Lec 1

My goal in these notes is to sketch the role of differential equations in modeling a single neuron. and synapse across two neurons.

1.1 First order equation

Consider an equation of the form

$$\dot{x} = ax, \ x(0) = x_0. \qquad (1\cdot1)$$

It is not too hard to see that its sol$^{\underline{n}}$ is given by

$$x(t) = x_0 e^{at} \qquad (1\cdot2)$$

We now modify (1·1) as

$$\dot{x} = ax + b(t) \qquad (1\cdot3)$$
$$x(0) = x_0$$

This time it is not so obvious but we can write

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)} b(\tau)\, d\tau. \qquad (1\cdot4)$$

If we choose $b(t)$ to be a step f^n

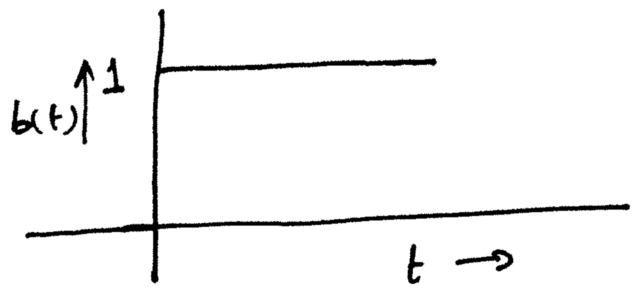

Fig 1·1: $b(t)$ chosen as a step f^n.

Assuming $x_0 = 0$ and $a < 0$ we have

$$x(t) = -\frac{1}{a}\left(1 - e^{at}\right), \qquad (1\cdot5)$$

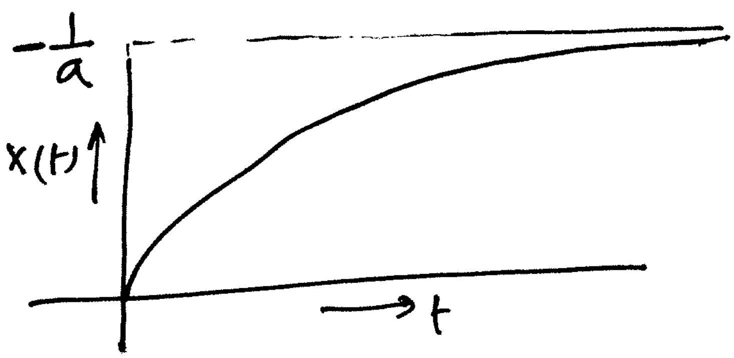

Fig 1·2: A sketch of $x(t)$

A sketch of $x(t)$ is given in Fig 1·2.

<u>Exercise (H.W. 1)</u>:– (1·1)

Solve the following equation

$$\dot{x} = \frac{1}{13}(-x + 5)\,;\; x(0) = 17 \qquad (1\cdot6)$$

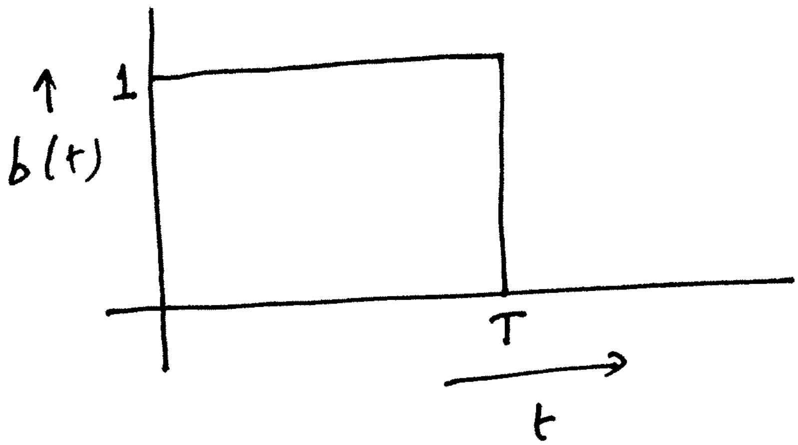

Fig 1.3: $b(t)$ is as in Fig 1.1 for $0 \leq t < T$ and goes back to 0 for $t \geq T$.

