Lec 10

Canonical forms

CONTROLLABLE CANONICAL FORM

Canonical Forms

1. Controllable canonical form:

Let us start this discussion with a dynamical system.

$$\dot{x} = Ax + bu$$

where A is a $n \times n$ matrix.
b is a $n \times 1$ vector.

Let us assume that

$$b, Ab, A^2b, \ldots, A^{n-1}b$$

are linearly independent.

$$\mathbb{R}^n \times \mathbb{R} \xrightarrow{(A,\, b)} \mathbb{R}^n \times \mathbb{R}^n$$

$$(v, p) \longmapsto (Av, bp)$$

We have a linear transformation that maps

$$U = \mathbb{R}^n \times \mathbb{R}$$

to

$$V = \mathbb{R}^n \times \mathbb{R}^n$$

$$T : U \longrightarrow V$$

$$(v, p) \longmapsto (Av, bp)$$

Let us now consider a basis of

U given by

$$\{A^{n-1}b, A^{n-2}b, \ldots, Ab, b\}, \{1\}$$

and a basis of

V given by

$$\{A^{n-1}b, A^{n-2}b, \ldots, Ab, b\}, \{A^{n-1}b, \ldots, Ab, b\}$$

Problem:

We would like to find a matrix representation of T. Let us call this (A^*, b^*).

$$\mathbb{R}^n \times \mathbb{R} \xrightarrow{(A,\, b)} \mathbb{R}^n \times \mathbb{R}^n$$

$$\mathbb{R}^n \times \mathbb{R} \xrightarrow{(A^*,\, b^*)} \mathbb{R}^n \times \mathbb{R}^n$$

(A^*, b^*) maps co-ordinates of (v, p) to co-ordinates of (Av, bp) with respect to the bases chosen in page 3.

$$
\begin{tikzcd}
(*,1) \arrow[r, mapsto] & (*, b) \arrow[d, mapsto] \\
(*,1) \arrow[u, mapsto] & \left(*, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}\right)
\end{tikzcd}
$$

To get b^*, note that '1' is mapped to $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ by b^* and hence

$$
b^* = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.
$$

$$\begin{tikzcd}
(b,*) \arrow[r, mapsto] & (Ab,*) \arrow[d, mapsto] \\
\left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, *\right) \arrow[u, mapsto] & \left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix}, *\right)
\end{tikzcd}$$

To get the last column of A^* note that

e_n is mapped to e_{n-1} by A^*

and hence

$$A^* = \begin{pmatrix} \vert & \vert & \vert & \vert & 0 \\ \vert & \vert & \vert & \vert & 0 \\ \vert & \vert & \vert & \vert & \vdots \\ \vert & \vert & \vert & \vert & 0 \\ \vert & \vert & \vert & \vert & 1 \\ \vert & \vert & \vert & \vert & 0 \end{pmatrix}$$

Define $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ i^{th} spot.

Proceeding as in page 6 we have

$$A^* = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & \ddots & \\ \vdots & \vdots & & 1 \\ 0 & 0 & & 0 \end{pmatrix}$$

where we only need to compute the 1st column of A^*.

Let us write down the characteristic polynomial of A as

$$p(\lambda) = \lambda^n - \alpha_1 \lambda^{n-1} - \alpha_2 \lambda^{n-2} - \cdots - \alpha_n$$

It follows from Cayley Hamilton Theorem that

$$A^n = \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \cdots + \alpha_n I$$

$$\Rightarrow A^n b = \alpha_1 A^{n-1} b + \alpha_2 A^{n-2} b + \cdots + \alpha_n b$$

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$$
\begin{tikzcd}
(A^{n-1}b, *) \arrow[r, mapsto] & (A^n b, *) \arrow[d, mapsto] \\
(e_1, *) \arrow[u, mapsto] & \left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}, * \right)
\end{tikzcd}
$$

From the above figure we infer that A^* maps the vector e_1 to the vector $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$

It follows that

$$A^* = \begin{pmatrix} \alpha_1 & 1 & & & \text{\Large 0} \\ \alpha_2 & 0 & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n-1} & 0 & 0 & \cdots & 1 \\ \alpha_n & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$b^* = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

