Sol^n to suggested problems.

1 Ans

(a) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

trace $A = 0 \Rightarrow a + d = 0 \Rightarrow a = -d$

$\therefore V_1 \triangleq \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} ; a, b, c \in \mathbb{R} \right\}$

verify that V_1 is closed under addition, scaler multiplication and satisfies all the usual properties of vector space.

write

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

V_1 is spanned by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Show that the above three matrices are l.i. as vectors in V_1. Hence they form a basis.

(b) $V_2 \triangleq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad = bc \right\}$.

Not closed under addition.

Let
$$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} : a_1 d_1 = b_1 c_1.$$

$$M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} : a_2 d_2 = b_2 c_2$$

$$M_1 + M_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

$\det(M_1 + M_2) =$

$$a_1 d_1 + a_2 d_2 + a_1 d_2 + a_2 d_1.$$

$$- b_1 c_1 - b_2 c_2 - b_1 c_2 - b_2 c_1.$$

$$= (a_1 d_2 + a_2 d_1) - (b_1 c_2 + b_2 c_1)$$

$$\neq 0 \text{ in general}.$$

Hence V_2 is not a vector space.

2 Ans: A & A^{-1} commutes i.e.

$$A \cdot A^{-1} = A^{-1} A$$

Hence $e^{(A + A^{-1})t} = e^{At} e^{A^{-1}t}$.

Likewise

A & A^3 commutes as well.

3 Ans:

a L is a linear transformation if

$$L(\alpha A + \beta B) = \alpha L(A) + \beta L(B)$$

for any $A, B \in \mathbb{R}^{2\times 2}$

& $\alpha, \beta \in \mathbb{R}$.

To verify ~~th~~is we compute

$$L(\alpha A + \beta B) = \frac{1}{2}\left[(\alpha A + \beta B) - (\alpha A^T + \beta B^T)\right]$$

$$= \frac{1}{2}\alpha\left[A - A^T\right] + \frac{1}{2}\beta\left[B - B^T\right]$$

$$= \alpha L(A) + \beta L(B).$$

Hence L is a linear transformation.

b) $\mathcal{N}(L) \triangleq \{A : L(A) = 0\}$.

$L(A) = 0$

$\Rightarrow A = A^T$.

ie A is a symmetric matrix.

$$\therefore \mathcal{N}(L) = \left\{ \begin{pmatrix} a & c \\ c & b \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

$\longrightarrow$ x $\longrightarrow$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$L(A) = \frac{1}{2}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix}$$ $\leftarrow$ a skew symmetric matrix.

$$\mathcal{R}(L) = \left\{ \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} : \omega \in \mathbb{R} \right\}.$$

(c)

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

A basis of $\mathcal{N}(L)$ is given by

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

$\dim \mathcal{N}(L) = 3 =$ nullity.

Likewise

$$\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = \omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

A bases of $R(L)$ is given by

$$\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

$\dim R(L) = 1 =$ rank.

d) Choose the following basis of $\mathbb{R}^{2\times 2}$

$$\left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{e_2}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{e_3}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{e_4} \right\}.$$

$$L(e_1) = 0$$

$$L(e_2) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{2} e_2 - \frac{1}{2} e_3 .$$

$$L(e_3) = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2} e_3 - \frac{1}{2} e_2 .$$

$$L(e_4) = 0 .$$

Let L be described by a matrix A

8

we have

$$A\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$A\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\1/2\\-1/2\\0\end{pmatrix}$$

$$A\begin{pmatrix}0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\-1/2\\1/2\\0\end{pmatrix}$$

