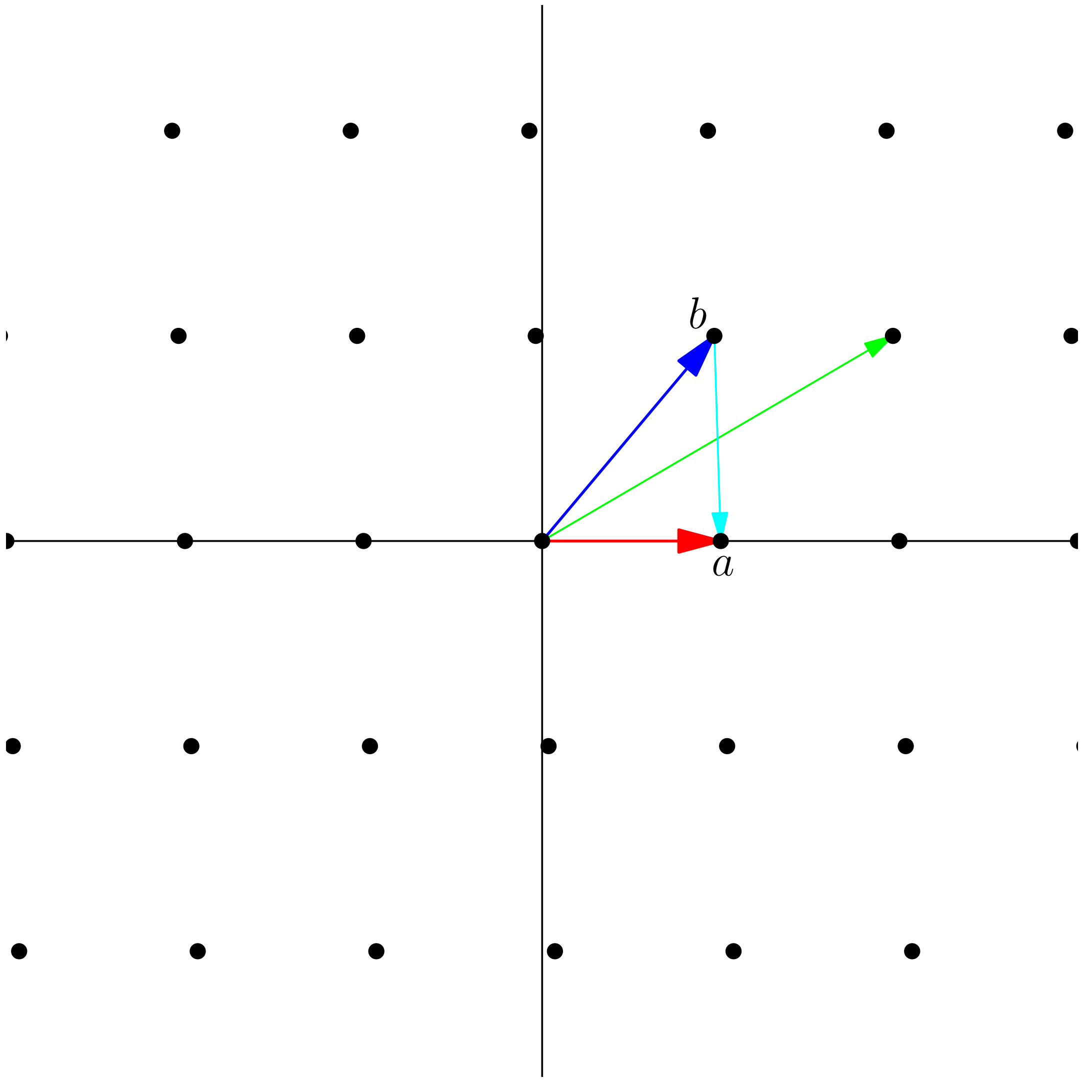
- Here are the cases

Case	Inequality	Lattice
1	$\|a\| = \|b\| = \|a-b\| = \|a+b\|$	Impossible
2	$\|a\| = \|b\| = \|a-b\| < \|a+b\|$	Hexagonal
3	$\|a\| = \|b\| < \|a-b\| = \|a+b\|$	Square
4	$\|a\| = \|b\| < \|a-b\| < \|a+b\|$	Centered Rect.
5	$\|a\| < \|b\| = \|a-b\| = \|a+b\|$	Impossible
6	$\|a\| < \|b\| = \|a-b\| < \|a+b\|$	Centered Rect.
7	$\|a\| < \|b\| < \|a-b\| = \|a+b\|$	Rectangular
8	$\|a\| < \|b\| < \|a-b\| < \|a+b\|$	Oblique

- Let's look at the Lattices.