If we let

$$P = \left(A^{n-1}b, A^{n-2}b, \dots, Ab, b\right)$$

It would follow that

$$A^* = P^{-1} A P$$

$$b^* = P^{-1} b$$

If we define a new state variable

$$z = P^{-1} x$$

we have

$$\dot{z} = P^{-1}\dot{x} = P^{-1}Ax + P^{-1}bu$$

$$\dot{z} = P^{-1}APz + P^{-1}bu$$

$$\boxed{\dot{z} = A^* z + b^* u}$$

$$\begin{pmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_n \end{pmatrix} = \begin{pmatrix} \alpha_1 & 1 & & \text{\Large 0} \\ \vdots & & \ddots & \\ & & & 1 \\ \alpha_n & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u$$

The above structure is called controllable canonical form. Every controllable pair (A, b) can be converted to the above form. Conversely the above form is always controllable.

— x —

OTHER CANONICAL FORMS

2. Other canonical forms:

Controllable canonical forms are not the only canonical form in town. We now ask a different question: start with the dynamical system

$$\dot{x} = Ax + bu$$

where A is $n \times n$

b is $n \times 1$

$\{b, Ab, \ldots, A^{n-1}b\}$ are linearly independent vectors.

Q: Find if possible a Q such that

$$Q^{-1} b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = b^*$$

$$Q^{-1} A Q =$$

$$\begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ \cdot & \cdot & \cdot & \cdots & \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_n & \alpha_{n-1} & \alpha_{n-2} & & \alpha_1 \end{pmatrix} \longleftarrow A^*$$

Note that the co-efficients $\alpha_1, \ldots, \alpha_n$ are coming from the characteristic polynomial of A.

writing

$$Q = (q_1\ q_2\ \dots\ q_n)$$

where q_j are columns of Q, we obtain

$$b = Qb^* = q_n \Rightarrow q_n = b$$

$$AQ = QA^* \Rightarrow$$

$$Aq_n = q_{n-1} + \alpha_1 q_n \Rightarrow q_{n-1} = Ab - \alpha_1 b$$

$$Aq_{n-1} = q_{n-2} + \alpha_2 q_n \Rightarrow q_{n-2} = A^2 b - \alpha_1 Ab - \alpha_2 b$$

$$\cdots$$

$$\cdots$$

$$Aq_2 = q_1 + \alpha_{n-1} q_n \Rightarrow q_1 = A^{n-1} b - \alpha_1 A^{n-2} b \dots$$
$$\dots - \alpha_{n-1} b$$

$$Aq_1 = \alpha_n q_n \Rightarrow Aq_1 = \alpha_n b$$

Note that this follows automatically from Cayley Hamilton Theorem.

The matrix Q is defined as

$$Q = (q_1 \; . \; . \; . \; q_n)$$

where

$$q_j = A^{n-j} b - \sum_{k=1}^{n-j} \alpha_k A^{n-j-k} b$$

As before, if we define

$$z = Q^{-1} x$$

we have

$$\dot{z} = Q^{-1}\dot{x} = Q^{-1}Ax + Q^{-1}bu$$
$$= Q^{-1}AQz + Q^{-1}bu$$

$$\dot{z} = A^* z + b^* u$$

$$\begin{pmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & 0 \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ 0 & 0 & 0 & & 1 \\ \alpha_n & \alpha_{n-1} & & & \alpha_1 \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u$$

$**$

Why study this new canonical form??

Ans: Pole Placement by state feedback

consider

$$u = \begin{pmatrix} k_n & k_{n-1} & \cdots & k_1 \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} + v$$

substitute this into (**), we get

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & & & 0 \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_n + k_n & \cdots & & \cdots & \alpha_1 + k_1 \end{pmatrix}}_{A_f} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} v$$

$k_1, \dots, k_n$ can be chosen to arbitrary assign the characteristic polynomial of the corresponding A_f.