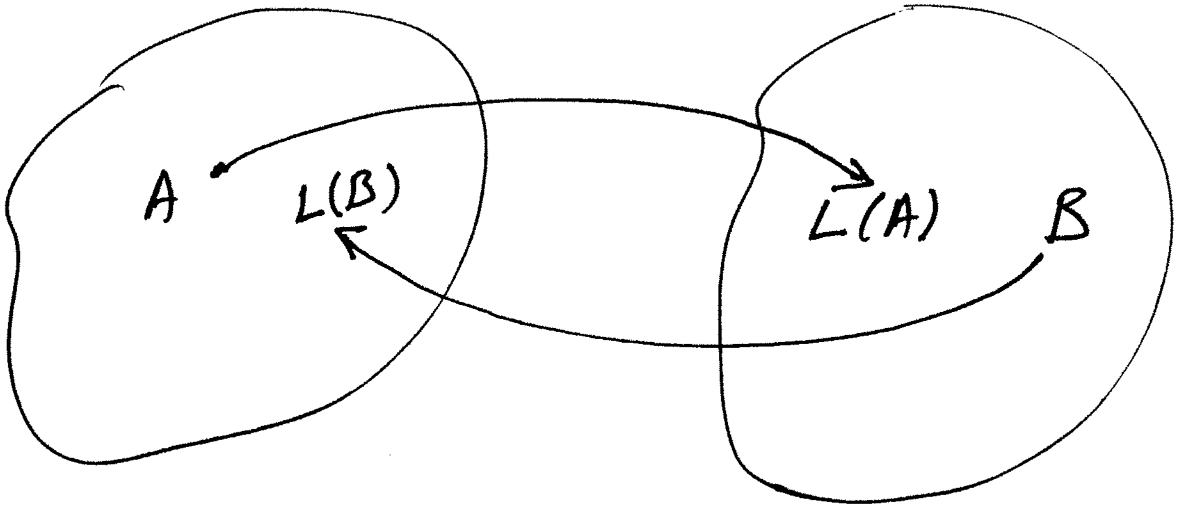
$$A\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Hence

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(e) Adj L has the matrix representation A^T.

However since $A^T = A$ it would follow that adjoint of L is L itself.



Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$

$L(A) = \frac{1}{2}\begin{pmatrix} 0 & a_2 - a_3 \\ a_3 - a_2 & 0 \end{pmatrix}$.

$B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$

$L(B) = \frac{1}{2}\begin{pmatrix} 0 & b_2 - b_3 \\ b_3 - b_2 & 0 \end{pmatrix}$

$$\langle A, L(B) \rangle = \frac{1}{2}\left[a_2(b_2 - b_3) + a_3(b_3 - b_2)\right]$$
$$= \frac{1}{2}\left[(a_2 - a_3)(b_2 - b_3)\right]$$

$$\langle L(A), B \rangle = \frac{1}{2}\left[b_2(a_2 - a_3) + b_3(a_3 - a_2)\right]$$
$$= \frac{1}{2}\left[(b_2 - b_3)(a_2 - a_3)\right].$$

(f)

$$\mathcal{N}(\operatorname{adj} L) = \left\{ \begin{pmatrix} a & c \\ c & b \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

$$\mathcal{R}(L) = \left\{ \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} : \omega \in \mathbb{R} \right\}.$$

$$\left\langle \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \right\rangle$$

$$= 0 + c\omega - c\omega + 0 = 0$$

$$\forall\, a, b, c, \omega \in \mathbb{R}.$$

4 Ans:

$$B(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \\ 0 & t \end{pmatrix}$$

$$BB^T = \begin{pmatrix} 1 & 0 \\ t & 1 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & t \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t & 0 \\ t & t^2+1 & t \\ 0 & t & t^2 \end{pmatrix}$$

$$\int_0^t BB^T(\sigma)\,d\sigma = \begin{pmatrix} t & t^2/2 & 0 \\ t^2/2 & t^3/3+t & t^2/2 \\ 0 & t^2/2 & t^3/3 \end{pmatrix}$$

$$= W(0, t).$$

$$\det\left[W(0,t)\right]$$

$$= t\left[t^6/9 + t^4/3 - t^4/4\right]$$

$$- t^2/2\left[t^5/6\right]$$

$$= t^7/9 + t^5\left[\frac{1}{12}\right] - t^7/12.$$

$$= \frac{1}{36}t^7 + \frac{1}{12}t^5 \cancel{\neq 0}$$

$$= \frac{1}{12}t^5\left[1 + \frac{1}{3}t^2\right] \neq 0$$

$$\forall\, t > 0.$$

For $t=1$

$$\det\left[W(0,1)\right] = \frac{1}{12}\left(1 + \frac{1}{3}\right).$$

$$= \frac{\cancel{4}}{3}\frac{1}{\cancel{12}_3} = \frac{1}{9}.$$

Hence

Range $[W(0,1)]$ is $\mathbb{R}^3$.

5 Ans:

Yes, we already did this in H.W. 2.

