Geometric Transformations and Wallpaper Groups

Lance Drager

Texas Tech University

Symmetries of Geometric Patterns (Discrete Groups of Isometries)

Discrete Groups of Isometries

- Let $G \subseteq \mathbb{E}$ be a group of isometries. If p is a point, the **orbit of p under G** is defined to be

$$\mathrm{Orb}(p) = \{gp \mid g \in G\}\,.$$

- Either $\mathrm{Orb}(p) \cap \mathrm{Orb}(q) = \emptyset$ or $\mathrm{Orb}(p) = \mathrm{Orb}(q)$.
  - Suppose $r \in \mathrm{Orb}(p) \cap \mathrm{Orb}(q)$, then $r = gp = hq$, so $q = g^{-1}hp = ap$ for some $a \in G$. Then $gp = gaq$ for all $g \in G$, so $\mathrm{Orb}(p) \subseteq \mathrm{Orb}(q)$. Similarly $\mathrm{Orb}(q) \subseteq \mathrm{Orb}(p)$.
- The orbits partition the plane.

Discrete Groups of Isometries

- **Definition**. G is **discrete** if, for every orbit Γ of G, there is a $\delta > 0$ such that

$$p, q \in \Gamma \text{ and } p \neq q \implies \mathrm{dist}(p, q) \geq \delta.$$

- Example: D_3 as the symmetries of an equilateral triangle is discrete.
- Example: The orthogonal group $\mathbb{O} \subseteq \mathbb{E}$ is not discrete.

The Point Group

- Let G be a discrete group of isometries. We can define a group mapping

$$\pi\colon G \to \mathbb{O}\colon (A \,|\, v) \mapsto A.$$

- $N(G) = \ker(\pi)$ is the subgroup of G consisting of all the translations $(I \,|\, v)$ in G. It is a normal subgroup called the **translation subgroup of** G.
- The image $K(G) = \pi(G) \subseteq \mathbb{O}$ is called the **point group of** G. Note that $K(G)$ is **not** a subgroup of G.

Three Kinds of Symmetry Groups

- If G is a discrete group of isometries, there are three cases for $N(G)$.
  - $N(G) = \{(I \,|\, 0)\}$. G fixes the origin. These groups are the symmetries of **rosette** patterns and are called **rosette groups**.
  - All the translation vectors in $N(G)$ are collinear. These groups are the symmetries of **frieze** patterns and are called **frieze groups**.
  - $N(G)$ contains non-collinear translations. These groups are the symmetries of **wallpaper** patterns and are called **wallpaper groups**.
- We will classify rosette groups and frieze groups. We'll describe the classification of wallpaper groups, but we won't go through all the details.

Classification of Rosette Groups

- Rosette groups are the symmetries of rosette patterns

Classification of Rosette Groups

- **Theorem**. A rosette group G is either a finite cyclic group generated by a rotation or one of the dihedral groups D_n.
  - Let $J \subseteq G$ be the subgroup of rotations.
  - It's possible $J = \{I\}$. In this case $J = C_1$ or can be just denoted by 1.
  - If J contains a nontrivial rotation, choose $R = R(\theta) \in J$, so that $0 < \theta < 360°$ and θ is as small as possible. There must be a smallest such θ because the group is discrete.

Classification of Rosette Groups

- Classification Continued.
  - **Claim**: $n\theta = 360^\circ$ for some positive integer n. Suppose not. Then there is an integer k so that $k\theta < 360^\circ < (k+1)\theta$. Hence $360^\circ = k\theta + \varphi$, where $0 < \varphi < \theta$. Then $I = R(360) = R(k\theta + \varphi) = R(\theta)^k R(\varphi) = R^k R(\varphi)$. But then $R(\varphi) = R^{-k} \in J$, which contradicts our choice of θ.
  - If $n\theta = 360^\circ$ then $R^n = I$. We say that R has **order** n, in symbols $o(R) = n$.

Classification of Rosette Groups

- Classification Continued.
  - **Claim**: $J = \langle R \rangle = \left\{ I, R, R^2, \ldots, R^{n-1} \right\}$. Suppose not. Then there is a rotation $R(\varphi) \in J$ ($0 < \varphi < 360^\circ$) where φ is not an integer multiple of θ. But then there is some integer k so that $\varphi = k\theta + \psi$, where $0 < \psi < \theta$. But then we have $R(\varphi) = R^k R(\psi)$, so $R(\psi) = R^{-k} R(\varphi) \in J$, which contradicts our choice of θ.
  - Thus J is a cyclic group, which we can denote by C_n or just n.
  - Note that if $m \in \mathbb{Z}$, then $R^m = R^k$ for some $k \in \{0, 1, \ldots, n-1\}$, namely $k = m \pmod{n}$.