Going back to the x-co-ordinate we write

$$u = \underbrace{(k_n \; k_{n-1} \cdots k_1) \; Q^{-1}}_{\begin{array}{c} \parallel \\ (f_n \; f_{n-1} \cdots f_1) = f \end{array}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + v$$

$$\boxed{u = f x + v} \leftarrow \text{feedback state}$$

we substitute this u into

$$\dot{x} = Ax + bu$$

and obtain

$$\dot{x} = (A + bf)x + bv$$

We claim that the characteristic polynomial of $A+bf$ can be arbitrarily chosen by appropriately chosing f provided (A, b) is controllable.

Theorem: (Pole placement by state feedback)

Let A be an $n \times n$ matrix and let b be a $n \times 1$ vector where

$$\{b, Ab, \ldots, A^{n-1}b\}$$

are linearly independent. There exists a $1 \times n$ vector f such that

$$A_f = A + bf$$

has an arbitrary characteristic polynomial

$$p(\lambda) = \lambda^n - \beta_1 \lambda^{n-1} - \beta_2 \lambda^{n-2} - \cdots - \beta_n .$$

KRONECKER INDICES AND CANONICAL FORMS

3. Kronecker Indices & Kronecker Canonical Forms

Let us now generalize the story of controllable canonical form to multiple inputs. We start this discussion with a dynamical system

$$\dot{x} = Ax + Bu$$

$x(t) \in \mathbb{R}^n$
$u(t) \in \mathbb{R}^m$

where

A is a $n \times n$ matrix

B is a $n \times m$ matrix

Note that $u(t)$ is not a scalar

Assume that

$$\operatorname{rank}\left(B, AB, \ldots, A^{n-1}B\right) = n.$$

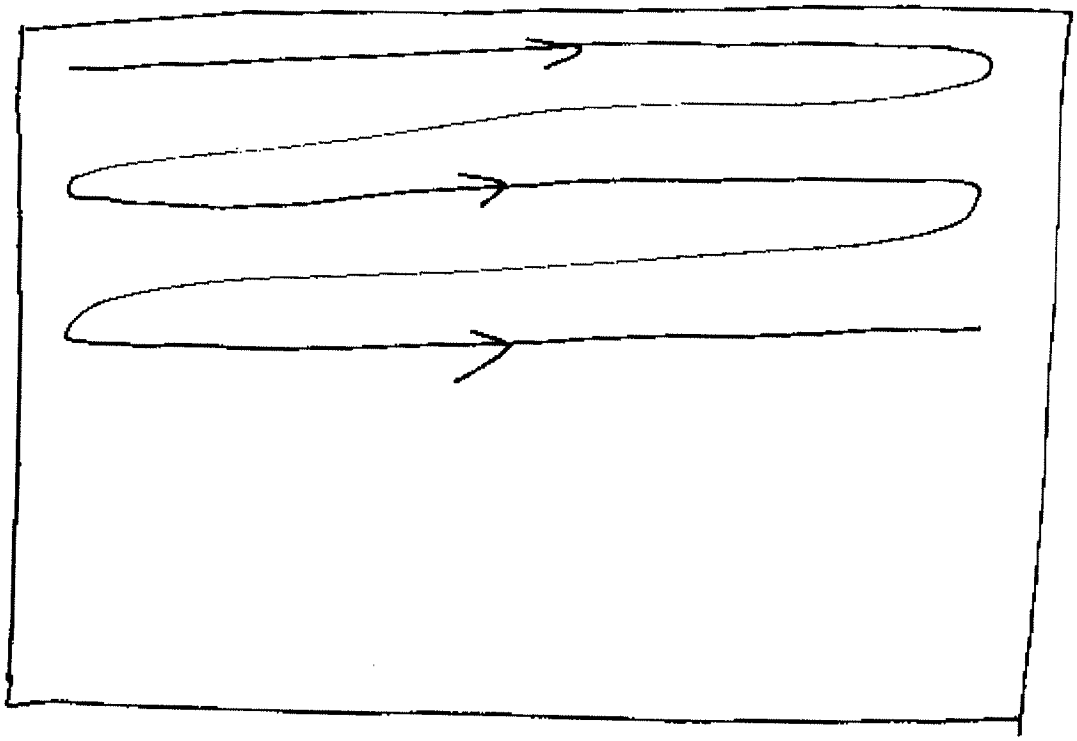
We can write

$$B = (b_1 \ b_2 \ \ldots \ b_m)$$

where $b_j \in \mathbb{R}^n$ are columns of the matrix B. We now make a crate of vectors as shown below:

b_1	b_2	b_3	$\cdots$	b_m
Ab_1	Ab_2	Ab_3	$\cdots$	Ab_m
A^2b_1	A^2b_2	A^2b_3	$\cdots$	A^2b_m
$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$A^{n-1}b_1$	$A^{n-1}b_2$	$A^{n-1}b_3$	$\cdots$	$A^{n-1}b_m$

This is a $n \times m$ crate of vectors in $\mathbb{R}^n$.



Look at the above crate rowwise starting from the 1st row, 2nd row etc from left to right.

Throw away any vector which are dependent with respect to vectors already looked at.

For example

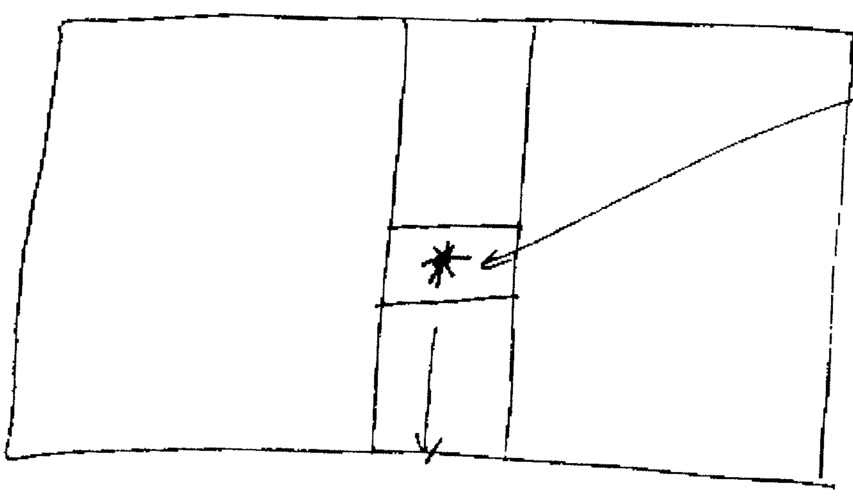
- if b_m is linearly dependent with respect to vectors $b_1, \ldots, b_{m-1}$ throw away b_m
- if Ab_3 is linearly dependent with respect to vectors $b_1, \ldots b_m, Ab_1, Ab_2$, then throw away Ab_3

proceed till the end of the crate.

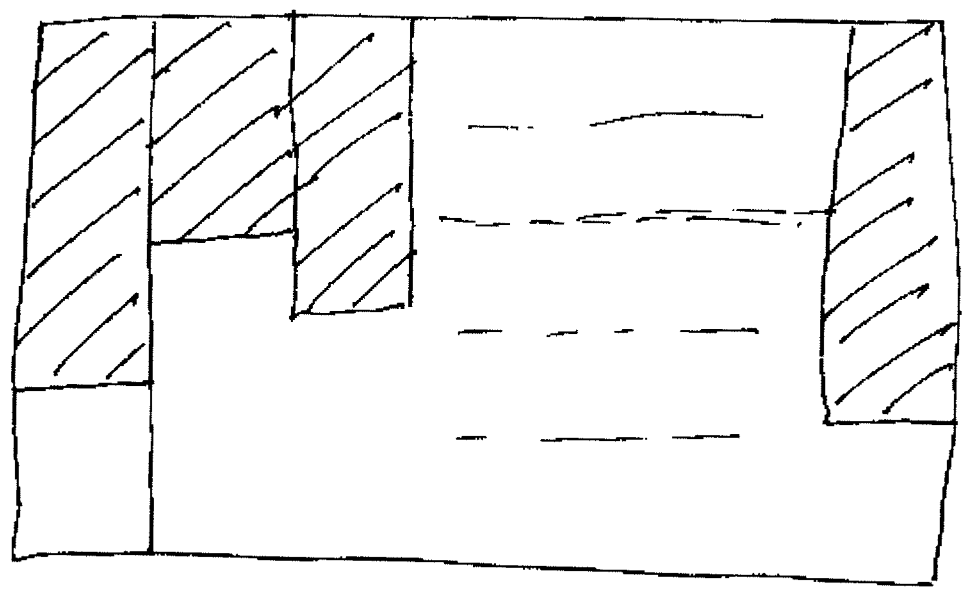
Theorem:

If $A^j b_k$ for some j and k is thrown away, then $A^{j+l} b_k$ for $l = 1, 2, \ldots, n-1-j$ would also be thrown away.

This theorem is easy to see and is left as an exercise.



If $*$ is thrown away all elements below $*$ are also thrown away.



The vectors that are kept would look like the shaded region above.

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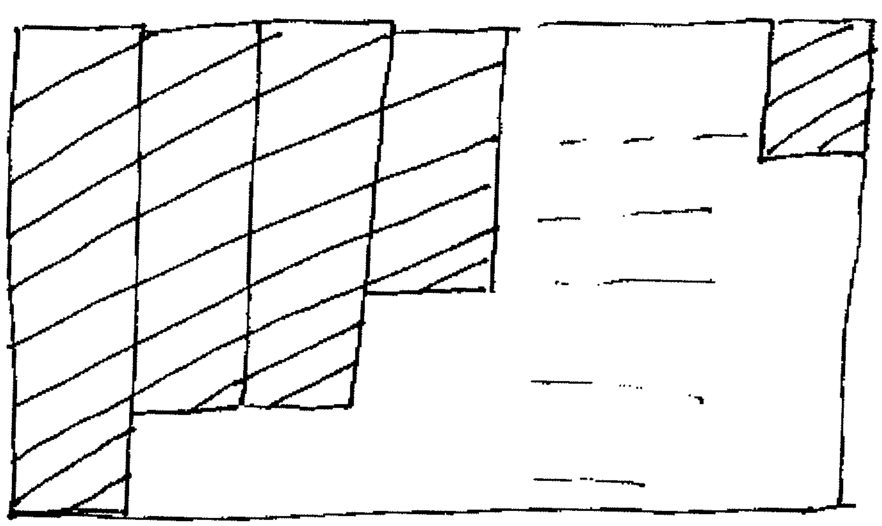
Upto reordering the columns of the crate, the crate of vectors that are kept would look like the following:



vectors in the j^{th} column

$\geq$ # vectors in the $j+1^{th}$ column.

$j = 1, 2, \ldots, m$.

Define

$$\mathcal{K}_j = \# \text{ vectors in the } j^{th} \text{ column.}$$

we have

$$\mathcal{K}_1 \geq \mathcal{K}_2 \geq \cdots \geq \mathcal{K}_m .$$

The m tuple of integers

$$(\mathcal{K}_1, \ldots, \mathcal{K}_m)$$

are called the Kronecker Indices.

and is uniquely defined given

(A, B).

Theorem:

If (A,B) is controllable then

$$\sum_{j=1}^{m} k_j = n$$

Proof: $\because$ (A,B) is controllable there are precisely n vectors in the crate that are independent. # vectors that are kept must be n. Hence

$$\sum_{j=1}^{m} k_j = n.$$

Example:

For $n=4$, $m=2$

the possible Kronecker indices are

① $(4, 0)$ $\qquad (\kappa_1, \kappa_2)$

② $(3, 1)$ $\qquad \kappa_1 \geq \kappa_2$

③ $(2, 2)$ $\qquad \kappa_1 + \kappa_2 = n$.