If $b(t)$ is as shown in Fig 1.3, we have

$$x(t) = -\frac{1}{a}\left(1 - e^{at}\right) \quad (1.7)$$

$$0 \leq t < T.$$

At $t = T$, $x(T) = -\frac{1}{a}\left(1 - e^{aT}\right) \triangleq x_T$.

For $t \geq T$, $x(t)$ is given by

$$x(t) = e^{a(t-T)} x_T$$

$$= e^{a(t-T)}\left(-\frac{1}{a}\right)\left(1 - e^{aT}\right)$$

$$= -\frac{1}{a} e^{a(t-T)} + \frac{1}{a} e^{at}$$

$$= \frac{1}{a} e^{at}\left[1 - e^{-aT}\right] \quad (1.8).$$

Writing (1·7), (1·8) together we have

$$x(t) = -\frac{1}{a}\left(1 - e^{at}\right), \quad 0 \leq t < T \qquad (1\cdot 9)$$
$$= \frac{1}{a} e^{at}\left(1 - e^{-aT}\right), \quad t \geq T.$$

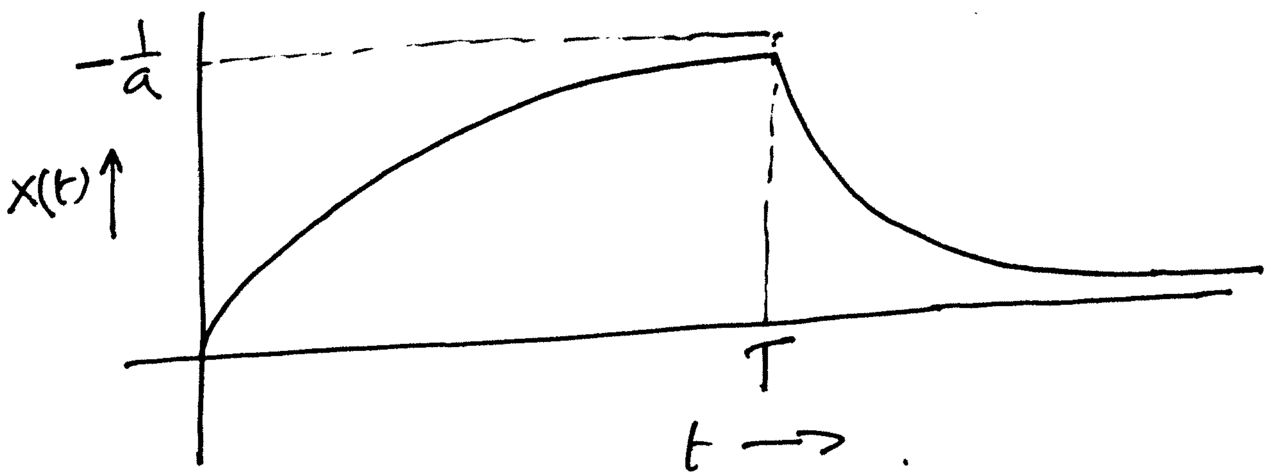

Fig 1·4: $x(t)$ rises exponentially to $-\frac{1}{a}$ before falling back to 0.

— x —

Exercise (H.W.2) : (1·2)

Solve the following equation

$$\dot{x} = \frac{1}{13}\left(-x + b(t)\right), \quad x(0) = 17 \qquad (1\cdot 10)$$

b(t) is given as follows

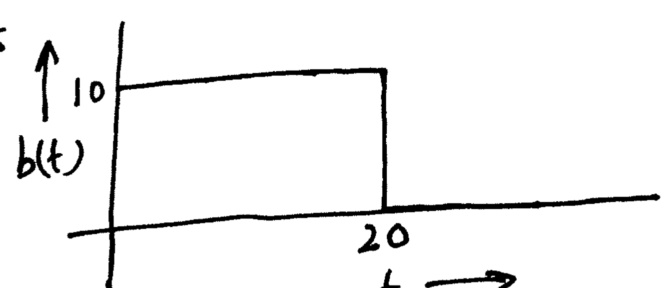

Fig 1·5:

Exercise 1.3 (H.W.1)

Redo Exercise 1.2 assuming

$$b(t) = (1 - e^{\alpha t})\,10 \quad \text{and} \quad x(0) = 0$$

where $\alpha < 0$. Sketch $x(t)$ for $\alpha = -5, -13$ and -30.