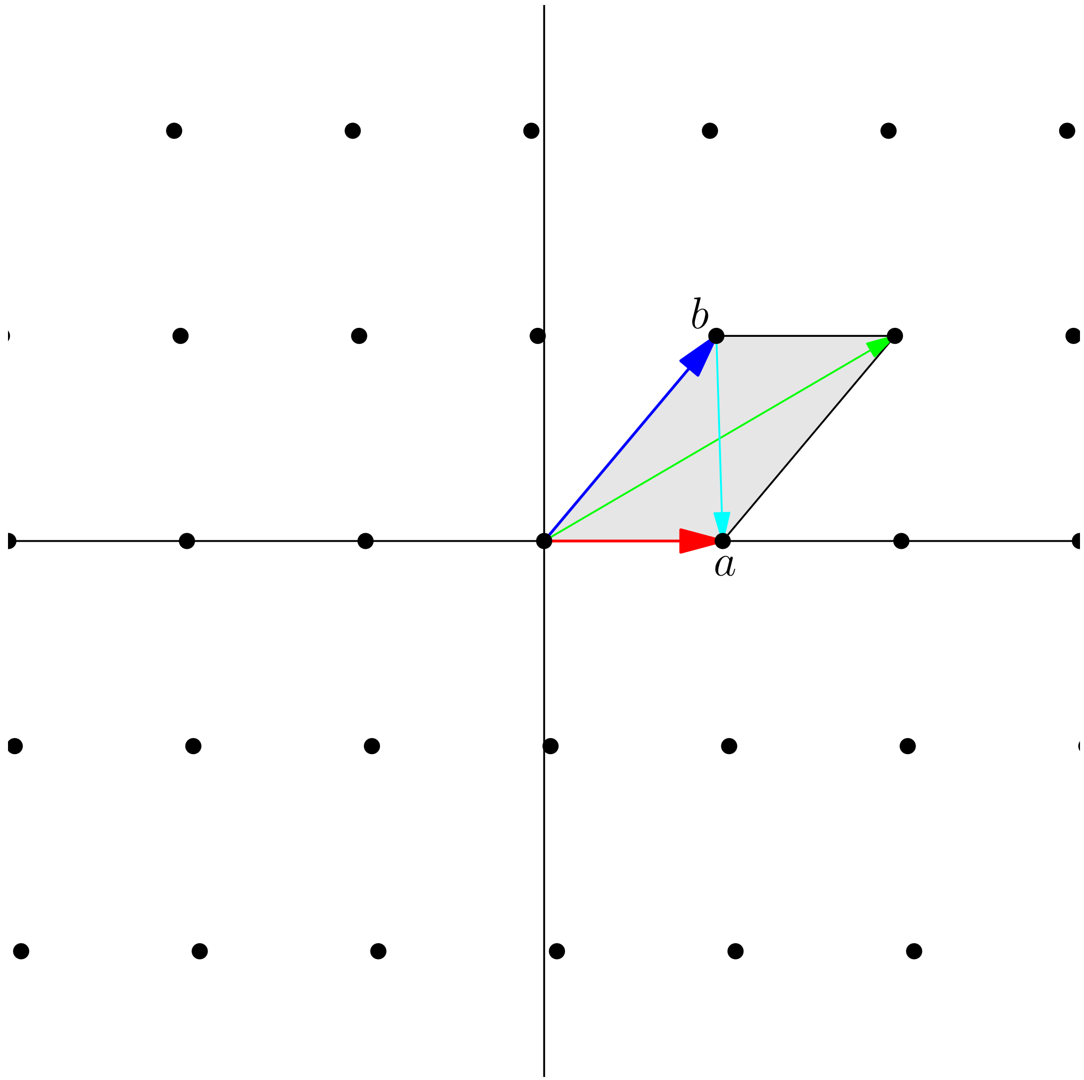
Classification of Lattices

Case 8, Oblique Lattice



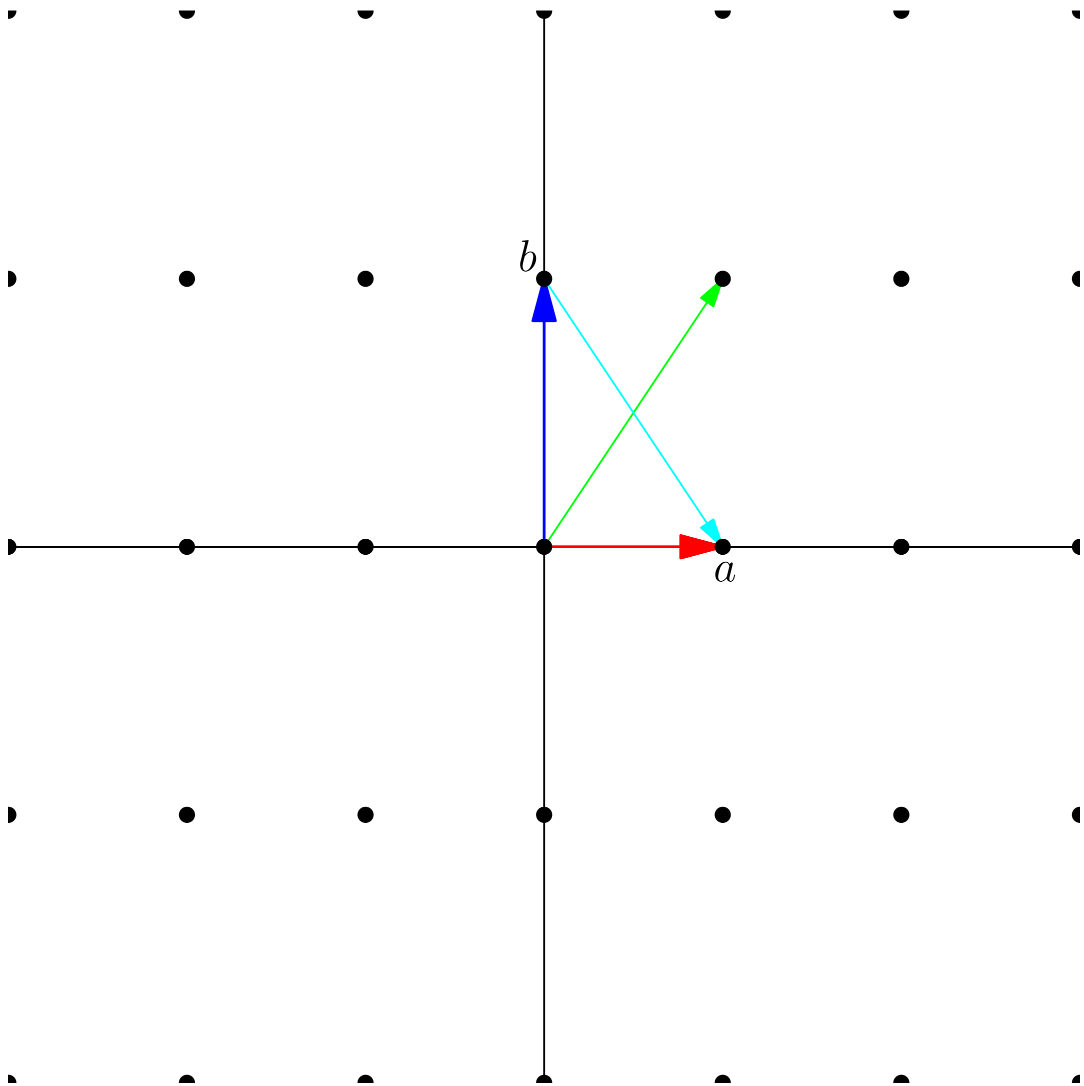
Classification of Lattices

Case 8, Oblique Lattice



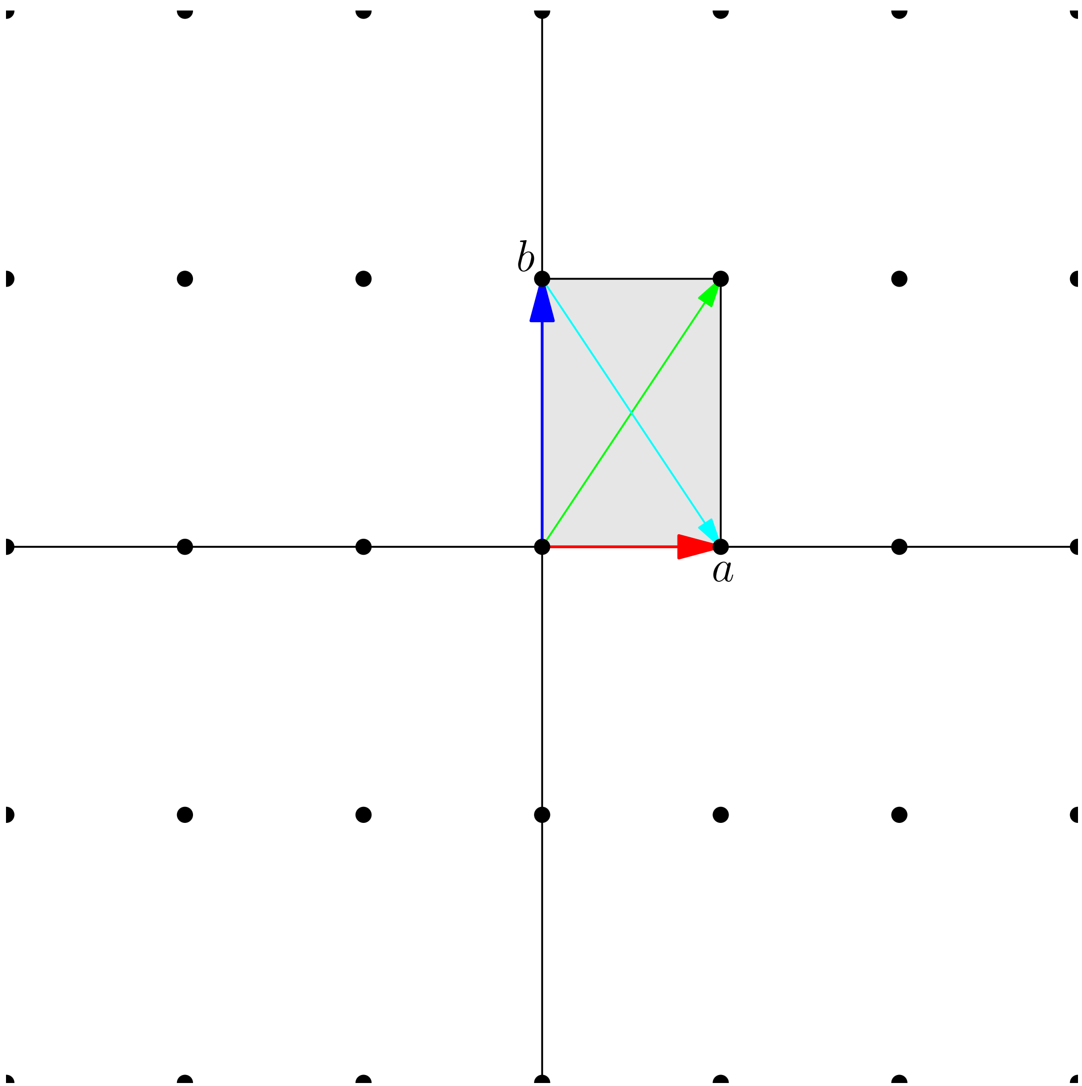
Classification of Lattices

Case 7, Rectangular Lattice