Classification of Rosette Groups

- Classification Continued.
  - If $J = G$ we are done. If not, G must contain a reflection.
  - Suppose that G contains a reflection $S = S(\varphi)$.
  - If $J = \{I\}$, then $G = \{I, S\}$ with $S^2 = I$. We can call this D_1 or $*1$.
  - If J is C_n, then G must contain the elements $R^k S = S(k\theta + \varphi)$ for $k = 0, 1, 2, \ldots n - 1$.
  - **Claim**: There is no reflection other that the $R^k S$ in G. Suppose not. Then $S(\psi) \in G$ for some $\psi \notin \{k\theta + \varphi \mid k \in \mathbb{Z}\}$. But $S(\psi)S(\varphi) = R(\psi - \varphi) \in G$, so we must have $\psi - \varphi = k\theta$ for some $k \in \mathbb{Z}$. But this implies $\psi = k\theta + \varphi$, a contradiction.

Classification of Rosette Groups

- Classification Continued.
  - **Claim**: $SR = R^{n-1}S$.

$$\begin{aligned} SR &= S(\varphi)R(\theta) \\ &= S(\varphi - \theta) \\ &= R(-\theta)S(\varphi) \\ &= R^{-1}S = R^{n-1}S. \end{aligned}$$

  - So, in this case, G is isomorphic to the dihedral group D_n, also denoted $*n$.
- Thus every rosette group is isomorphic to C_n (a.k.a. n) or the dihedral group D_n (a.k.a. $*n$). The names n and $*n$ are due to Conway.

The Lattice of an Isometry Group

- If G is a discrete group of isometries, the **lattice** L **of** G is the orbit of the origin under the translation subgroup $N(G)$ of G. In other words,

$$L = \left\{ v \in \mathbb{R}^2 \mid (I \,|\, v) \in G \right\}.$$

- if $a, b \in L$ then $ma + nb \in L$ for all $m, n \in \mathbb{Z}$, since $(I \,|\, ma) = (I \,|\, a)^m$ and $(I \,|\, nb)$ are in G and $(I \,|\, ma)(I \,|\, nb)0 = ma + nb$.
- **Theorem**. Let L be the lattice of G and let $K = K(G) \subseteq \mathbb{O}$ be the point group. Then $KL \subseteq L$, i.e. if $A \in K$ and $v \in L$ then $Av \in L$.

The Lattice of an Isometry Group

- We can make the following general computation, conjugating a translation $(I \,|\, v)$ by an isometry $(A \,|\, a)$

$$\begin{aligned}
(A \,|\, a)(I \,|\, v)(A \,|\, a)^{-1} &= (A \,|\, a)(I \,|\, v)(A^{-1} \,|\, -A^{-1}a) \\
&= (A \,|\, a)(A^{-1} \,|\, v - A^{-1}a) \\
&= (AA^{-1} \,|\, a + A(v - A^{-1}a)) \qquad (1) \\
&= (I \,|\, a + Av - a) \\
&= (I \,|\, Av),
\end{aligned}$$

so we get a translation with the translation vector changed by the matrix A.

The Lattice of an Isometry Group

- If $v \in L$ then $(I \,|\, v) \in G$ and if $A \in K$ then $(A \,|\, a) \in G$ for some vector a. Then the conjugate of $(I \,|\, v)$ by $(A \,|\, a)$ is in G. By the computation above $(I \,|\, Av) \in G$, so $Av \in L$.

Frieze Groups

- Let G be a Frieze group, i.e., all the vectors in the lattice L are collinear. These groups are the symmetry groups of Frieze patterns.