6 $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

char polynomial

$$p(\lambda) = \det \begin{pmatrix} \lambda-1 & -2 \\ 0 & \lambda-1 \end{pmatrix}$$

$$= (\lambda-1)^2$$

$\lambda = 1, 1 \leftarrow$ repeated eigenvalues at 1.

writing

$$f(A) = \alpha_0 I + \alpha_1 A.$$

$$f(\lambda) = \alpha_0 + \alpha_1 \lambda.$$

$$\frac{df}{d\lambda} = \alpha_1.$$

substituting $\lambda = 1$ we have

$$\alpha_0 + \alpha_1 = f(1).$$

$$\alpha_1 = f'(1)$$

For $f(A) = A^{99}$.

$$\alpha_0 + \alpha_1 = 1^{99} = 1.$$

$$\alpha_1 = 99\lambda^{98}\Big|_{\lambda=1} = 99.$$

$$\Rightarrow \alpha_1 = 99, \ \alpha_0 = -98$$

$$\therefore A^{99} = \begin{pmatrix} -98 & 0 \\ 0 & -98 \end{pmatrix} + \begin{pmatrix} 99 & 198 \\ 0 & 99 \end{pmatrix} = \begin{pmatrix} 1 & 198 \\ 0 & 1 \end{pmatrix}$$

For $f(A) = e^{At}$

we have

$$\alpha_0 + \alpha_1 = e^t.$$

$$\alpha_1 = te^t.$$

$$\alpha_0 = -te^t + e^t = e^t(1-t).$$

$$e^{At} = \begin{pmatrix} e^t(1-t) & 0 \\ 0 & e^t(1-t) \end{pmatrix}$$

$$+ \begin{pmatrix} te^t & 2te^t \\ 0 & te^t \end{pmatrix}$$

$$= \begin{pmatrix} e^t & 2te^t \\ 0 & e^t. \end{pmatrix}$$

7 Ans

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$p(\lambda) = \det \begin{pmatrix} \lambda-1 & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda-1 \end{pmatrix}$$

$$= (\lambda-1)^2 \lambda$$

Eigenvalues at $\lambda = 0, \lambda = 1, \lambda = 1$

$$f(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

$$f(\lambda) = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2$$

$$f'(\lambda) = 0 + \alpha_1 + 2\alpha_2 \lambda$$

$$f(1) = \alpha_0 + \alpha_1 + \alpha_2$$

$$f'(1) = \alpha_1 + 2\alpha_2$$

$$f(0) = \alpha_0$$

For $f(A) = A^{103}$

$$f(\lambda) = \lambda^{103}$$

$$f(1) = 1$$

$$f'(1) = 103$$

$$f(0) = 0$$

Hence $\alpha_0 = 0$

$$\left.\begin{aligned} \alpha_1 + \alpha_2 &= 1 \\ \alpha_1 + 2\alpha_2 &= 103 \end{aligned}\right\} \alpha_2 = 102$$

$$\alpha_1 = -101$$

$$\therefore A^{103} = \alpha_1 A + \alpha_2 A^2$$

where $\alpha_1 = -101$, $\alpha_2 = 102$.

$$\alpha_1 A = \begin{pmatrix} -101 & -101 & 0 \\ 0 & 0 & -101 \\ 0 & 0 & -101 \end{pmatrix}$$

$$A^2 =$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\alpha_2 A^2 = \begin{pmatrix} 102 & 102 & 102 \\ 0 & 0 & 102 \\ 0 & 0 & 102 \end{pmatrix}$$

Hence

$$A^{103} = \begin{pmatrix} 1 & 1 & 102 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

For $f(A) = e^{At}$

$$f(\lambda) = e^{\lambda t}$$

$$f'(\lambda) = te^{\lambda t}.$$

$$f(1) = e^{t}$$

$$f'(1) = te^{t}$$

$$f(0) = 1$$

$$\alpha_0 = f(0) = 1$$

$$\alpha_0 + \alpha_1 + \alpha_2 = f(1) = e^{t}$$

$$\Rightarrow \alpha_1 + \alpha_2 = e^{t} - 1$$

$$\alpha_1 + 2\alpha_2 = te^{t}$$

$$\Rightarrow$$

$$\alpha_2 = te^{t} - e^{t} + 1$$

$$\alpha_1 = -te^{t} + e^{t} - 1 + e^{t} - 1$$

$$= -te^{t} + 2e^{t} - 2$$

$$e^{At} =$$

$$\alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} +$$

$$\begin{pmatrix} \alpha_1 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1 \\ 0 & 0 & \alpha_1 \end{pmatrix} + \begin{pmatrix} \alpha_2 & \alpha_2 & \alpha_2 \\ 0 & 0 & \alpha_2 \\ 0 & 0 & \alpha_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+\alpha_1+\alpha_2 & \alpha_1+\alpha_2 & \alpha_2 \\ 0 & 1 & \alpha_1+\alpha_2 \\ 0 & 0 & 1+\alpha_1+\alpha_2 \end{pmatrix}$$

