# CONTROL-I

## Midterm II

This is a take home exam.
You can start working on this exam any time.
There is no time limit.
Submit your answers in my "mail box"
"not in my office" by

Monday $19^{th}$ Nov.
10:00 AM.

The exam will be collected by
10:01 AM.

You can consult any books or notes
"but not each other"

① Consider a $5^{th}$ order linear state space system

$$\dot{x} = Ax + Bu$$

where

$$A = \begin{pmatrix} \alpha_1 & 1 & 0 & \beta_1 & 0 \\ \alpha_2 & 0 & 1 & \beta_2 & 0 \\ \alpha_3 & 0 & 0 & \beta_3 & 0 \\ \alpha_4 & 0 & 0 & \beta_4 & 1 \\ \alpha_5 & 0 & 0 & \beta_5 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We want to calculate a matrix $P$ such that

$$P^{-1}AP = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ x & x & x & x & x \\ 0 & 0 & 0 & 0 & 1 \\ x & x & x & x & x \end{pmatrix}, \quad P^{-1}B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & x \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$F =$ $\qquad$ $G =$

(a) Calculate $F, G, P$ where '$x$' is replaced by a $f^{n}$ of $\alpha_i, \beta_i$ or some fixed numbers such as $0, 1$.

(b) Calculate $\mathcal{F}_1$ such that

$$F + G\mathcal{F}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

(c) Calculate $\mathcal{F}$ such that

$A + B\mathcal{F}$ has the same characteristic polynomial as $F + G\mathcal{F}_1$.

(d) Calculate $A + B\mathcal{F}$ anyway.

(2) Generalize the main theorem of Lec 11 (page 19) to multiple input multiple output systems as follows:

Consider the dynamical system

$$\dot{x} = Ax + Bu, \quad y = Cx$$

where
$$\left.\begin{array}{l} A \text{ is } n \times n \\ B \text{ is } n \times m \\ C \text{ is } p \times n \end{array}\right\} \text{ matrices.}$$

Prove the following main theorem:

<u>Main Theorem:</u>

The following two statements are equivalent:

(I) $\dot{x} = Ax + Bu, \; y = Cx$; is controllable and observable.

(II) The hankel matrix.

$$H = \begin{pmatrix} CB & CAB & CA^2B & \cdots \\ CAB & CA^2B & CA^3B & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has rank $n$.

Remark: Note ~~that~~ $H$ is a block hankel matrix, ie $CB, CAB, CA^2B \ldots$ are all $p \times m$ blocks.

③ Let

$$\dot{x} = Ax + bu$$

$$y = cx$$

be a controllable and unobservable single input single output system, where

$A$ is $n \times n$, $b$ is $n \times 1$, $c$ is $1 \times n$.

Assume that

$$\operatorname{rank} \begin{pmatrix} c \\ cA \\ cA^2 \\ \vdots \\ cA^{n-1} \end{pmatrix} = n_1 < n$$

ⓐ Construct a matrix $P$ such that

$$P^{-1}AP = \begin{array}{c} \\ n_1 \\ \\ \end{array} \begin{array}{c} n_1 \\ \left(\begin{array}{c|c} F_{11} & 0 \\ \hline F_{21} & F_{22} \end{array}\right) \end{array} \qquad P^{-1}b = \begin{array}{c} n_1 \\ n-n_1 \end{array} \left(\begin{array}{c} g_1 \\ \hline g_2 \end{array}\right)$$

$$cP = \big(\underbrace{c_1}_{n_1} \;\big|\; \underbrace{0}_{n-n_1}\big)$$

(b) If $x = Pz$, we have

$$\dot{z}_1 = F_{11} z_1 + g_1 u$$

$$\dot{z}_2 = F_{21} z_1 + F_{22} z_2 + g_2 u$$

$$y = c_1 z_1$$

where $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

Show that

$$c_1 (sI - F_{11})^{-1} g_1 = c (sI - A)^{-1} b.$$

ie

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx \end{aligned} \quad \& \quad \begin{aligned} \dot{z}_1 &= F_{11} z_1 + g_1 u \\ y &= c_1 z_1 \end{aligned} \quad (*)$$

have the same transfer function.

(c) Show that the Hankel matrix H from the two systems in (*) are the same.

What is the rank of H.

(d) Is it true that

$$\dot{z} = F_{11} z_1 + g_1 u$$
$$y = c_1 z_1$$

is controllable and observable.