Frieze Groups

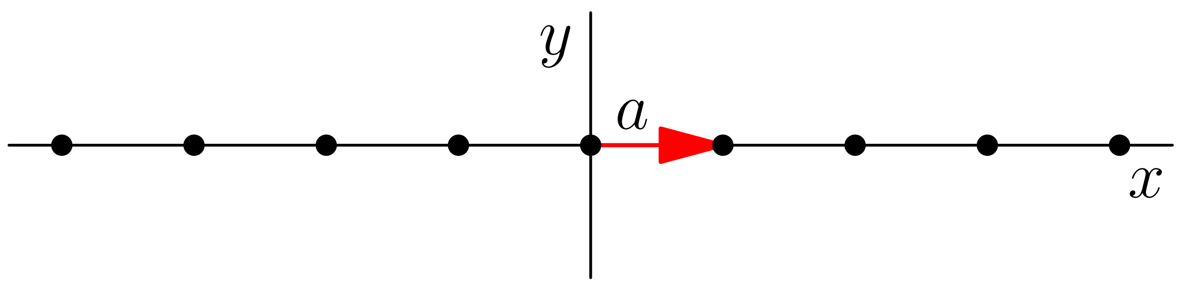
- A wallpaper border.

Frieze Groups

- We want to classify the Frieze Groups up to **geometric** isomorphism. We consider two Frieze Groups G_1 and G_2 to be in the same geometric isomorphism class if there is a group isomorphism $\varphi\colon G_1 \to G_2$ that preserves the type of the transformations.
- Let G be a Frieze group. Choose $a \in L$ so that $\|a\| > 0$ is a small as possible. This is possible because G is discrete.
- **Claim**: $L = \{na \mid n \in \mathbb{Z}\}$. Suppose not. Then there is a c in L so that $c \notin \mathbb{Z}a$. But we must have $c = sa$ for some scalar s, where $s \notin \mathbb{Z}$. We can write $s = k + r$ where k is an integer and $0 < |r| < 1$. Thus, $c = sa = ka + ra$. Since $ka \in L$, $ra = c - ka \in L$. But $\|ra\| = |r|\,\|a\| < \|a\|$, contradicting our choice of a.

Frieze Groups

- If $T \in \mathbb{E}$, the conjugation $\psi_T(S) = TST^{-1}$ is a group isomorphism $\mathbb{E} \to \mathbb{E}$ which preserves type. Hence $\psi_T(G)$ is geometrically isomorphic to G. Choosing T to be a rotation, we may as well assume that a points in the positive x-direction. Thus, our lattice looks like



Frieze Groups

- The point group $K \subseteq \mathbb{O}$ of G must preserve the lattice L. There are only four possible elements of K:
  - The identity I
  - A half turn, i.e., rotation by 180°. Call it T for turn.
  - Reflection through the x-axis, call it H.
  - Reflection through the y-axis, call it V.
- With this preparation, let's describe the classification of frieze groups.

Classification of Frieze Groups

- **Theorem**. There are exactly 7 geometric isomorphism classes of frieze groups.
- A system for naming these groups has been standardized by crystallographers. Another (better) naming system has been invented by Conway, who also invented English names for these classes.

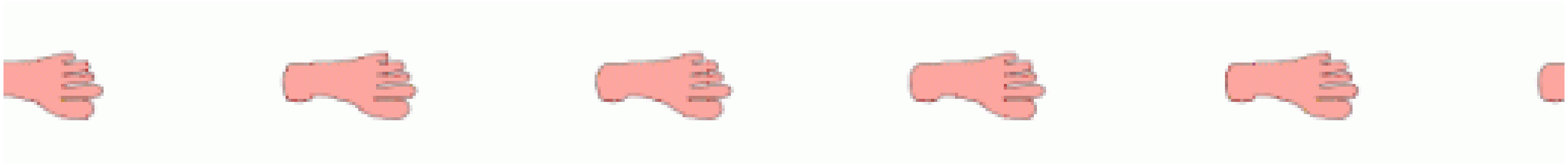
Crystallographic Names

- The crystallographic names consist of two symbols.
  1. First Symbol
     - 1 No vertical reflection.
     - m A vertical reflection.
  2. Second Symbol
     - 1 No other symmetry.
     - m Horizontal reflection.
     - g Glide reflection (horizontal).
     - 2 Half-turn rotation.