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1.2 Responses of a simple model neuron

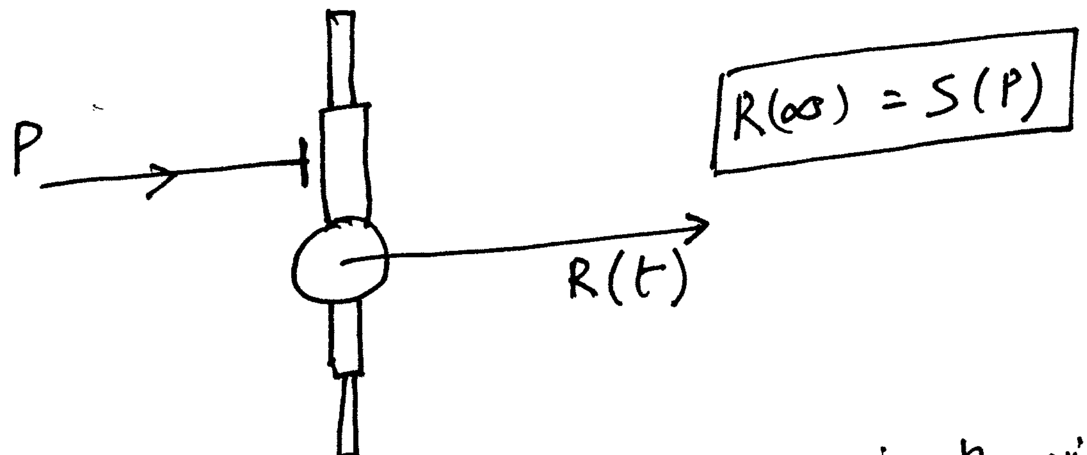

Fig 1.5: A simple sketch of a neuron in the visual system. P is the stimulus intensity and $R(t)$ is the firing rate.

One of the basic property of any neuron is that when it is subjected to a stimulus of intensity P, sufficiently strong, it fires repeatedly as long as the stimulus is present.

Recordings from several different levels of the visual system have confirmed this fact: All neurons could be described by a single equation known as the Naka-Rushton equation, which is described as follows —

$$S(P) = \begin{cases} \dfrac{M P^N}{\sigma^N + P^N} & \text{for } P \geq 0 \\ 0 & \text{for } P < 0 \end{cases} \qquad (1.11)$$

M: maximum spike rate for very intense stimuli.

σ: point at which S(P) reaches half of its maximum.
σ: semisaturation constant.

N: Determines the maximum slope of the function or how sharp the transition is between threshold and saturation.

Typical values of N reported by Sclar:

Lateral Geniculate $N = 1.4$
Visual cortical neurons $N = 2.4$
Middle temporal cortex $N = 3$

Exercise 1·4 (H.W. 1)

(a) consider

$$S(P) = \frac{100\, P^N}{50^N + P^N}, \quad P \geq 0$$

For $N = 1, 2, 4$ plot $S(P)$ as a f^{n} of P.

(b) Consider

$$S(P) = \frac{50\, P^4}{15^4 + P^4}, \quad P \geq 0$$

plot $S(P)$ as a function of P.

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We now write down a differential equation describing the response $R(t)$ of a single neuron to an arbitrary stimulus P.

$$\dot{R}(t) = \frac{1}{\tau}\left(-R(t) + S(P)\right) \qquad (1\cdot 12)$$

This equation is of the form (1·3) and we know how to solve it. In fact, for constant P we obtain

$$R(t) = \left(1 - e^{-t/\tau}\right) S(P) \qquad (1\cdot 13)$$

τ is a positive constant and it is easy to see that $R(\infty) = \frac{1}{\tau} S(P)$. Thus $S(P)$ is the equilibrium rate of the neuron.