$$e^{At} =$$

$$\begin{pmatrix} e^t & e^t - 1 & te^t - e^t + 1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{pmatrix}$$

8 Ans:

$$\underline{x}(t) = e^{At}\underline{x}(0) +$$

$$\int_0^t e^{A(t-\tau)} b\, u(\tau)\, d\tau.$$

$$e^{At}\underline{x}(0) = \begin{pmatrix} e^t - 1 \\ 1 \\ 0 \end{pmatrix}$$

$$e^{A(t-\tau)} b\, u(\tau) =$$

$$e^{At} e^{-A\tau} b\, u(\tau)$$

— x —

$$e^{-A\tau} b =$$

$$\begin{pmatrix} -\tau e^{-\tau} - e^{-\tau} + 1 \\ e^{-\tau} - 1 \\ e^{-\tau} \end{pmatrix}$$

$$\int_0^t e^{-A\tau} b\, u(\tau)\, d\tau =$$

$$\begin{bmatrix} \int_0^t -\tau e^{-\tau} d\tau - \int_0^t e^{-\tau} d\tau + \int_0^t d\tau \\ \int_0^t e^{-\tau} d\tau - \int_0^t d\tau . \\ \int_0^t e^{-\tau} d\tau . \end{bmatrix}$$

$$\int_0^t d\tau = t.$$

$$\int_0^t e^{-\tau} d\tau = \left.\frac{e^{-\tau}}{-1}\right|_0^t.$$

$$= -e^{-t} + 1 = 1 - e^{-t}.$$

$$\int_0^t \tau e^{-\tau} d\tau =$$

$$\left.\tau \frac{e^{-\tau}}{-1}\right|_0^t + \int_0^t e^{-\tau} d\tau.$$

$$= -te^{-t} + 1 - e^{-t}.$$

$$\underline{\qquad\qquad} \times \underline{\qquad\qquad}$$

$$\int_0^t e^{-A\tau} b\,u(\tau)\,d\tau =$$

$$\begin{pmatrix} t e^{-t} - 1 + e^{-t} - 1 + e^{-t} + t \\ 1 - e^{-t} - t \\ 1 - e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} t e^{-t} + 2 e^{-t} + t - 2 \\ 1 - e^{-t} - t \\ 1 - e^{-t} \end{pmatrix}$$

$$\int_0^t e^{A(t-\tau)} b u(\tau)\, d\tau =$$

$$\begin{pmatrix} e^t & e^t-1 & te^t-e^t+1 \\ 0 & 1 & e^t-1 \\ 0 & 0 & e^t \end{pmatrix}\begin{pmatrix} te^{-t}+2e^{-t}+t-2 \\ 1-e^{-t}-t \\ 1-e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} t+2+te^t-2e^t+\cancel{e^t}\cancel{-1}-\cancel{te^t}\cancel{-1}+\cancel{e^{-t}}+\cancel{t} \\ \quad +\cancel{te^t}\cancel{-e^t}\cancel{+1}-\cancel{t}+\cancel{1}-\cancel{e^{-t}} \\ \hline \cancel{1}-\cancel{e^{-t}}-t+e^t\cancel{-1}-1+\cancel{e^{-t}} \\ \hline e^t-1 \end{pmatrix}$$

$$= \begin{pmatrix} t(1+e^t)+2(1-e^t) \\ e^t-t-1 \\ e^t-1 \end{pmatrix}$$

Finally

$$\mathbb{X}(t) = \begin{pmatrix} e^t - 1 + t + te^t + 2 - 2e^t \\ e^t - t \\ e^t - 1 \end{pmatrix}$$

