Control Theory I

Midterm 1

- No calculators.
- open notes.
- All questions equal weight.
- Total time 90 min.

1. Let us consider an automomous linear o.d.e. given by

$$\dot{\underline{X}} = A \underline{X}$$

where

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let P be a non singular matrix such that

$$\underline{X} = P Z$$

and where

$$\dot{Z} = B Z$$

$$B = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

(i) Compute the eigenvalues and generalized eigenvectors v_1, v_2, v_3 of A. (15 pts)

(ii) Define

$$P = (v_1 \mid v_2 \mid v_3)$$

compute

$$P^{-1}$$ (5 pts)

(iii) Verify that

$$P^{-1} A P = B$$ (10 pts)

2 Let A be a 2×2 matrix given by

$$A = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}$$

(i) compute

$$\exp\left(At^2\right) \quad (20\ pts).$$

(ii) consider the time varying o.d.e given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = 2t \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $x_1(0) = x_2(0) = 1$.

Calculate

$$x_1(t), x_2(t) \quad (10\ pts)$$

3 consider the linear dynamical system

$$\dot{x}_1 = x_2 + 2u \qquad x_1(0) = x_2(0) = 0 \quad *$$
$$\dot{x}_2 = 3u$$

(i) Write down * in a state variable form

$$\dot{\underline{x}} = A\underline{x} + bu \qquad (10 \text{ pts})$$

and calculate

$$\operatorname{rank}(b, Ab)$$

(ii) compute the controllability gramian matrix

$$\int_0^1 e^{-A\tau} b b^T e^{-A^T\tau}\, d\tau \qquad (10 \text{ pts})$$

and check its rank

(iii) Find a vector $\eta \in \mathbb{R}^2$ such that if we choose

$$u(\tau) = b^T e^{-A^T\tau} \eta$$

as a control for (*), then the sol$^{\underline{n}}$ for

$$\dot{x}_1 = x_2 + 2b^T e^{-A^T\tau} \eta$$
$$\dot{x}_2 = 3b^T e^{-A^T\tau} \eta$$

would hit

$$x_1(1) = 1 \quad , \quad x_2(1) = 0$$

at $t = 1$. (10 pts)