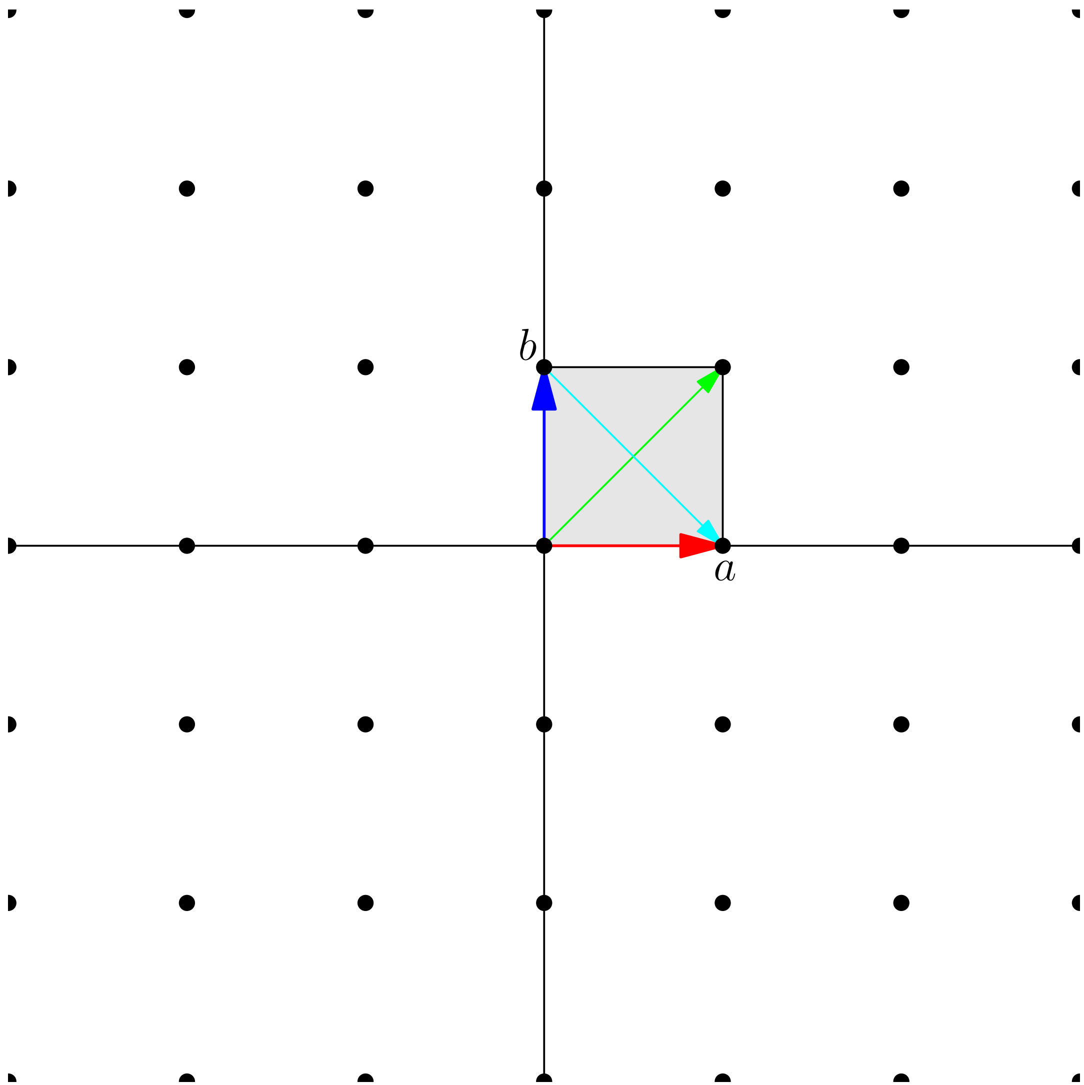
Classification of Lattices

Case 7, Rectangular Lattice



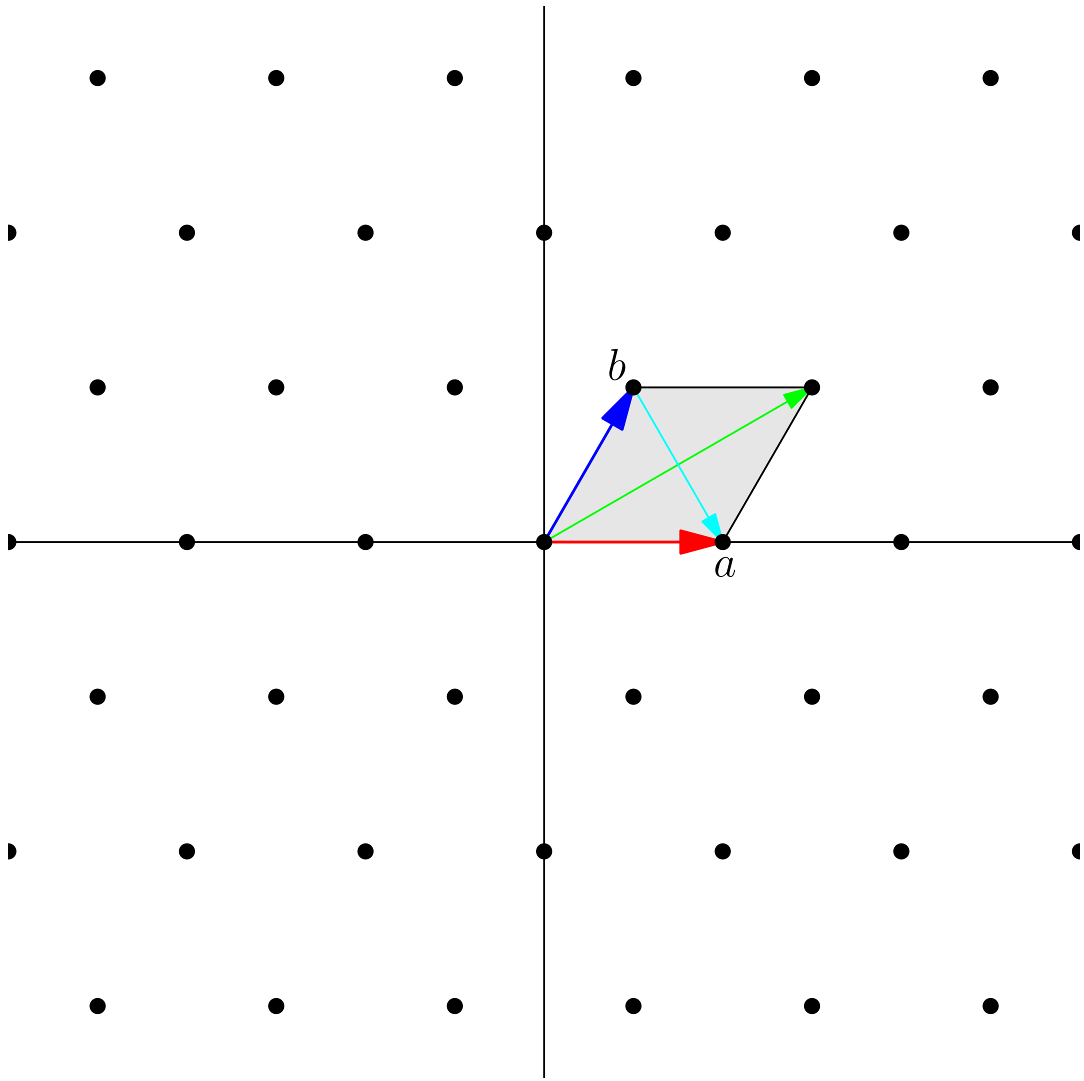
Classification of Lattices

Case 3, Square Lattice



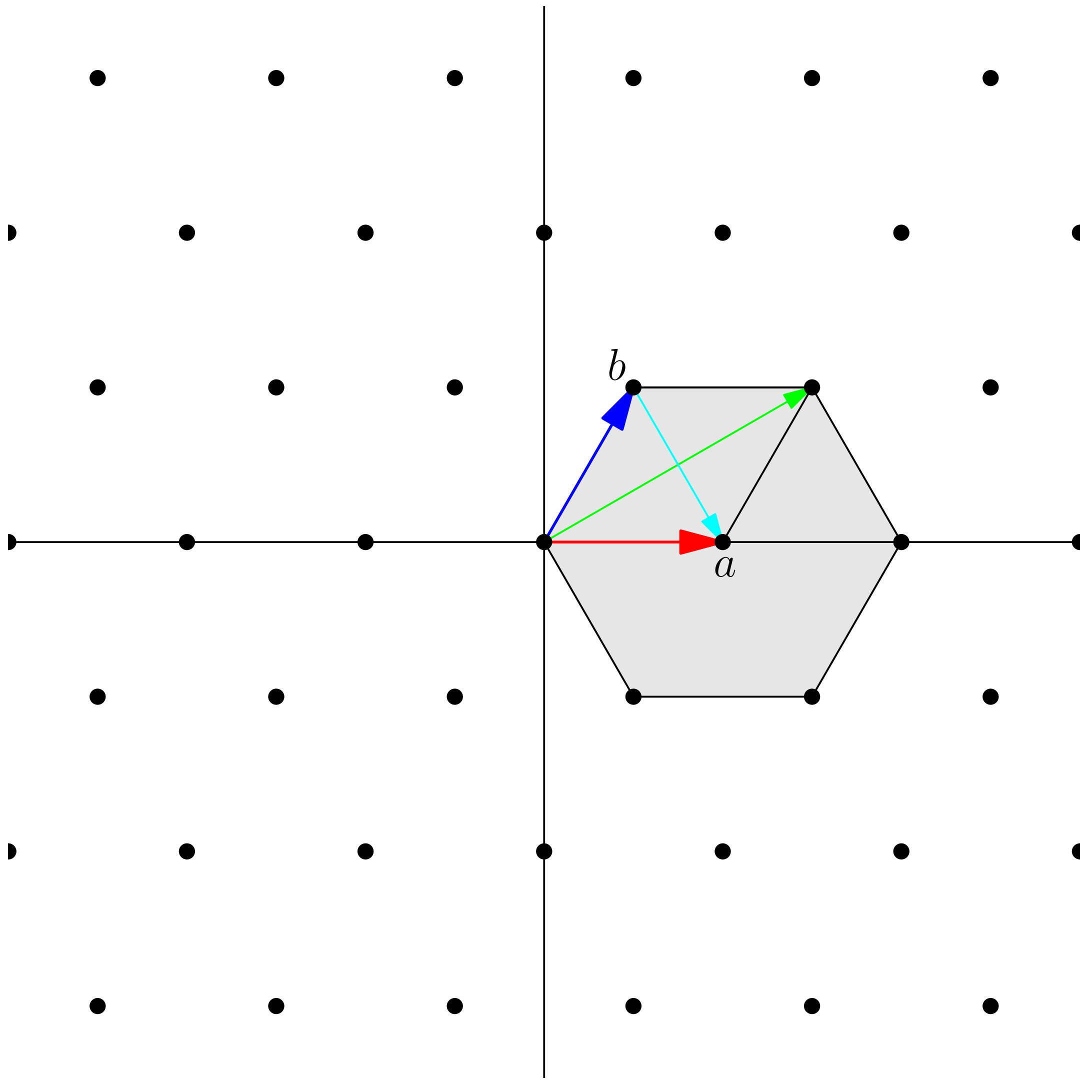
Classification of Lattices

Case 2, Hexagonal Lattice



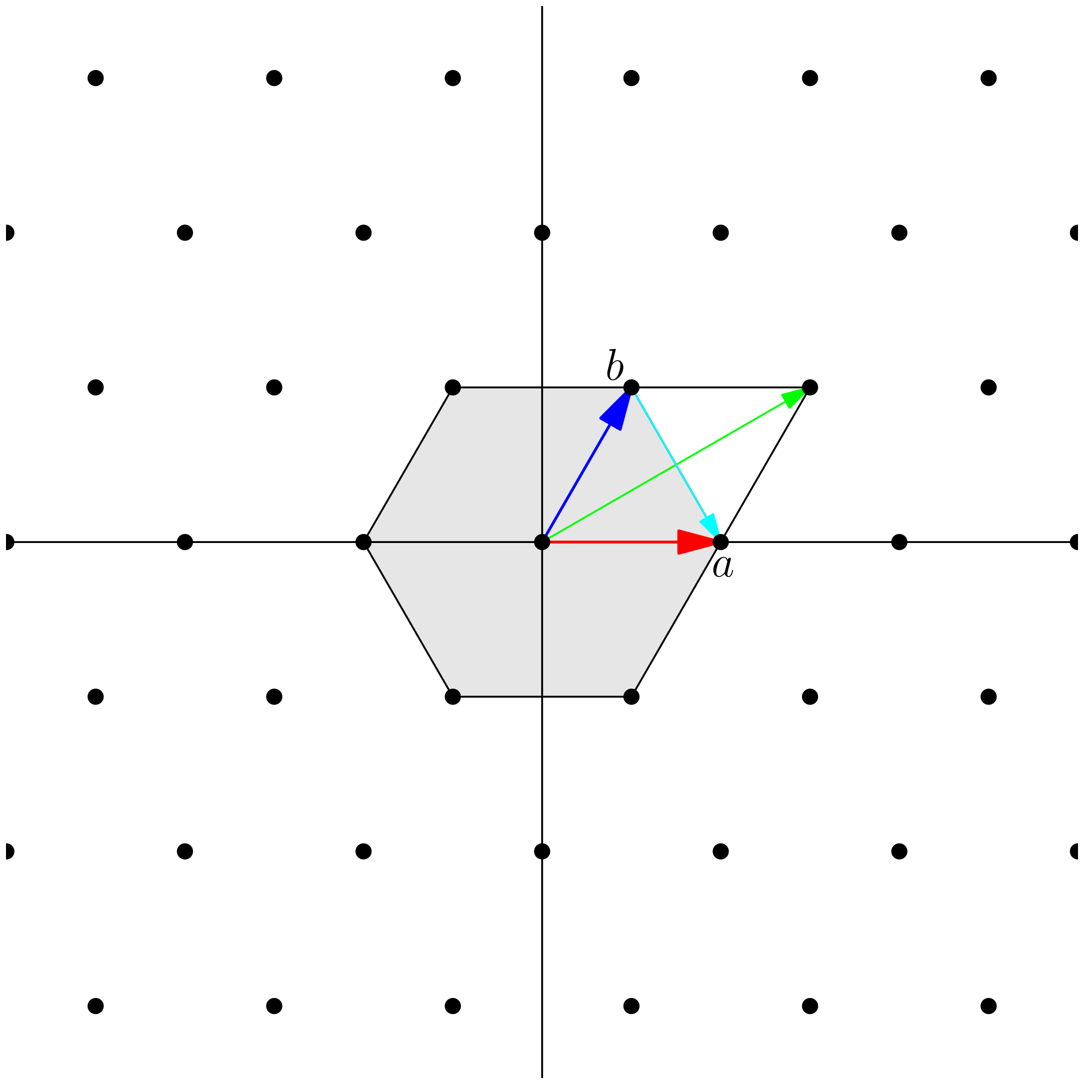