$$= \begin{pmatrix} te^t + t + 1 - e^t \\ e^t - t \\ e^t - 1 \end{pmatrix}$$

$$\overline{W}(0,1) = \int_0^1 \begin{pmatrix} -\tau e^{-\tau} - e^{-\tau} + 1 \\ e^{-\tau} - 1 \\ e^{-\tau} \end{pmatrix} \left(-\tau e^{-\tau} - e^{-\tau} + 1 \;\vdots\; e^{-\tau} - 1 \;\vdots\; e^{-\tau} \right) d\tau .$$

$$= \int_0^1 \left(\begin{array}{c|c|c} (-\tau e^{-\tau} - e^{-\tau} + 1)^2 & (-\tau e^{-\tau} - e^{-\tau} + 1)(e^{-\tau} - 1) & (-\tau e^{-\tau} - e^{-\tau} + 1) e^{-\tau} \\ \hline (-\tau e^{-\tau} - e^{-\tau} + 1)(e^{-\tau} - 1) & (e^{-\tau} - 1)^2 & (e^{-\tau} - 1) e^{-\tau} \\ \hline (-\tau e^{-\tau} - e^{-\tau} + 1) e^{-\tau} & (e^{-\tau} - 1) e^{-\tau} & e^{-2\tau} \end{array} \right) d\tau$$

9 **Ans:**

The proof is same as the scalar case ie when $A(t)$ is a scalar.

For $A(t) = \begin{pmatrix} t & t^2 \\ t^2 & t \end{pmatrix}$

$$\int_{t_0}^{t} A(\sigma)\, d\sigma = \begin{pmatrix} \frac{\cancel{\sigma}^2}{2} & \frac{\cancel{\sigma}^3}{3} \\ \frac{\cancel{\sigma}^3}{3} & \frac{\cancel{\sigma}^2}{2} \end{pmatrix}\Bigg|_{t_0}^{t}$$

$$= \begin{pmatrix} \frac{t^2 - t_0^2}{2} & \frac{t^3 - t_0^3}{3} \\ \frac{t^3 - t_0^3}{3} & \frac{t^2 - t_0^2}{2} \end{pmatrix} = M(t$$

$$\phi(t, t_0) = I + M + \frac{M^2}{2!} + \cdots$$

$$\det \phi(t,t_0) = e^{\int_{t_0}^{t} 2\sigma \, d\sigma}$$

$$= e^{\sigma^2 \big|_{t_0}^{t}} = e^{(t^2 - t_0^2)}$$

10 $$\phi(t,0) = e^{\int_0^t A(\sigma)\, d\sigma}$$

$$= e^{\begin{pmatrix} t^2/2 & t^3/3 \\ t^3/3 & t^2/2 \end{pmatrix}}$$

Let $A = \begin{pmatrix} t^2/2 & t^3/3 \\ t^3/3 & t^2/2 \end{pmatrix}$

$$e^A = \alpha_0 I + \alpha_1 A$$

char poly

$$p(\lambda) = \begin{vmatrix} \lambda - t^2/2 & -t^3/3 \\ -t^3/3 & \lambda - t^2/2 \end{vmatrix}$$

~~An alternative (better) approach~~

problem is to calculate

$$e^{\begin{pmatrix} t^2/2 & t^3/3 \\ t^3/3 & t^2/2 \end{pmatrix}}$$

write

$$M = \begin{pmatrix} t^2/2 & t^3/3 \\ t^3/3 & t^2/2 \end{pmatrix} = t^2/2\, I + t^3/3 \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{Q}.$$

$$e^M = e^{t^2/2\, I}\, e^{t^3/3\, Q}.$$

$$e^{t^2/2\, I} = \begin{pmatrix} e^{t^2/2} & 0 \\ 0 & e^{t^2/2} \end{pmatrix}$$

Calculate

$$e^{t^3/3\, Q}.$$

Q has eigenvalues at $\lambda_1 = 1$, $\lambda_2 = -1$.

writing

$$e^{t^3/3\, Q} = \alpha_0 I + \alpha_1 Q$$

we have

$$\left.\begin{aligned} e^{t^3/3\; 1} &= \alpha_0 + \alpha_1 \\ e^{-t^3/3} &= \alpha_0 - \alpha_1 \end{aligned}\right\} \text{ solve for } \alpha_0 \;\&\; \alpha_1$$

Hence obtain $e^{t^3/3\, Q}$.