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VECTOR INVARIANTS OF $\mathsf{Syl}_p(\mathsf{GL}(n, \mathbb{F}_q))$ AND THEIR HILBERT IDEALS

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Abstract. We describe the Hilbert ideal of the vector invariants of a p-Sylow subgroup of the general linear group.

1. Introduction

Let $\mathbb{F} = \mathbb{F}_q$ be a finite field of characteristic p and order $q = p^s$. Consider the general linear group of $d \times d$ matrices over this field, $\mathsf{GL}(d, \mathbb{F})$.

The group $\mathsf{GL}(d, \mathbb{F})$ acts on the vector space $W = \mathbb{F}^d$ by matrix multiplication, which induces an action on the dual space and hence on the full symmetric algebra on the dual, denoted by $\mathbb{F}[W]$. Its ring of polynomial invariants is the Dickson algebra, denoted by $\mathcal{D}(d) = \mathbb{F}[W]^{\mathsf{GL}(d,\mathbb{F})}$. Moreover for any subgroup $G \subseteq \mathsf{GL}(d, \mathbb{F})$ we obtain

$$\mathcal{D}(d) \hookrightarrow \mathbb{F}[W]^G \hookrightarrow \mathbb{F}[W]$$

a chain of Noetherian commutative $\mathbb{F}$-algebras, see [7] for more background on invariant theory of finite groups.

Consider a finite group P and a faithful representation

$$\rho_1 : P \hookrightarrow \mathsf{GL}(d, \mathbb{F})$$

afforded by the upper triangular matrices

$$M = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \in \mathsf{GL}(d, \mathbb{F}).$$

The group $\rho_1(P) \cong P$ is a p-Sylow subgroup of the general linear group. Denote by $x_1, \dots, x_d$ the standard dual basis of W^*. Then its ring of invariants can be written as the polynomial algebra

$$\mathbb{F}[x_1, \dots, x_d]^P = \mathbb{F}[c_{\text{top}}(x_1), \dots, c_{\text{top}}(x_d)]$$

where $c_{\text{top}}(x_i)$ denotes the top orbit Chern class of the basis element x_i, i.e., the product of all linear forms in the set $\{gx_i | g \in \rho_1(P)\}$, see, e.g., Example 2 in Section 4.5 in [7]

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In this article we consider the n-fold vector invariants of P, i.e., we embed the group P into $\mathsf{GL}(dn, \mathbb{F})$

$$\rho_n : P \hookrightarrow \mathsf{GL}(dn, \mathbb{F})$$

afforded by the block diagonal matrices

$$\mathsf{block}(\underbrace{M, \ldots, M}_{n \text{ times}}) = \begin{bmatrix} M & & 0 \\ & \ddots & \\ 0 & & M \end{bmatrix}$$

for all $M \in \rho_1(P)$. Denote by $V = W^{\oplus n}$ the corresponding dn-dimensional vector space. We denote the standard dual basis of V^* by $x_{11}, \ldots, x_{1d}, x_{21}, \ldots x_{2d}, \ldots, x_{nd}$.

Recall that the Hilbert ideal of the ring of invariants $\mathbb{F}[V]^P$ is defined as the ideal in the ambient ring of polynomials generated by all invariants of positive degree

$$\mathfrak{H}(\rho_n(P)) = (\overline{\mathbb{F}[V]^P})\mathbb{F}[V].$$

In this paper we prove the following result:

Theorem 1.1. *The Hilbert ideal $\mathfrak{H}(\rho_n(P))$ is generated by the top orbit Chern classes of the basis elements x_{ji}, $j = 1, \ldots, n$ and $i = 1, \ldots, d$.*

Indeed, in the case of $d = 2$, this result follows from the description of the ring of invariants:

Theorem 1.2. *The ring of invariants $\mathbb{F}[V]^P$ is generated by*

$$c_{\text{top}}(x_{j1}) \quad j = 1, \ldots, n,$$

and the elements in the ideal $I = (x_{12}, \ldots, x_{n2})\mathbb{F}[V] \cap \mathbb{F}[V]^P$.