Conway Names

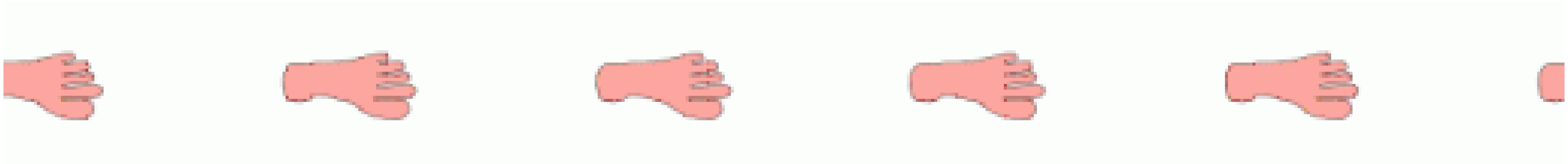
- In the Conway system two rotations are of the same class if their rotocenters differ by a motion in the group. Two reflections are of the same class if their mirror lines differ by a motion in the group.
- Conway Names
  - ∞ We think of the translations as "rotation" about a center infinity far away in the up direction, or down direction. This "rotation" has order ∞.
  - 2 A class of half-turn rotations.
  - $*$ Shows the presence of a reflection. If a 2 or ∞ comes after the $*$, then the rotocenters are at the intersection of mirror lines.
  - x Indicates the presence of a glide reflection.