Exercise 1·5 (A.W.1):

Solve the following equation for the response rate $R(t)$ of a neuron for each of the following values of the post synaptic potential, $P = 10, 20, 30$. Plot your results on a single graph for times up to 100 ms assuming $R(0) = 5$ in each case

$$\dot{R} = \frac{1}{20}\left(-R + \frac{50\,P^4}{15^4 + P^4}\right) \qquad (1.14).$$

Exercise 1·6 (H.W.1)

4 In this problem we will explore a piecewise linear approximation to action potential generation. There are two ionic channels one for Na^+ ($E_{Na} = 50\,mV$) and one for K^+ ($E_K = -90\,mV$), resulting in the equation

$$\dot{V} = -4\left[g_{Na}(V-50) + g_K(V+90)\right] + I_{input}, \quad (1 \cdot 15)$$

where $g_k = \cdot 6$ and $g_{Na} = \cdot 1$ as the resting conductance.

a) Find the resting potential or equilibrium and use this as the initial condition $V(0)$.

b) For $0 \leq t < 1$ ms, choose $g_{Na} = 5$, $g_k = \cdot 6$.

c) For $1 \leq t < 4$ ms, choose $g_k = 2$, $g_{Na} = 0$

d) For $t \geq 4$, choose $g_k = \cdot 6$, $g_{Na} = \cdot 1$

Solve for $V(t)$ and plot. Choose I_{input} in the range $0 < I_{input} < 10$

1·3 Excitatory & Inhibitory post synaptic potentials

Ion channels are governed by Ohm's Law, which states that

$$I = g(V - E) \quad (1{\cdot}16)$$

where I is the ionic current across the nerve membrane, g is the conductance measured usually in nano siemens, V is the voltage difference across the membrane in millivolts.

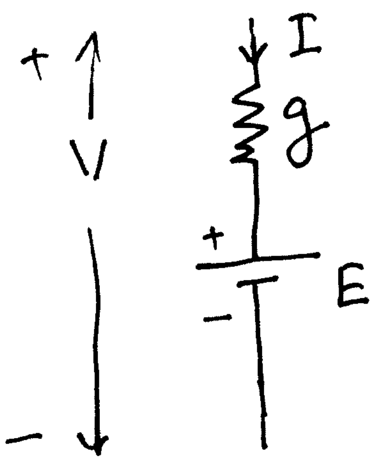

Fig 1·6: Ohm's Law describing (1·16).

For any ionic species, the equilibrium potential E is determined by the Nernst Eqn

$$E = \frac{RT}{zF} \ln\left(\frac{C_{out}}{C_{in}}\right) \quad (1{\cdot}17)$$

where

z is the charge on the ion in question

C_{out} & C_{in} are the respective concentrations of the ion outside and inside the cell

R & F are respectively the thermodynamic gas constant and the Faraday's constant.

T is the temperature in degree Kelvin.

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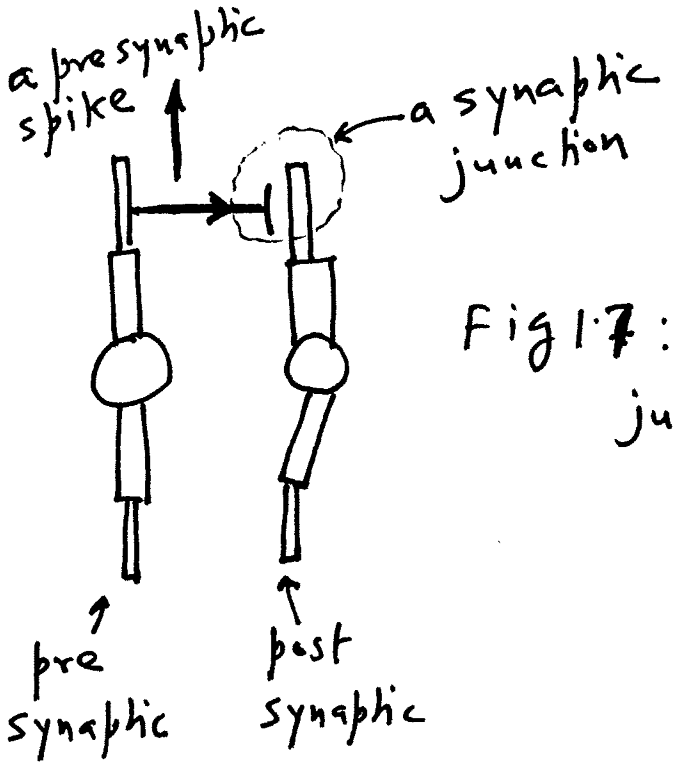