Ever since Weyl's First Main Theorem of Invariant Theory vector invariants have been extensively studied. We mention some of the (for our paper) most relevant results: In [4] Grosshans studied Weyl's result over algebraically closed fields of finite characteristic. Richman computed in [9] the generating set of the ring of invariants for the case $p = q = 2$ and $d = 2$. Campbell and Hughes proved in [2] Richman's conjecture on the generating set for the case $p = q$ and $d = 2$. In [3] Campbell, Shank and Wehlau produced a SAGBI basis for the case $p = q$ and $d = 2$. In Sezer's and Ünlü's paper [8] we find a description of a reduced Gröbner basis of the Hilbert ideal for $p = q = 2$ and $d = 2$.

In the next section we choose a term order and prove some technical preliminary results. In Section 3 we prove Theorem 1.2 and deduce Theorem 1.1 for the case $d = 2$. This serves as an induction start. The induction is completed in Section 4 proving Theorem 1.1 in general. In Section 5 we explain the significance of the ideal I of Theorem 1.2: It is the radical of the image of the transfer.

2. Choosing a Good Term Order

We denote the variables as $x_{11}, \ldots, x_{1d}, x_{21}, \ldots, x_{2d}, \ldots, x_{n1}, \ldots, x_{nd}$ and order them as follows

$$x_{11} > x_{21} > \cdots > x_{n1} > x_{12} > \cdots > x_{n2} > \cdots > x_{1d} > \cdots > x_{nd}.$$

This induces a lexicographic term order on the elements of $\mathbb{F}[V]$. We denote by $LT(-)$ the leading term of $-$. The following results motivate this choice of order.

Lemma 2.1. *Let $m \in \mathbb{F}[x_{11}, \ldots, x_{nd}]$ be a monomial. Then*

$$LT(gm) = m \quad \forall g \in P.$$

Moreover, $gm = m + h$ for some $h \in (x_{12}, \ldots, x_{n2}, \ldots, x_{1d}, \ldots, x_{nd})\mathbb{F}[V]$.

Proof. Let $m = x_{11}^{\alpha_{11}} \cdots x_{nd}^{\alpha_{nd}}$. Let $\rho_n(g) = \mathsf{block}(\underbrace{M, \ldots, M}_{n \text{ times}})$ where

$$M = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1d} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{d-1,d} \\ 0 & \cdots & 0 & 1 \end{bmatrix} \in \rho_1(P)$$

be an arbitrary element of $\rho_n(P)$. Then

$$gm = \prod_{j,i} (x_{ji} + a_{i,i+1}x_{j,i+1} + \cdots + a_{id}x_{jd})^{\alpha_{ji}}.$$

Expanding this expression gives the desired result. □

Lemma 2.2. *If $f \in \mathbb{F}[V]^P$ has a term $x_{11}^{\alpha_{11}} x_{21}^{\alpha_{21}} \cdots x_{n1}^{\alpha_{n1}}$, then α_{j1} is divisible by q^{d-1} for all $j = 1, \ldots, n$.*

Proof. We prove this by induction on n. If $n = 1$ we have an explicit description of the ring of invariants (see introduction) and we note that the top orbit Chern class

$$c_{\text{top}}(x_{11}) = x_{11}^{q^{d-1}} + \text{other terms}$$

is the only generator with a term $x_{11}^{\alpha_{11}}$.

Next, let $n > 1$. We consider the term

$$m = x_{11}^{\alpha_{11}} x_{21}^{\alpha_{21}} \cdots x_{n1}^{\alpha_{n1}}.$$

In case that there is a j_0 such that $\alpha_{j_0 1} = 0$ we obtain our desired statement by induction hypothesis. So assume that $\alpha_{j1} \neq 0$ for all $j = 1, \ldots, n$. We sort the invariant f by monomials $x_{n1}^{\alpha_{n1}} \cdots x_{nd}^{\alpha_{nd}}$ and obtain

$$f = \sum_I f_I x_{n1}^{\alpha_{n1}} \cdots x_{nd}^{\alpha_{nd}}$$

where the sum runs over d-tuples $I = (\alpha_{n1}, \ldots, \alpha_{nd})$. Note that