The 7 Frieze Groups



Hop, 11, $\infty\infty$

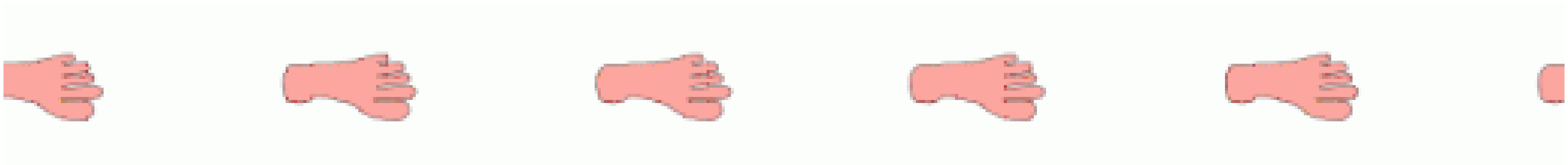
The 7 Frieze Groups



Hop, 11, $\infty\infty$

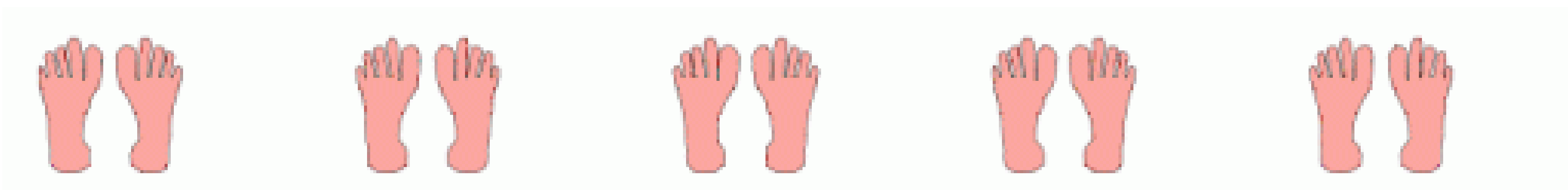
Jump, $1m$, $\infty *$

The 7 Frieze Groups



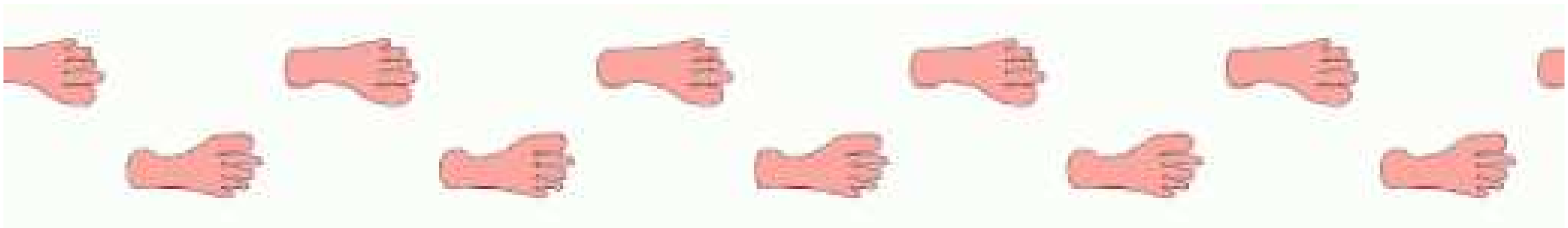
Hop, 11, $\infty\infty$

Jump, $1m$, $\infty *$



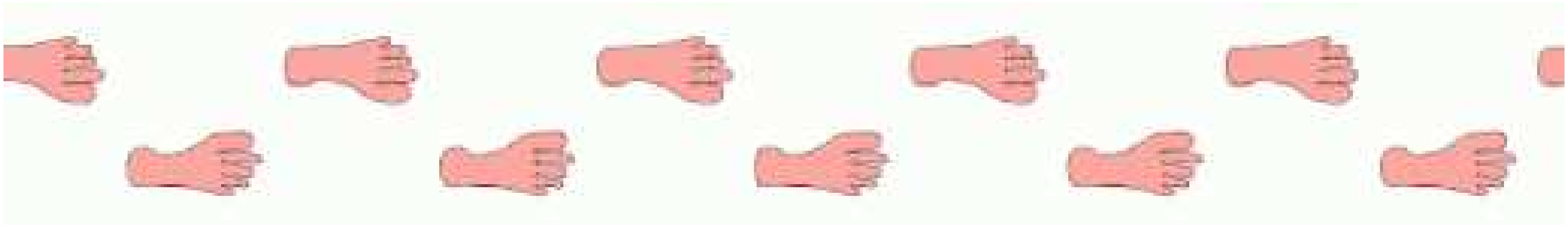
Sidle, $m1$, $*\infty\infty$

The 7 Frieze Groups

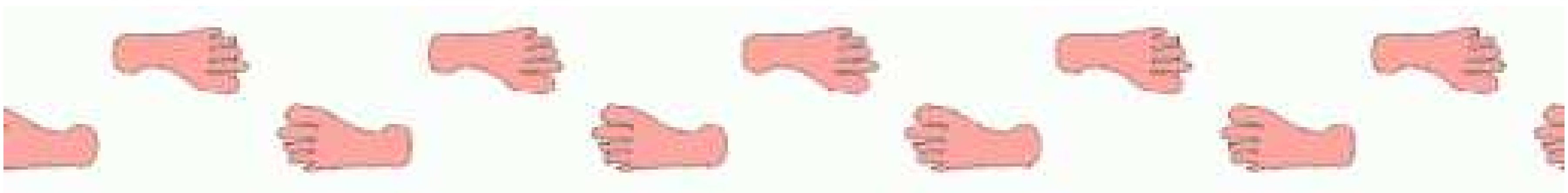


Step, $1g$, ∞x

The 7 Frieze Groups

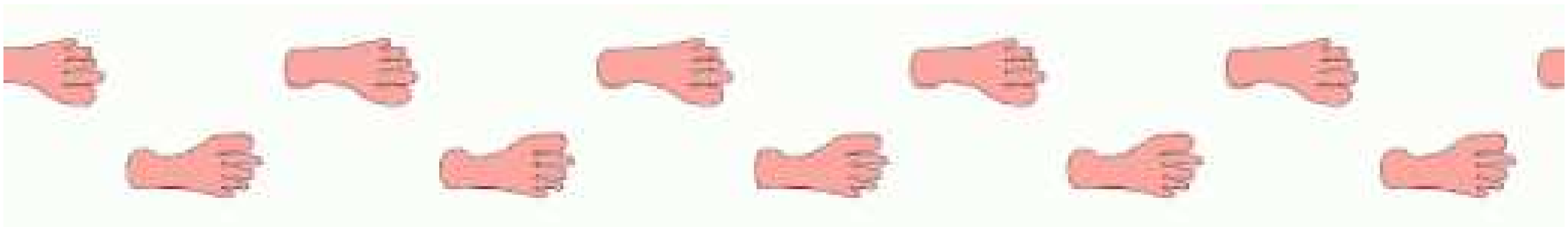


Step, $1g$, ∞x

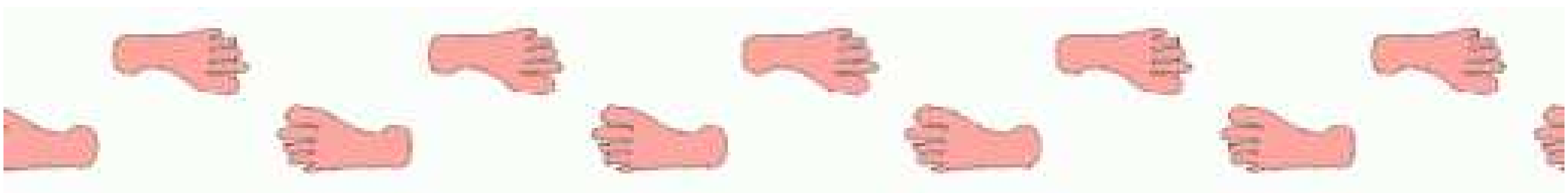


Spinhop, 12, 22∞