Fig 1.7: Figure shows a synaptic junction between two neurons.

To understand the effect of a synapse, in Fig 1·7 two neurons are shown. The presynaptic neuron has produced a single spike. If V is the membrane potential of the post synaptic membrane it satisfies an equation of the form

$$\dot{V}(t) = -\frac{1}{\tau}\Big\{ g_\ell (V - E_\ell) + g_e (V - E_e) + g_i (V - E_i) \Big\} \qquad (1{\cdot}18)$$

g_ℓ and E_ℓ are the conductance and equilibrium potential for the leakage current.

We assume $g_\ell = 1\,nS$, $E_\ell = -75\,mV$

$\tau = 12{\cdot}5\,ms$ for human neurons.

g_e, E_e : refer to Excitatory Postsynaptic Potential EPSP ion channels

g_i, E_i : refer to Inhibitory Postsynaptic Potental IPSP ion channels.

We choose $E_e = 0$, $E_i = -75\,mV$, so that (1·18) takes the form

$$\dot{V}(t) = -\frac{1}{12\cdot 5}\left\{(V+75) + g_e V + g_i (V+75)\right\} \qquad (1\cdot 19)$$

Effect of a spike across a EPSP junction

In this case $g_i = 0$ and let T be the time of spike

$$\cancel{\begin{aligned} g_e &= 0 & t &< \text{time of spike} \\ &= 2 & t &> \text{time of spike} \\ &= 0 & t & \end{aligned}}$$

$$\begin{aligned} g_e &= 0, & t &< T \\ &= 2, & T \le t &< T+1 \\ &= 0 & t &\geq T+1 \end{aligned} \qquad (1\cdot 20)$$

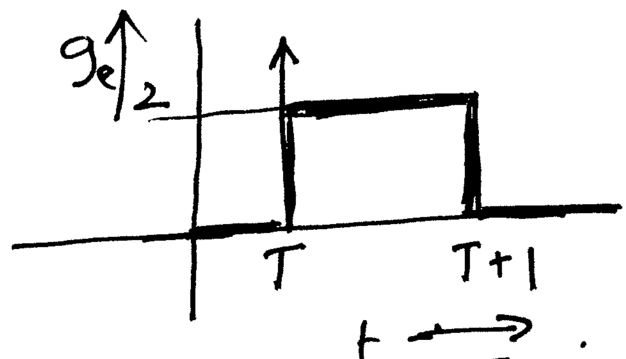


Fig 1·8

For mathematical simplicity assume $T = 0$. We would like to solve (1·19) assuming $g_i = 0$ and g_e satisfies (1·20).

If we assume that $V(t)$ is in equilibrium, it follows that $V(0) = -75$. In the interval $0 \leq t < 1$, we have

$$\dot{V} = -\frac{1}{12.5}\left\{V + 75 + 2V\right\}$$

$$= -\frac{3}{12.5}\left\{V + 25\right\} \qquad (1.21)$$

$$V(t) = -50e^{-.24t} - 25 \qquad (1.22)$$

for $0 \leq t < 1$.

It follows from (1.22) that $V(1) = -64.3\,mV$.