$$f_I = f_I(x_{11}, \ldots, x_{1d}, \ldots, x_{n-1,1}, \ldots, x_{n-1,d}).$$

Our monomial m appears in $f_{I_0} x_{n1}^{\alpha_{n1}}$ for $I_0 = (\alpha_{n1}, 0, \ldots, 0)$. By Lemma 2.1 $x_{n1}^{\alpha_{n1}}$ cannot be a nontrivial translate of any monomial. Therefore, f_{I_0} has to be an invariant. In particular we can assume by induction that $\alpha_{11}, \ldots, \alpha_{n-1,1}$ are divisible by q^{d-1}.

Switching the roles of n and, say, $n-1$ in this argument allows us to conclude that all α_{j1}, $j = 1, \ldots, n$ are divisible by q^{d-1}. □

3. The case of 2×2-matrices

In this section we prove Theorem 1.1 for the case $d = 2$, which serves as an induction start as it will become apparent in Section 4. We note that the result of this section was proven in [1] for the cases $q = 2, 4$, $n = 2, 3$, in addition to the papers mentioned in the introduction.

Consider the p-Sylow subgroup of $\mathsf{GL}(2, \mathbb{F})$ given as follows:

$$\rho_1 : P \hookrightarrow \mathsf{GL}(2, \mathbb{F})$$

where

$$P \cong \rho_1(P) = \{M \in \mathsf{GL}(2, \mathbb{F}) | M = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} | a \in \mathbb{F}\} \subseteq \mathsf{GL}(2, \mathbb{F}).$$

It is an elementary abelian p-group of rank s. Its ring of invariants is given by

$$\mathbb{F}[x, y]^P = \mathbb{F}[x^q - xy^{q-1}, y]$$

where we chose the standard dual basis x, y for V^*. Note that this is a polynomial algebra generated by the top orbit Chern classes of the basis elements:

$$c_{\text{top}}(x) = \prod_{g \in P} gx = x^q - xy^{q-1} \quad c_{\text{top}}(y) = y.$$

Next consider the 2-fold vector invariants of P, i.e., we look at the faithful representation of P

$$\rho_2 : P \hookrightarrow \mathsf{GL}(4, \mathbb{F})$$

afforded by the block diagonal matrices

$$\begin{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

where $a \in \mathbb{F}$. Its ring of invariants is given by

$$\mathbb{F}[x_1, y_1, x_2, y_2]^P = \mathbb{F}[c_{\text{top}}(x_1), y_1, c_{\text{top}}(x_2), y_2, Q_{12}]/(r)$$

where

$$Q_{12} = x_1 y_2 - x_2 y_1$$

and

$$r = Q_{12}^q - c_{\text{top}}(x_1) y_2^q + c_{\text{top}}(x_2) y_1^q - Q_{12} y_1^{q-1} y_2^{q-1}$$

see [6].[1] Next consider the n-fold vector invariants of P:

$$P \cong \rho_n(P) = \{ \begin{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \end{bmatrix} | a \in \mathbb{F}\} \subseteq \mathsf{GL}(2n, \mathbb{F}).$$

We denote the standard dual basis as $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ and note that by choice of our order we have

$$x_1 > x_2 > \cdots > x_n > y_1 > \cdots > y_n.$$

[1]This article treats only the case where $q = p$. However, the proof works in the general case.

Theorem 3.1. *The ring of invariants* $\mathbb{F}[V]^P$ *is generated by*

$$c_{\text{top}}(x_j), j = 1, \dots n$$

and the elements in the ideal $I = (y_1, \dots, y_n)\mathbb{F}[V] \cap \mathbb{F}[V]^P$.

Proof. Let A be the $\mathbb{F}$ algebra generated by $c_{\text{top}}(x_j)$, $j = 1, \dots, n$ and the elements in the ideal $(y_1, \dots, y_n)\mathbb{F}[V] \cap \mathbb{F}[V]^P$. By construction A is a subalgebra of the invariants $\mathbb{F}[V]^P$.