Classification of Lattices

Case 2, Hexagonal Lattice



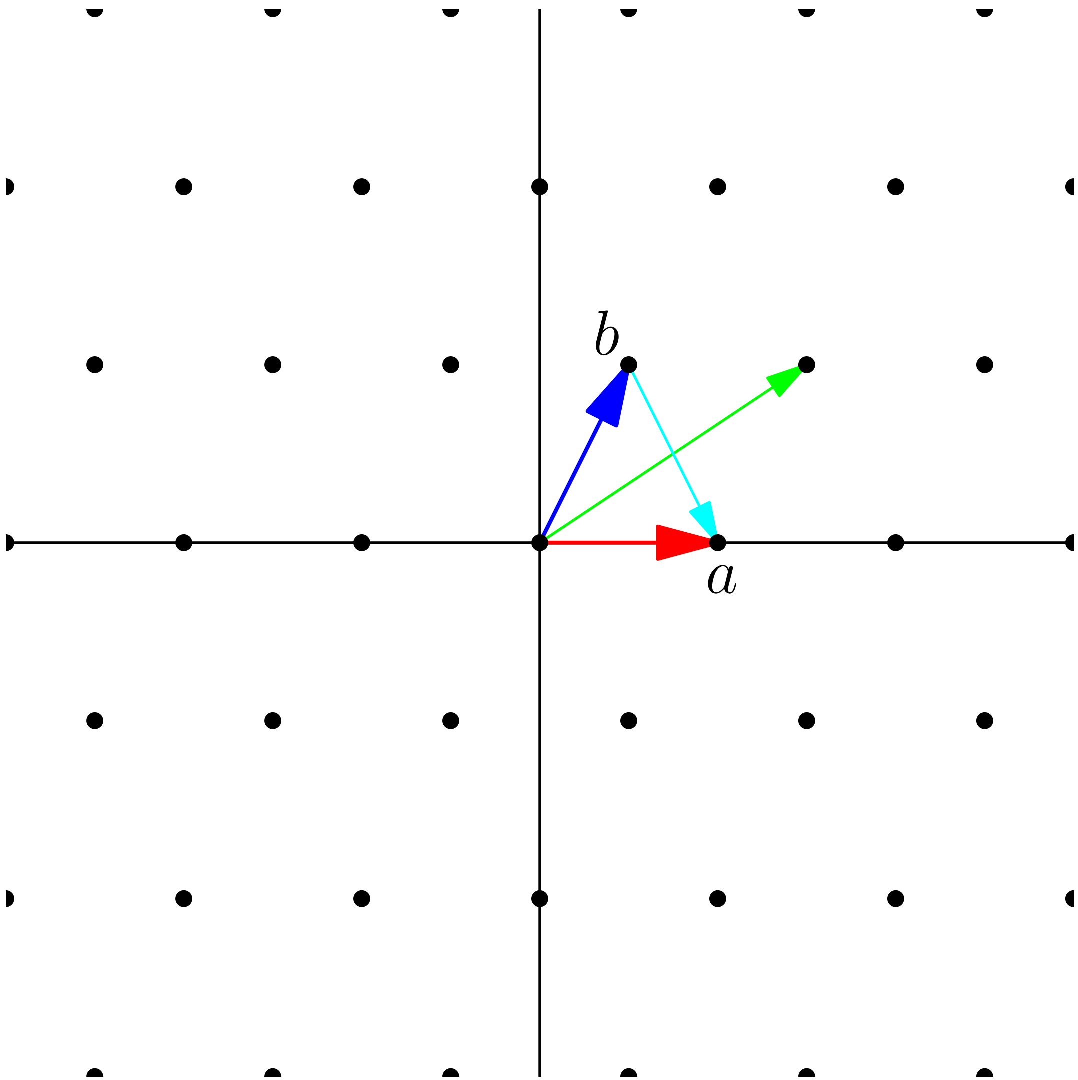
Classification of Lattices

Case 2, Hexagonal Lattice



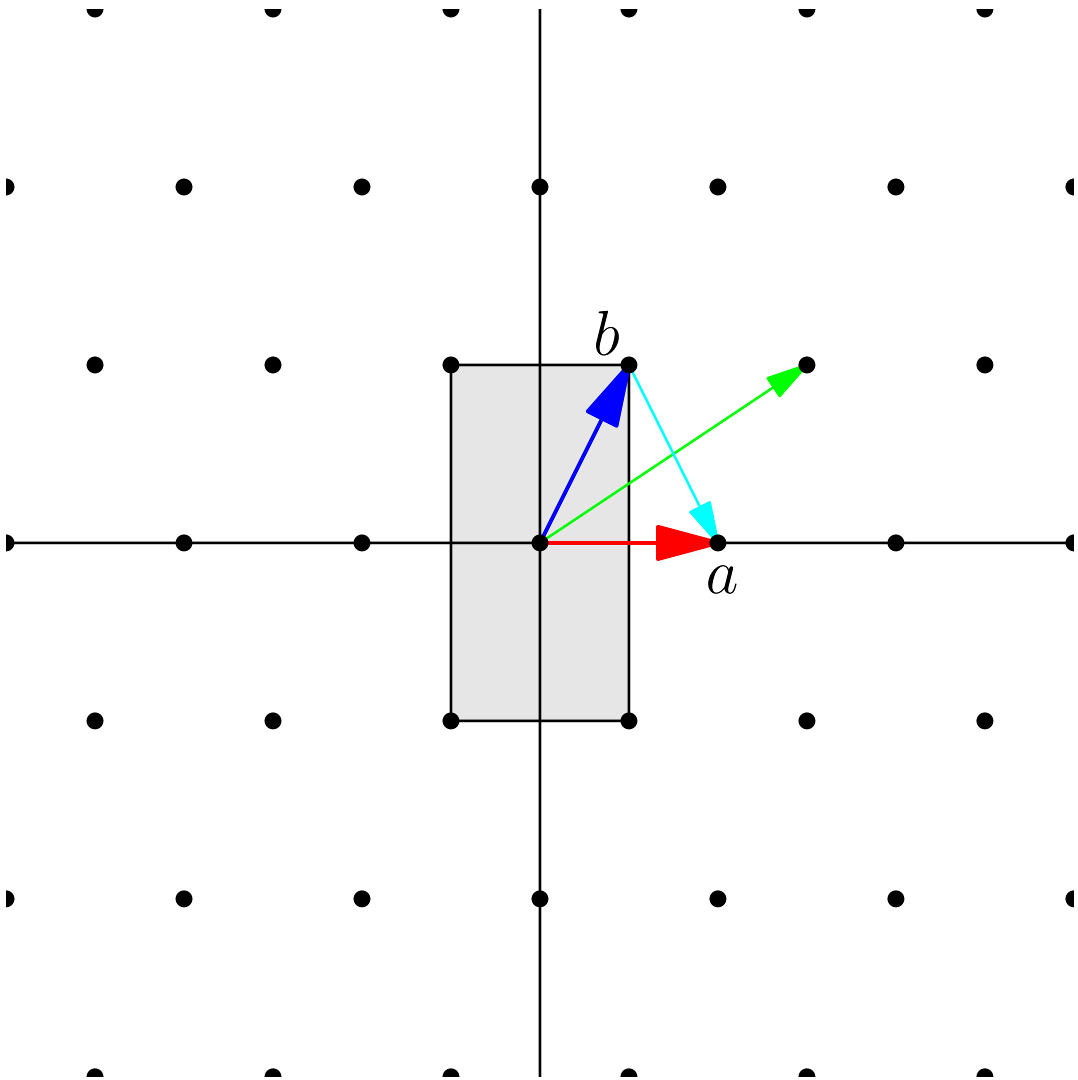
Classification of Lattices

Case 6, Centered Rectangular



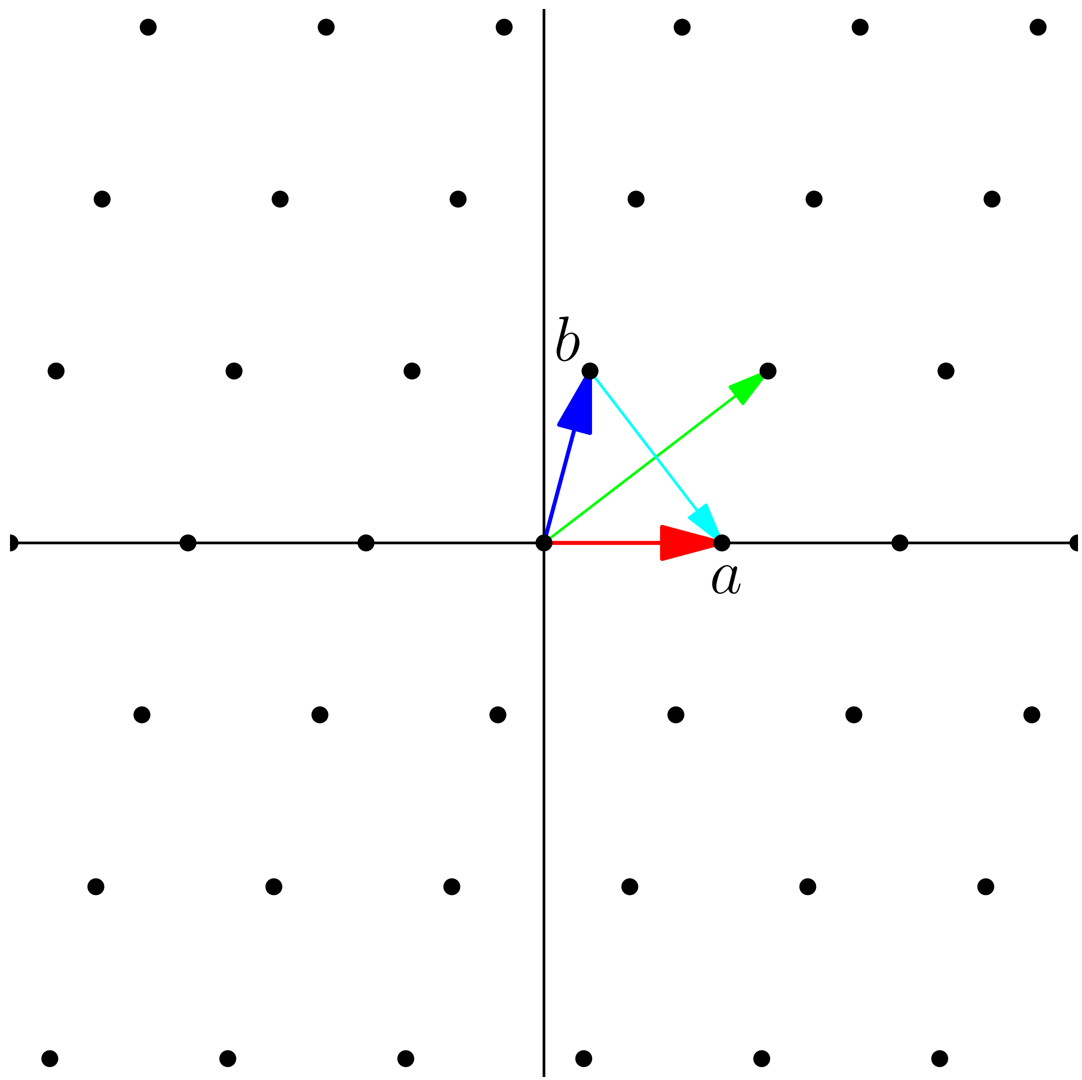