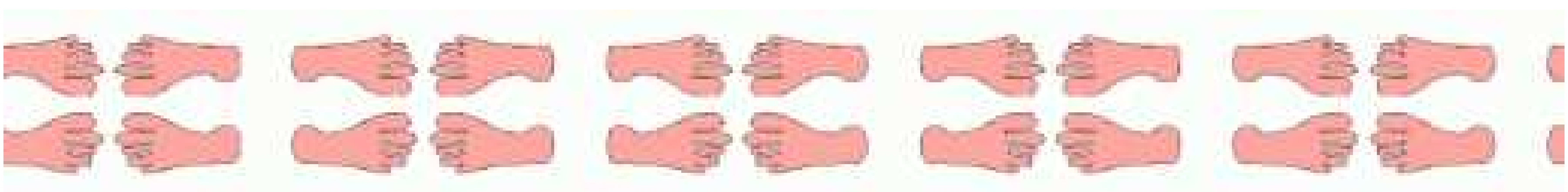
The 7 Frieze Groups



Step, $1g$, ∞x

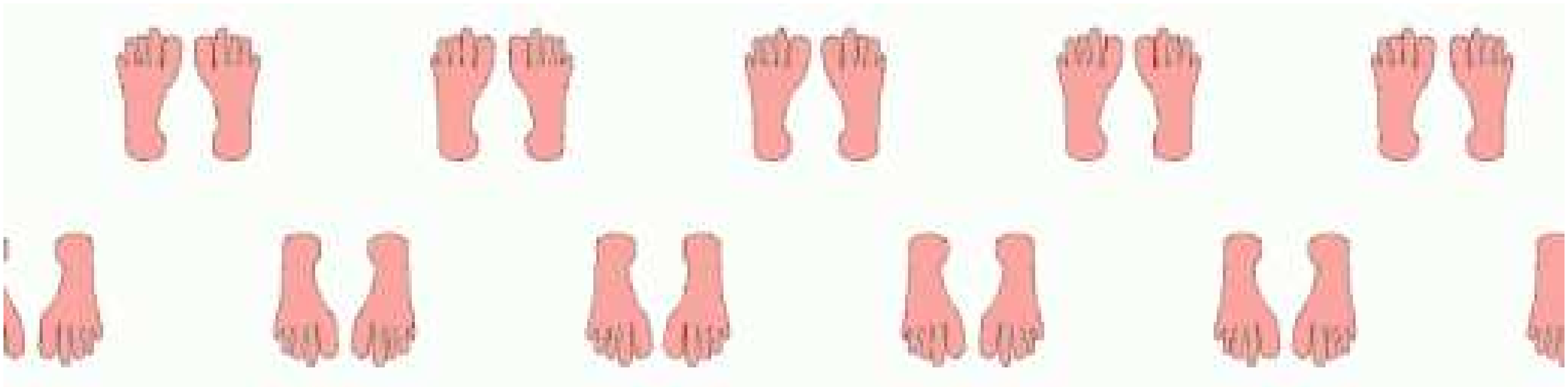


Spinhop, 12, 22∞



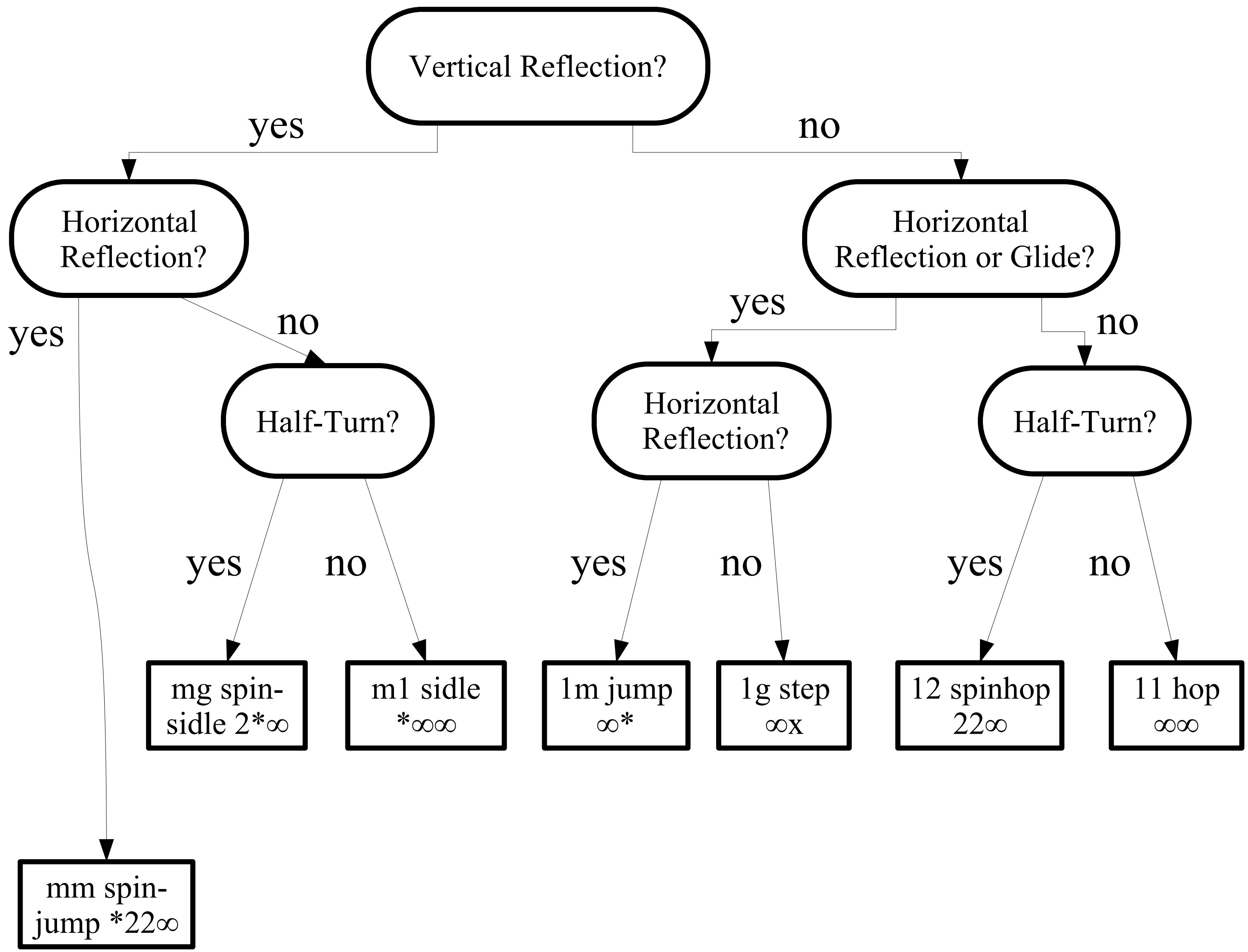
Spinjump, mm, $*22\infty$

The 7 Frieze Groups



Spinsidle, mg, $2 * \infty$

How to Classify Frieze Groups



Proof of the Classification

- Let G be our frieze group, as set up above. The possible elements of the point group K are the half-turn T, reflection in the x-axis H, reflection in the y-axis V, and the identity matrix I. Of course, we have explicit matrices for these transformations

$$T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- It's easy to check
  - $HT = TH = V$.
  - $VT = TV = H$.
  - $HV = VH = T$.

  Thus, if two of these are in K, so is the third.

Proof of the Classification

- The 5 possibilities for K are
  I. $\{I\}$.
  II. $\{I, T\}$.
  III. $\{I, V\}$.
  IV. $\{I, H\}$.
  V. $\{I, T, H, V\}$.
- **Case I**. The group contains only translations. This is Hop (11, $\infty\infty$).
- **Case II**. Since $T \in K$, we must have $(T \mid v) \in G$ for some v. This is a rotation around some point. We can conjugate our group by a translation to get an isomorphic group that contains $(T \mid 0)$ (the translation subgroup doesn't change).