Any invariant f such that each of its terms is divisible by one of the y_j's is in I.

Next, let $f \in \mathbb{F}[V]^P$ be an invariant not in I. Then f contains a term $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. By Lemma 2.2 we have that all the α_j's are divisible by q. Set $\alpha_j = qk_j$, then

$$f - c_{\text{top}}(x_1)^{k_1} \cdots c_{\text{top}}(x_n)^{k_n}$$

is an invariant such that the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is replaced by an element of the ideal $(y_1, \dots, y_n)\mathbb{F}[V]$, because

$$c_{\text{top}}(x_1)^{k_1} \cdots c_{\text{top}}(x_n)^{k_n} = \prod_{j=1}^{n} (x_j^q - x_j y_j^{q-1})^{k_j} = \prod_{j=1}^{n} (x_j^{qk_j}) + h$$

where $h \in (y_1, \dots, y_n)\mathbb{F}[V]$. Successively we obtain an invariant in $(y_1, \dots, y_n)\mathbb{F}[V]$ and hence in I. □

Corollary 3.2. *The Hilbert ideal is generated by the top orbit Chern classes of the basis elements* $x_1, \dots, x_n, y_1, \dots, y_n$.

Proof. The Hilbert ideal is generated by all invariants of positive degree, i.e., it is generated by the orbit Chern classes $c_{\text{top}}(x_1), \dots, c_{\text{top}}(x_n)$ and the elements in the ideal $(y_1, \dots, y_n)\mathbb{F}[V] \cap \mathbb{F}[V]^G$. Since the y_j's are top orbit Chern classes (and in particular invariant) we are done. □

4. The General Case $d > 2$

We start by proving a refinement of Lemma 2.2 for the general case.

Lemma 4.1. *Let* $f \in \mathbb{F}[V]^P$ *be an invariant with a term*

$$m = x_{11}^{\alpha_{11}} \cdots x_{nd}^{\alpha_{nd}}.$$

Then there exists a pair j_0, i_0 *such that* $\alpha_{j_0 i_0} \geq q^{d-i_0}$.

Proof. We proceed by induction on d.

Let $d = 2$. If $x_{j_0 2}$ divides m for some $j_0 = 1, \dots, n$ we are done. Otherwise,

$$m = x_{11}^{\alpha_{11}} \cdots x_{n1}^{\alpha_{n1}}$$

and our result follows from Lemma 2.2. Thus let $d > 2$.

If

$$m = x_{11}^{\alpha_{11}} x_{21}^{\alpha_{21}} \cdots x_{n1}^{\alpha_{n1}}$$

then we know by Lemma 2.2 that all the α_{j1}'s are divisible by q^{d-1} as desired.

So consider monomials

$$m = x_{11}^{\alpha_{11}} \cdots x_{nd}^{\alpha_{nd}}$$

such that there exists an exponent $\alpha_{j_1 i_1} \neq 0$ for $i_1 \in \{2, \dots, d\}$ and some j_1.

The group $\rho_n(P)$ contains subgroups P_r consisting of block diagonal matrices

$$\mathsf{block}(\underbrace{M, \dots, M}_{n \text{ times}})$$

with

$$M = \begin{bmatrix} 1 & a_{1,2} & & 0 & & \cdots & a_{1,d-1} \\ & 1 & a_{2,3} & \vdots & & \cdots & a_{2,d-1} \\ & & \ddots & 0 & & & \vdots \\ & & & 1 & 0 & \cdots & 0 \\ & & & & \ddots & & \\ & & & & & \ddots & a_{d-1,d} \\ & & & & & & 1 \end{bmatrix}$$

i.e., the rth column and the rth row are zero except at the r, r spot where there is a 1. We note that for all $r = 1, \ldots d$ the group P_r is isomorphic to the p-Sylow subgroup of $\mathsf{GL}(d-1, \mathbb{F})$. The inclusion of groups induces an embedding of the invariants of P into those of P_r.