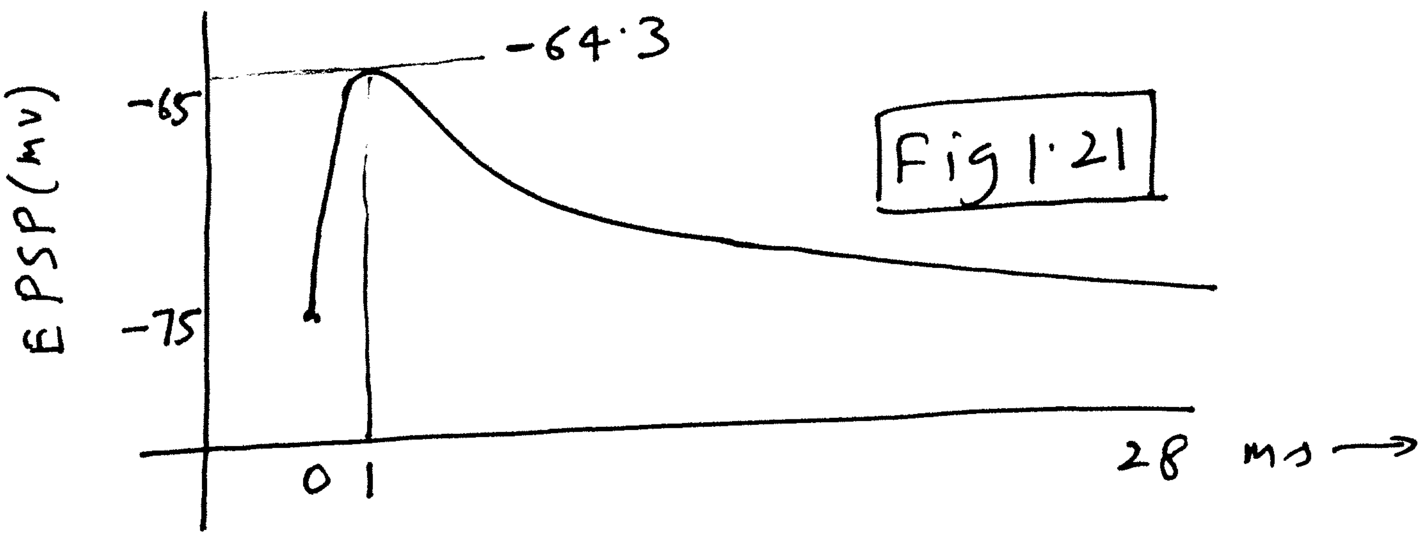
For $t \geq 1$ we have

$$\dot{V} = -\frac{1}{12.5}\left\{V + 75\right\}, \quad V(1) = -64.3 \qquad (1.23)$$

Solving (1.23) we obtain

$$V(t) = 10.7e^{-.08(t-1)} - 75. \qquad (1.24)$$

for $t \geq 1$. The solutions (1.22) and (1.24) are plotted in Fig 1.9 as shown.



Effect of a spike across an IPSP junction

(in addition to a spike across an EPSP junction)

When a spike occurs at $t = T$ across an IPSP junction, g_i changes as follows

$$g_i = 0, \quad t < T$$
$$= 12, \quad T \leq t < T+1 \quad (1{\cdot}25).$$
$$= 0, \quad t \geq T+1$$

It follows from (1·19) that when we have one spike at an EPSP jn and one spike at IPSP junction; both occurring precisely at