Classification of Lattices

Case 6, Centered Rectangular



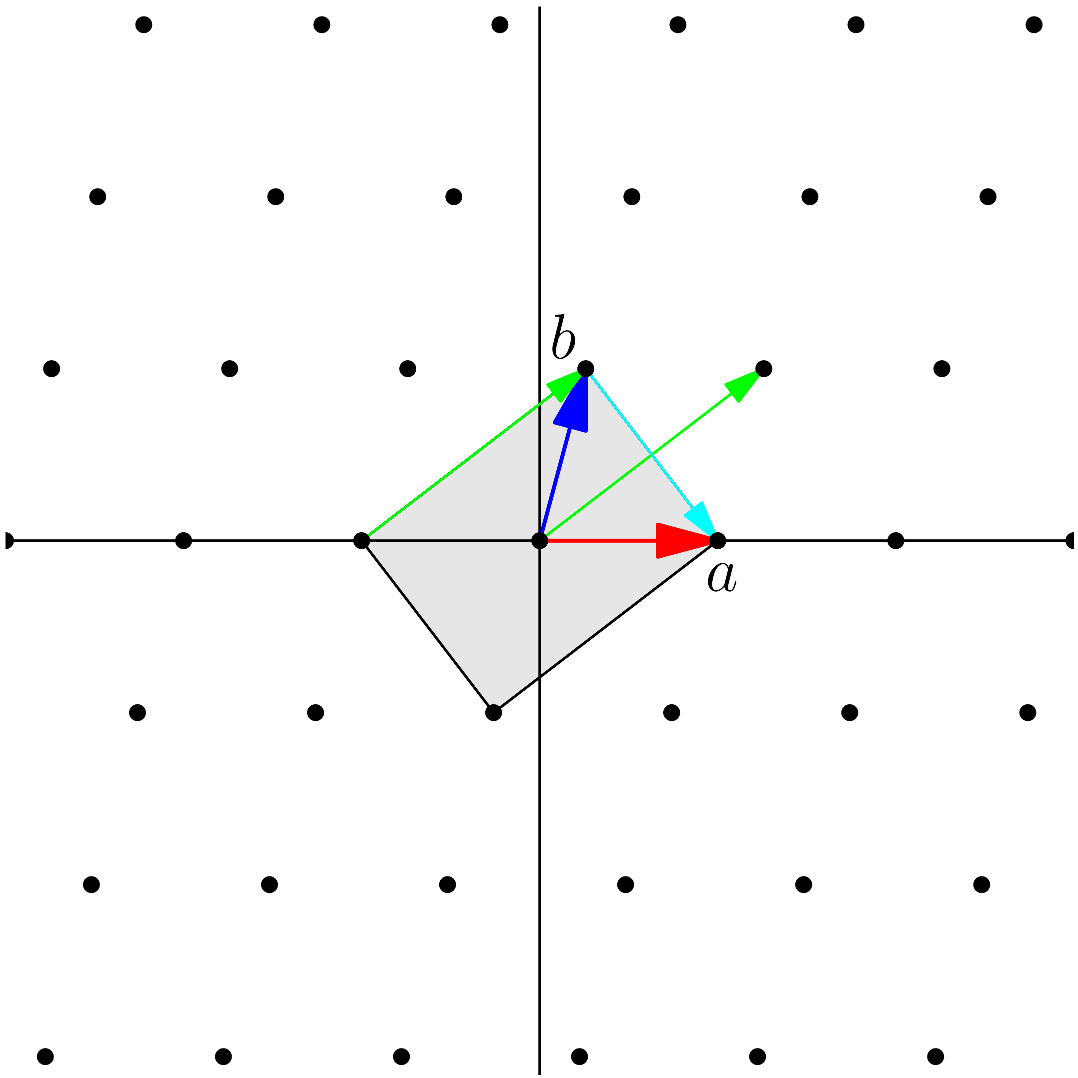
Classification of Lattices

Case 4, Centered Rectangular



Classification of Lattices

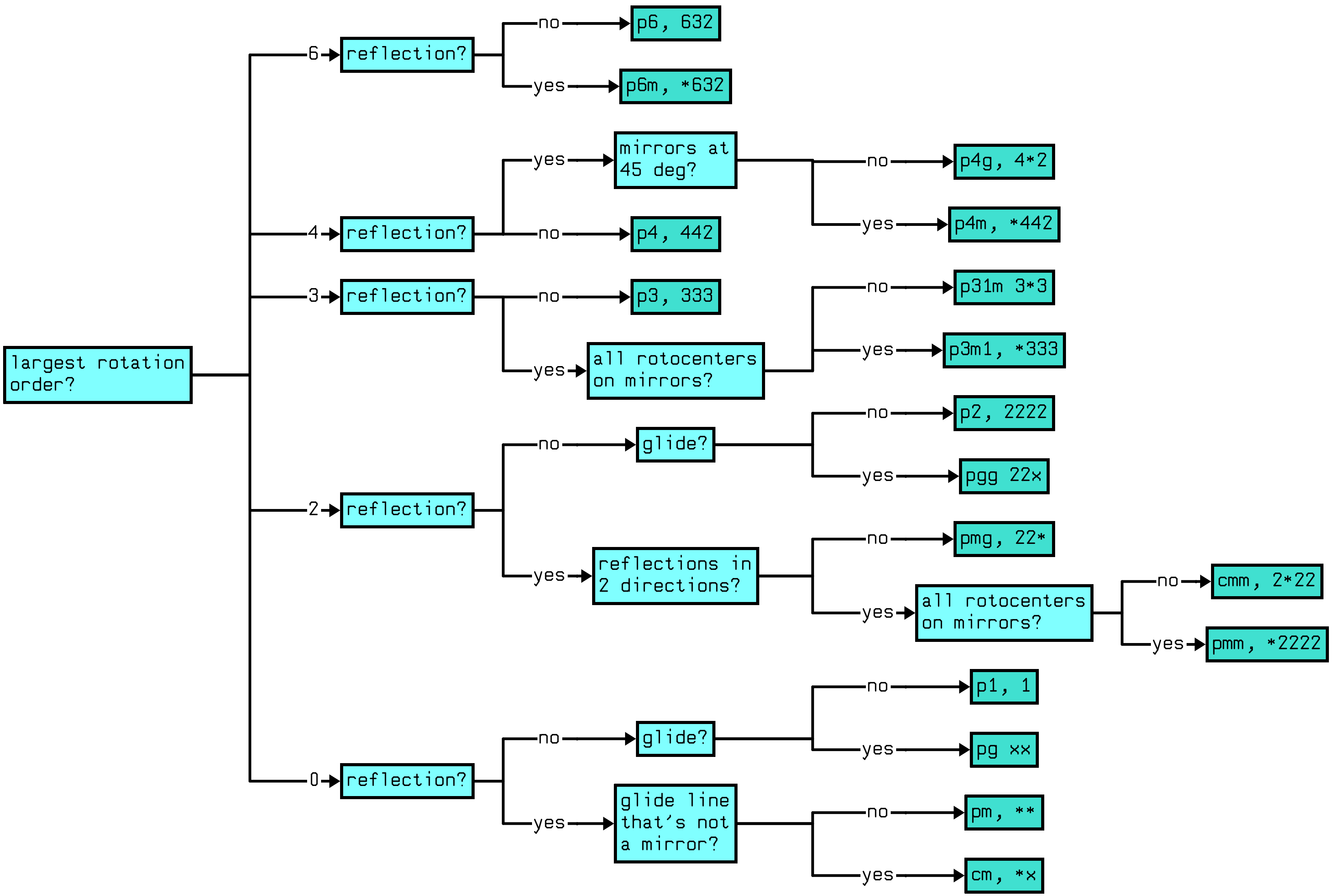
Case 4, Centered Rectangular



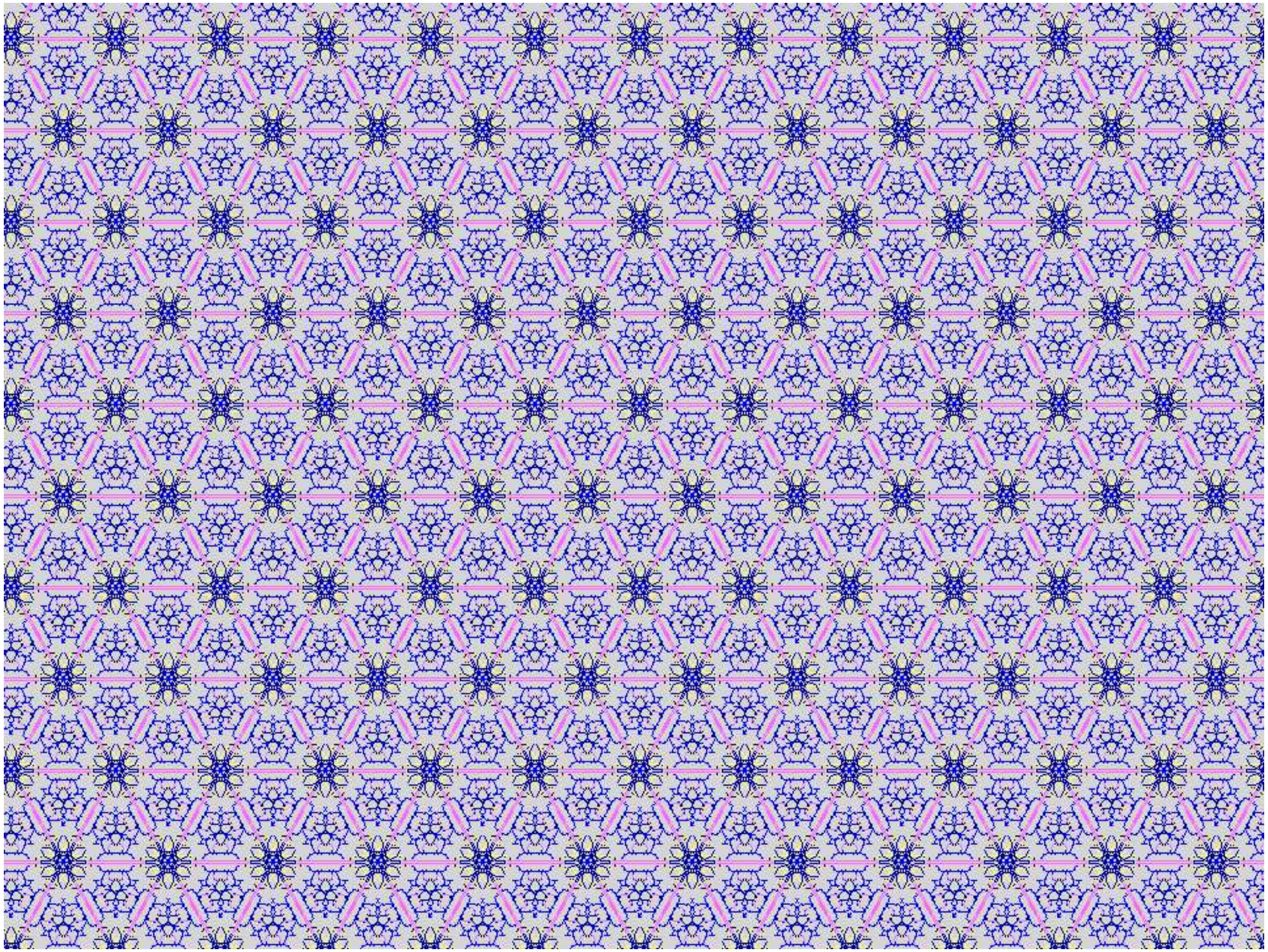
Classifying Wallpaper Groups

- The Conway notation uses the same symbols as before. In the crystallographic notation a “p” stands for a primitive cell (a and b are sides of the cell) and c stands for a centered cell. The crystallographic notation is an abbreviation of a longer, logical system. (See web links for an explanation.)