Let us consider the group P_1. Then f as well as $x_{11}, x_{21}, \ldots, x_{n1}$ are invariant under P_1. Sorting by monomials in the x_{j1}'s we obtain

$$f = \sum_I f_I x_{11}^{\alpha_{11}} \cdots x_{n1}^{\alpha_{n1}}$$

where the sum runs over n-tuples $I = (\alpha_{11}, \ldots, \alpha_{n1})$. Note that the polynomials f_I are P_1-invariant. Thus by induction hypothesis we can assume that in each of the monomials appearing in a f_I there exists a $j_0 \in \{1, \ldots, n\}$ and an $i_0 \in \{2, \ldots d\}$ such that

$$\alpha_{j_0 i_0} \geq q^{d-i_0}$$

unless $f_I \in \mathbb{F}$.

□

We are ready to prove Theorem 1.1 in general.

Theorem 4.2. *The Hilbert ideal is generated by the top orbit Chern classes of the basis elements x_{ij}, $i = 1, \ldots, n$ and $j = 1, \ldots, d$.*

Proof. By construction

$$J = (c_{\text{top}}(x_{ji}), \forall i, j) \subseteq \mathfrak{H}(\rho_n(P)).$$

To show the reverse inclusion, let $F \in \mathfrak{H}(\rho_n(P))$. Then

$$F = \sum_{r=1}^{u} H_r f_r$$

for some nontrivial P-invariants f_r and some $H_r \in \mathbb{F}[V]$. We proceed by induction on term order. The smallest monomial in any degree δ is x_{nd}^{δ} which is invariant as well as in our proposed ideal J. Let

$$LT(F) = x_{11}^{\beta_{11}} \cdots x_{nd}^{\beta_{nd}} > x_{nd}^{\beta_{11}+\cdots+\beta_{nd}}.$$

Without loss of generality we can assume that the leading term of F appears in $H_1 f_1$:

$$x_{11}^{\beta_{11}} \cdots x_{nd}^{\beta_{nd}} = \gamma h_1 x_{11}^{\alpha_{11}} \cdots x_{nd}^{\alpha_{nd}}$$

for some $\gamma \in \mathbb{F}^{\times}$, and some terms $h_1 \in H_1$ and $x_{11}^{\alpha_{11}} \cdots x_{nd}^{\alpha_{nd}}$ in f_1. By Lemma 4.1 there exist $j_0 i_0$ such that

$$\beta_{j_0 i_0} \geq \alpha_{j_0 i_0} \geq q^{d-i_0}.$$

Thus

$$F - c_{\mathrm{top}}(x_{j_0 i_0}) x_{j_0 i_0}^{\beta_{j_0 i_0} - q^{d-i_0}} \prod_{ji \neq j_0 i_0} x_{ji}^{\beta_{ji}} < F.$$

Since the top orbit Chern classes are in the Hilbert ideal, we find by induction on term order that the LHS is in J. Furthermore, the top orbit Chern classes are in J, and therefore $F \in J$. □

Observe that this result shows the following:

- The maximal degree of a generator of the Hilbert ideal is q^{d-1} which is far less that the order of P.
- The Hilbert ideal does not characterize the group P as any group between $\rho_n(P)$ and $\rho(\times_n P)$ has the same orbit Chern classes of the basis elements and hence the same Hilbert ideal, where the representation

$$\rho : \times_n P \hookrightarrow \mathsf{GL}(dn, \mathbb{F})$$

is afforded by the matrices

$$\mathsf{block}(\underbrace{I, \dots, I, M, I, \dots, I}_{n})$$

where $I \in \mathsf{GL}(d, \mathbb{F})$ is the identity matrix, and $M \in \rho_1(P)$ appears in block j for $j = 1, \dots n$. We will show in [5] that this phenomenon (and indeed a more general statement) remains valid for large classes of groups and representations.