Proof of the Classification

- **Case II continued**. We now have the origin as a rotocenter. The group contains exactly the elements $(I \mid na)$ and $(T \mid na)$ for $n \in \mathbb{Z}$. To find the rotocenter p of this rotation, we have to solve $(I - T)p = na$, but

  $$(I - T)(na/2) = na/2 - (n/2)Ta = na/2 - (n/2)(-a) = na,$$

  so the rotocenters are at "half lattice points" $na/2$. There are two classes of rotations, one containing the rotation at $a/2$ and the other the rotation at a. This is Spinhop (12, 22∞).

Proof of the Classification

- **Case III**. In this case we have the reflection matrix $V \in K$. Note $Ve_1 = -e_1$ and $Ve_2 = e_2$. We must have some isometry $(V \mid b) \in G$, and we can write $b = \alpha e_1 + \beta e_2$. If $\beta \neq 0$, then $(V \mid b)$ is a glide. But this is impossible, because then $(V \mid b)^2$ would be translation in the vertical direction, which is not in the group. Thus, $(V \mid b) = (V \mid \alpha e_1)$ is a reflection. By conjugating the whole group by a translation, we may as well assume the reflection $(V \mid 0)$ is in the group.
  Now the group consists of the elements $(I \mid na)$ and $(V \mid na)$, which is a reflection through the vertical line that passes through $na/2$. Thus we have vertical reflections at half lattice points. There are two classes of mirror lines. The group is Sidle ($m1$,$*\infty\infty$).

Proof of the Classification

- **Case IV**. We have $K = \{I, H\}$. Now things start to get interesting! The matrix H could come either from a reflection or from a glide. Recall $He_1 = e_1$ and $He_2 = -e_2$.
  - Suppose H comes from a reflection. By conjugating with a translation, we may assume the mirror line is the x-axis, i.e., $(H \mid 0)$ is in the group. The group contains the elements $(I \mid na)$ and $(H \mid na)$, the latter being uninteresting glides. The group is Jump ($1m$, $\infty *$).

Proof of the Classification

- Case IV. continued.
  - Suppose H comes from a glide (but not a reflection). We can conjugate the group to make the glide line the x-axis, so we will have a glide of the form $(H \,|\, se_1)$. But then $(H \,|\, se_1)^2 = (I \,|\, 2se_1)$ is in the group, so $2se_1$ must be a lattice point, say $2se_1 = ma$. If m was even, se_1 would be a lattice point ka but then we would have $(I \,|\, ka)^{-1}(H \,|\, ka) = (H \,|\, 0)$ in the group, a contradiction. Thus m is odd, say $m = 2k + 1$. Then $(H \,|\, 2se_1) = (H \,|\, a/2 + ka)$. Multiplying by a translation, we get the glide $(H \,|\, a/2)$ in the group. The group contains the elements $(I \,|\, na)$ and $(H \,|\, a/2 + na)$. The group is Step ($1g$, ∞x).

Proof of the Classification

- **Case V**. The point group is $K = \{I, V, H, T\}$. Again, V must come from a reflection, which we can assume is $(V \mid 0)$. Again, there are two cases: H comes from a reflection and H comes from a glide.
  - Suppose H comes from a reflection. By conjugating with a vertical translation we can assume the mirror is the x-axis, so we have $(H \mid 0)$, and we'll still have $(V \mid 0)$. Then we have $(H \mid 0)(V \mid 0) = (T \mid 0)$, the half-turn around the origin. The group elements are $(I \mid na)$; $(H \mid na)$, which are uninteresting glides; $(V \mid na)$, which give vertical mirrors a half lattice points; and $(T \mid na)$, which are half-turns with the rotocenters at half-lattice points. The group is Spinjump, (mm, $*22\infty$).

Proof of the Classification

- Case V, continued.
  - Assume H comes from a glide. As above, we can assume $(V \mid 0)$ is in the group. As in Case IV, we can assume the glide line is the x-axis and that our glide is $(H \mid a/2)$. We then have $(H \mid a/2)(V \mid 0) = (T \mid a/2)$. The group elements are $(I \mid na)$; the glides $(H \mid a/2 + na)$, which are not very interesting; the reflections $(V \mid na)$ which have their mirrors at half lattice points; and the half-turns $(T \mid a/2 + na)$, which have their rotocenters at the points $a/4 + na/2$, for example $a/4, 3a/4, 5a/4, 7a/4 \ldots$. The group is Spinsidle (mg, $2 * \infty$).