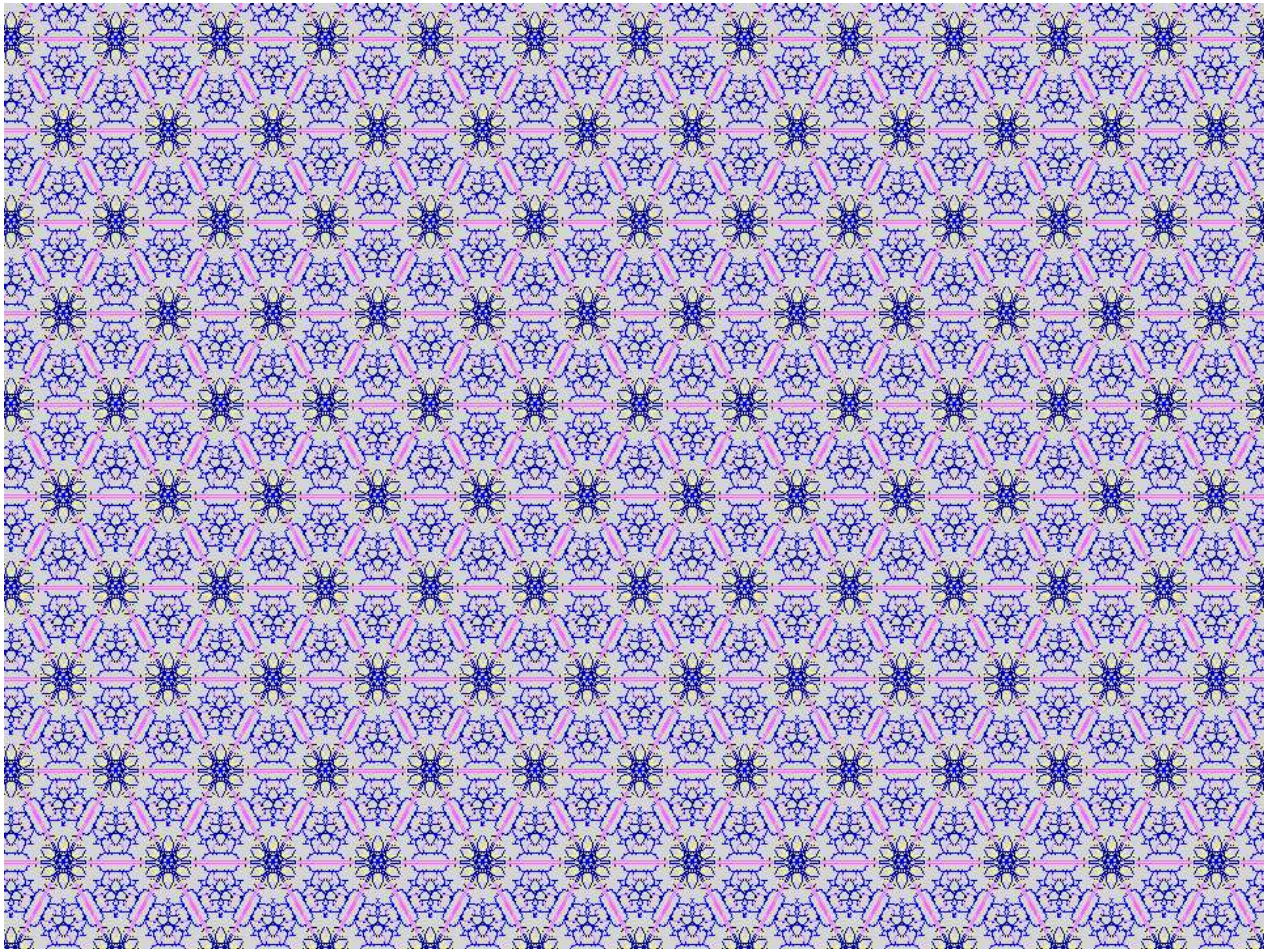
Classifying Wallpaper Groups



Classify



Classify



$*632$

Classify

Classify

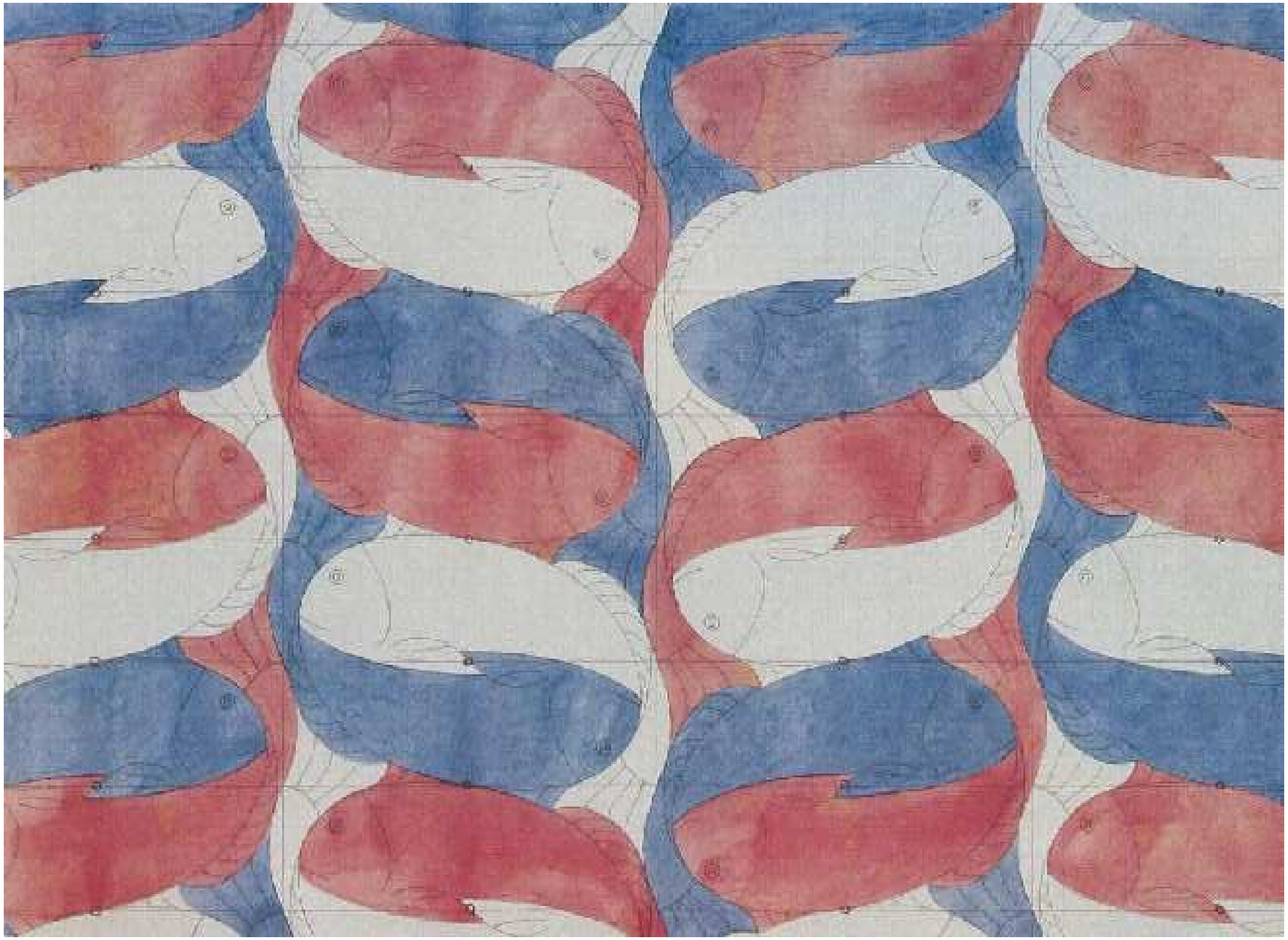
$3 * 3$

Classify