5. The Transfer Variety of P

Recall that the transfer is given by

$$\mathrm{Tr}^P : \mathbb{F}[V] \longrightarrow \mathbb{F}[V]^P, \; f \mapsto \sum_{g \in P} gf.$$

It is an $\mathbb{F}[V]^P$-module map and as such its image is an ideal in $\mathbb{F}[V]^P$. We denote by ∂_g the twisted differential given by

$$\partial_g = 1 - g : V^* \longrightarrow V^*,$$

for $g \in P$. We denote

$$I_g = (\mathrm{Im}(\partial_g)) \subseteq \mathbb{F}[V].$$

By work of M. Feshbach, see, e.g., Theorem 6.4.7 in [7], we know that

$$\mathrm{Rad}(\mathrm{Im}\mathrm{Tr}^P) = \bigcap_{g, |g| = p} (I_g \cap \mathbb{F}[V]^P) \subseteq \mathbb{F}[V]^P.$$

Furthermore, the height of the image of the transfer is

$$\mathrm{height}(\mathrm{Im}\mathrm{Tr}^P) = \dim_{\mathbb{F}}(V) - \max\{\dim_{\mathbb{F}} V^g \mid |g| = p\}.$$

Apparently, an element $g \in \rho_n(P)$ of order p whose fixed point set has maximal dimension is given by

$$g_0 = \mathsf{block}(\underbrace{M, \dots, M}_{n \text{ times}}),$$

where M is an identity matrix with an additional 1 in the $1, d$ spot. Thus the height of the image of the transfer is $dn - (d-1)n = n$.

Furthermore, note that

$$\operatorname{Im}(\partial_{g_0}) = \operatorname{span}_{\mathbb{F}}\{x_{1d}, \ldots, x_{nd}\}$$

Thus

$$I_{g_0} = (x_{1d}, \ldots, x_{nd}) \subseteq \mathbb{F}[V]$$

is a prime ideal of height n. By the Krull relations it follows that $I_{g_0} \cap \mathbb{F}[V]^P$ is a minimal isolated prime ideal of ImTr^P.

More generally we claim the following.

Proposition 5.1. *The radical of the image of the transfer of P is given by*

$$\operatorname{Rad}(\operatorname{ImTr}^P) = \bigcap_{\mathbf{a}} (l_{\mathbf{a},1}, \ldots, l_{\mathbf{a},n}) \cap \mathbb{F}[V]^P$$

where $\mathbf{a} = (a_2, \ldots, a_d) \in \mathbb{F}^{d-1} \setminus \{\mathbf{0}\}$ and $l_{\mathbf{a},j} = a_2 x_{j2} + \cdots + a_n x_{jn}$.

Proof. We note that any element $g_{\mathbf{a}} = \mathsf{block}(\underbrace{M, \ldots, M}_{n \text{ times}})$ where $\mathbf{a} = (a_2, \ldots, a_d) \in \mathbb{F}^{d-1} \setminus \{\mathbf{0}\}$ and

$$M = \begin{bmatrix} 1 & a_2 & a_3 & \cdots & a_d \\ & \ddots & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ & & & & 1 \end{bmatrix}$$

has order p. The ideal $I_{g_{\mathbf{a}}}$ associated to this element is one of the ideals mentioned in the statement:

$$I_{g_{\mathbf{a}}} = (l_{\mathbf{a},1}, \ldots, l_{\mathbf{a},n}).$$

Finally, let $g = \mathsf{block}(\underbrace{M, \ldots, M}_{n \text{ times}})$ be an arbitrary element of order p and set

$$M = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1d} \\ & \ddots & \ddots & \vdots \\ & & \ddots & a_{d-1,d} \\ & & & 1 \end{bmatrix}$$

Then I_g is the ideal in $\mathbb{F}[V]$ generated by the linear forms

$$a_{12}x_{j2} + \cdots + a_{1d}x_{jd}, \ldots, a_{d-1,d}x_{jd} \quad \forall j = 1, \ldots, n.$$

However, $I_g \supset I_{g_{\mathbf{a}}}$ for $\mathbf{a} = (a_{12}, \ldots, a_{1d})$. □

Observe that for the case $d = 2$ we obtain

$$\operatorname{Rad}(\operatorname{ImTr}^P) = (x_{12}, \ldots, x_{n2}) \cap \mathbb{F}[V]^P$$

as the ideals I_g are equal for all $g \in P$ of order p, and hence the radical of the image of the transfer is prime of height n.

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