$t=0$, we have

$$\dot{V} = -\frac{1}{12.5}\left\{(V+75)+2V+12(V+75)\right\} \qquad (1.26)$$

$$V(0) = -75$$

for $0 \leq t \leq 1$. Solving (1.26) we obtain

$$V(t) = -10e^{-1.2t} - 65. \qquad (1.27)$$

We obtain

$$V(1) = -68 \text{ mV} \qquad (1.28)$$

$$0 \leq t \leq 1$$

$$= 7e^{-.08(t-1)} - 75, \quad t \geq 1$$

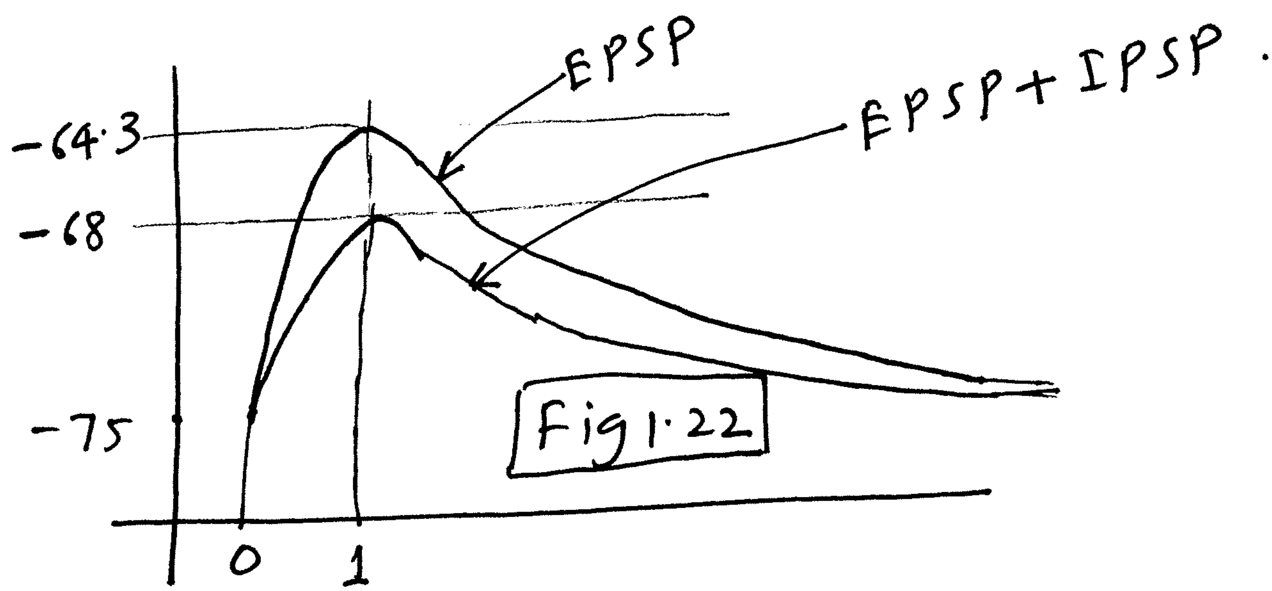


Fig 1.22

with a spike at the IPSP junction, the peak has dropped from -64.3 to -68 mV.

Exercise 1.7 (H.W.1) :-

In this exercise we study the effect of N simultaneous excitatory synaptic spikes all occur at the same time, $t=0$.

Solve

$$\dot{V} = -\frac{1}{12.5}\left\{(V+75)+2NV\right\} \quad 0 \le t < 1. \tag{1.29}$$

$$V(0) = -75.$$

Calculate $V(t)$ analytically and write down $V(1)$. Plot the peak EPSP value $V(1)$ for $1 \le N \le 12$. Do you see the EPSPs adding linearly or are there any saturation effects.

Exercise 1·8 (H.W.1)

In this exercise we study the effect of N simultaneous inhibitory synaptic spikes and 1 excitatory spike at time $t=0$.

Solve

$$\dot{V} = -\frac{1}{12.5}\left\{(V+75)+2V+12N(V+75)\right\} \qquad 0 \leq t \leq 1 \quad (1\cdot30)$$

$$V(0) = -75\,mV.$$

Calculate $V(t)$ analytically and write down $V(1)$. Plot the peak EPSP value $V(1)$ for $1 \leq N \leq 8$ and comment if it is linear or non-linear.

Exercise 1·9 (H.W.1)

An excitatory synapse at $t=0$ is followed by an inhibitory synapse at $t=1$. Starting with $V(0) = -75\,mV$, solve (1·19) for $0 \le t \le 1$, $1 \le t \le 2$ and $t \geq 2$. Finally plot $V(t)$ as a function of t.

(a) Does the inhibitory synapse affect the peak EPSP value $V(1)$?

(b) How does the inhibitory synapse affect $V(t)$?