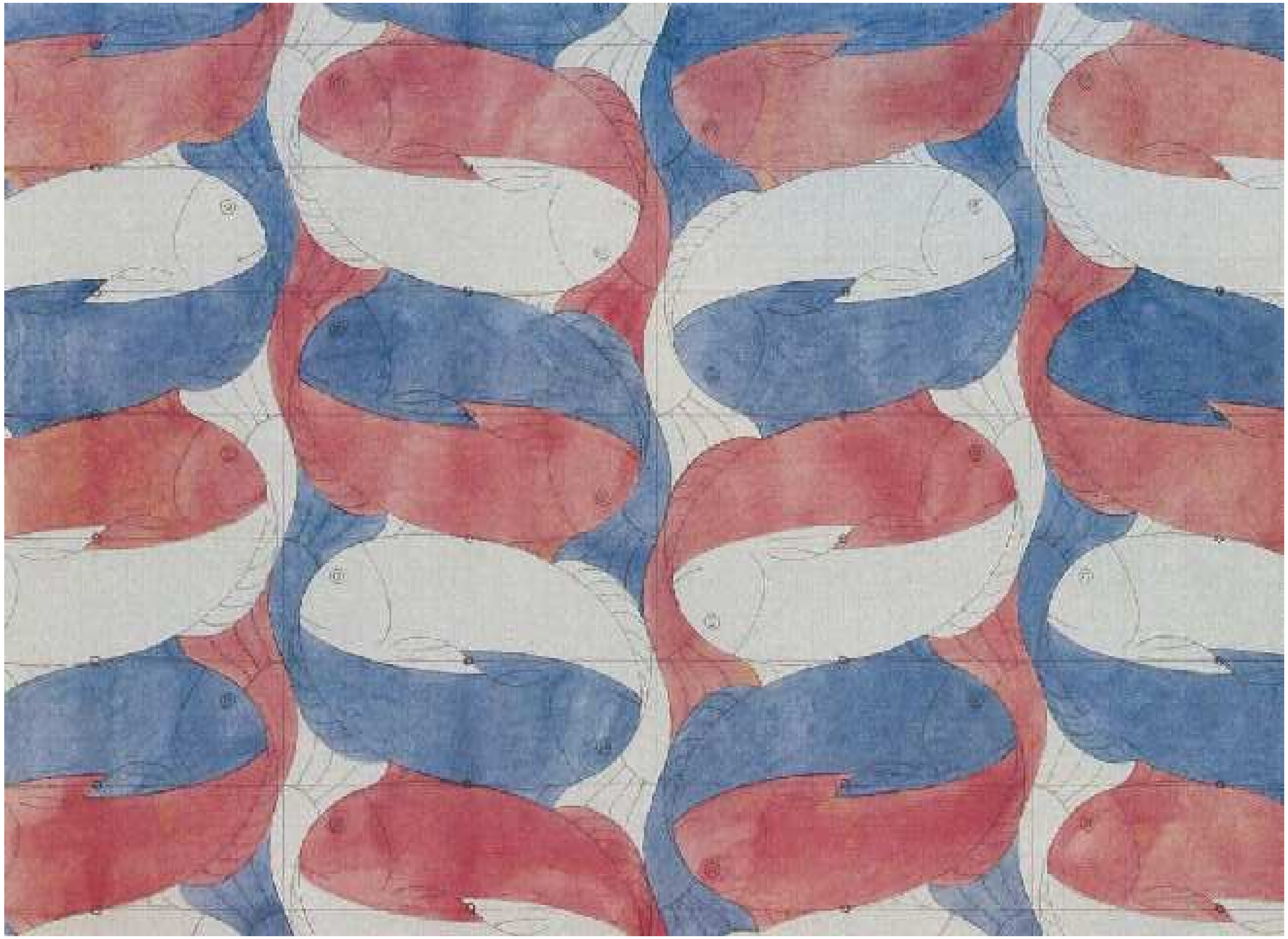
Classify

$*2222$

Classify

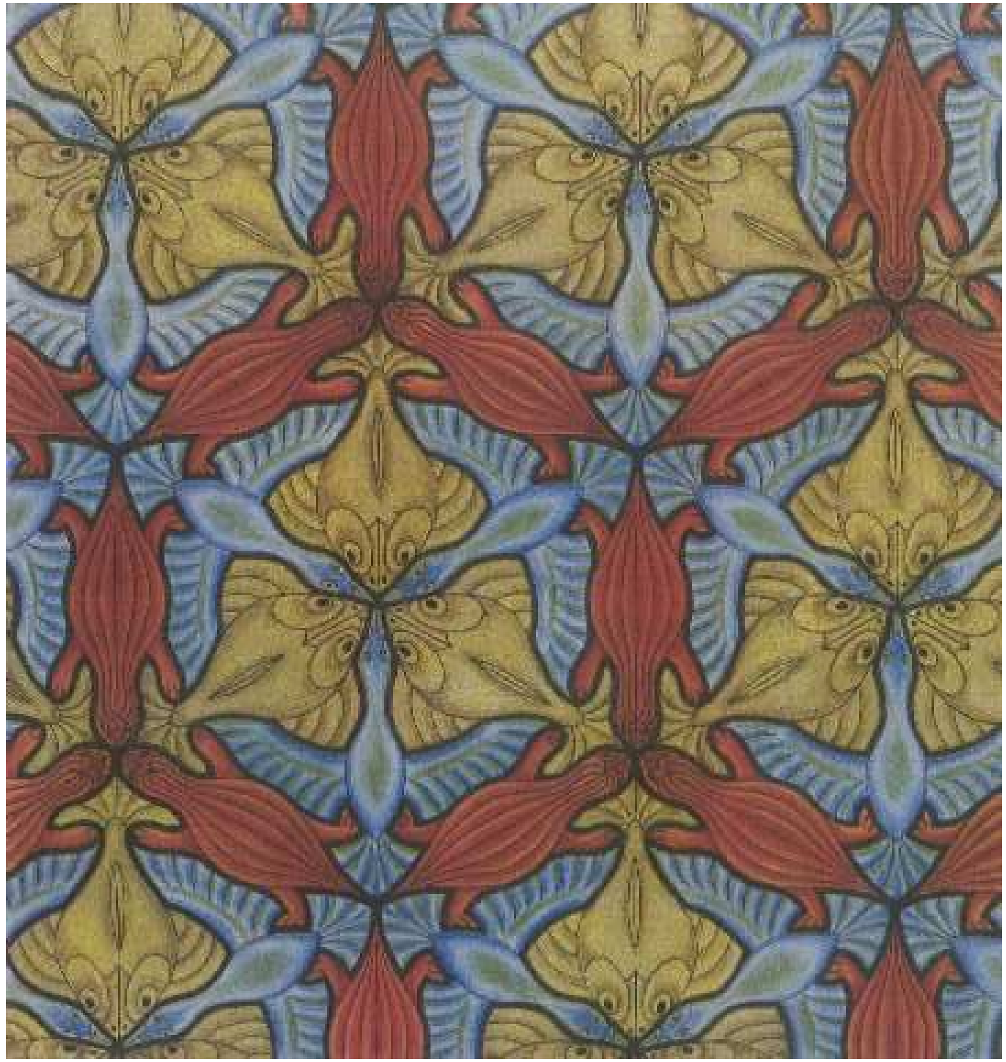


Classify

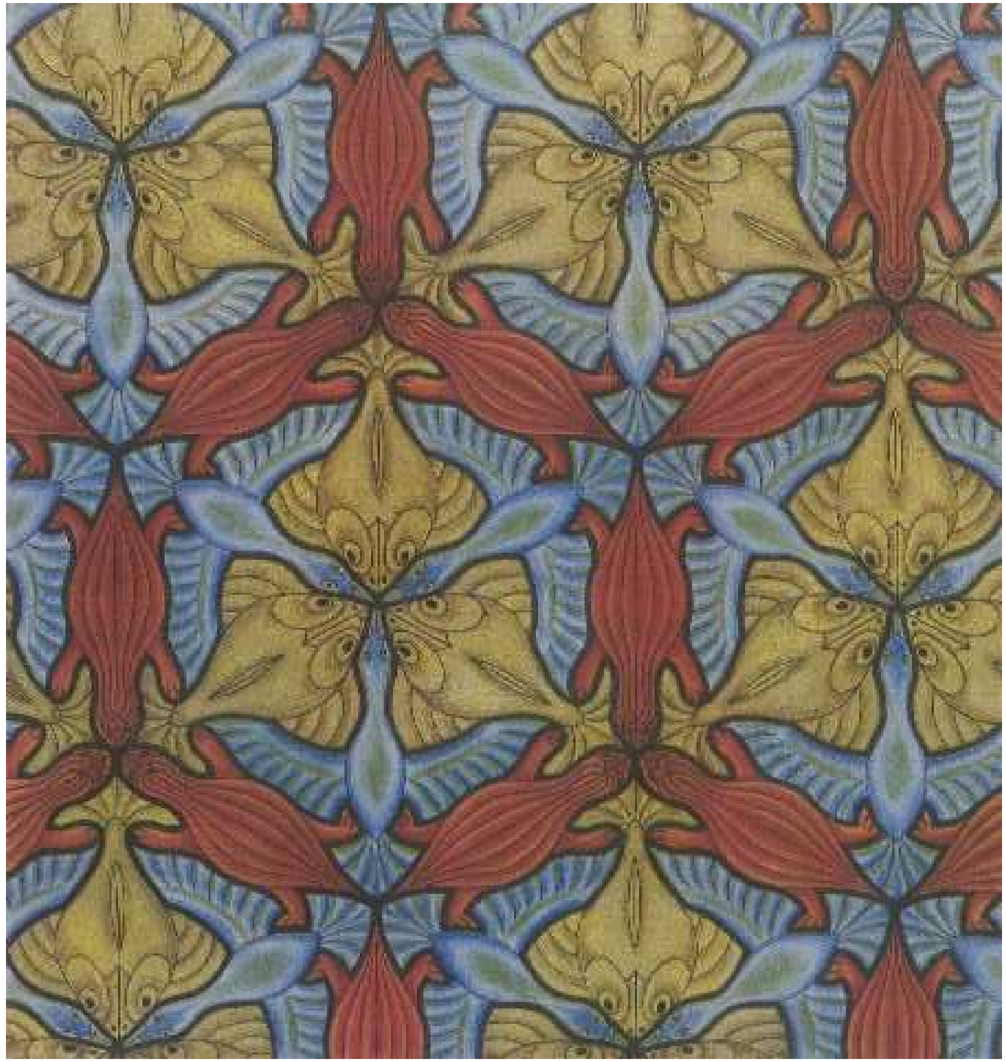


$22*$

Classify



Classify



$*333$