Example

for $n=6$, $m=3$

the possible Kronecker indices are

1. $(6, 0, 0)$
2. $(5, 1, 0)$
3. $(4, 1, 1)$
4. $(4, 2, 0)$
5. $(3, 3, 0)$
6. $(3, 2, 1)$
7. $(2, 2, 2)$

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We are looking for a partition of n into m integers

$$x_1, \ldots, x_m$$

such that

$$x_1 \geq x_2 \geq \cdots \geq x_m$$

and

$$x_1 + x_2 + \cdots + x_m = n.$$

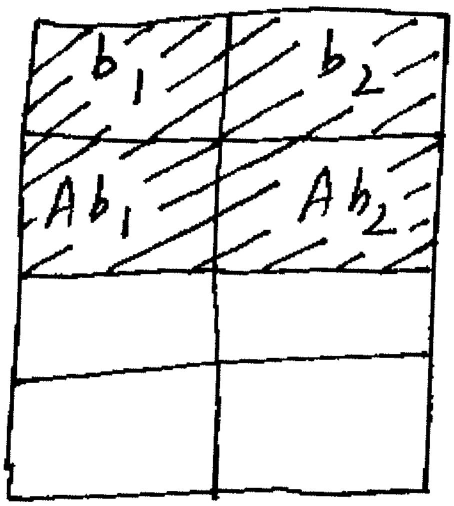
Example:

Assume $n=4$, $m=2$

$$\varkappa_1 = \varkappa_2 = 2.$$

write $B = (b_1 \;\; b_2)$

and the crate as.



We assume that

$$A^2 b_1 = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 Ab_1 + \alpha_4 Ab_2 .$$

$$A^2 b_2 = \beta_1 b_1 + \beta_2 b_2 + \beta_3 Ab_1 + \beta_4 Ab_2$$

Define

$$P = \begin{pmatrix} Ab_1 & b_1 & Ab_2 & b_2 \end{pmatrix}$$

$$\dot{x} = AX + Bu$$

We compute

$$P^{-1}AP \quad \text{and} \quad P^{-1}B$$

as follows.

$$P^{-1}B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$P^{-1}AP = \left(\begin{array}{cc|cc} \alpha_3 & 1 & \beta_3 & 0 \\ \alpha_1 & 0 & \beta_1 & 0 \\ \hline \alpha_4 & 0 & \beta_4 & 1 \\ \alpha_2 & 0 & \beta_2 & 0 \end{array}\right)$$

Define

$$z = P^{-1} x \quad \text{as before,}$$

we obtain

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \left(\begin{array}{cc|cc} \alpha_3 & 1 & \beta_3 & 0 \\ \alpha_1 & 0 & \beta_1 & 0 \\ \hline \alpha_4 & 0 & \beta_4 & 1 \\ \alpha_2 & 0 & \beta_2 & 0 \end{array}\right) \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

Controllable canonical form.

$$+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

The above system is controllable always.

Note that

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$$

are uniquely defined on page 32 and does not come from the characteristic polynomial of A.

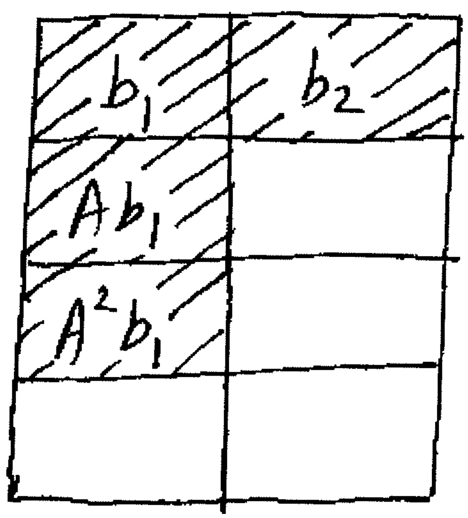
Example

Assume $n=4$, $m=2$

$\kappa_1 = 3 \qquad \kappa_2 = 1$

write $B = (b_1 \ \ b_2)$

and the create as



we assume that

$$Ab_2 = \beta_1 b_1 + \beta_2 b_2 + \beta_3 Ab_1 .$$

$$A^3 b_1 = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 Ab_1 + \alpha_4 A^2 b_1$$

Proceeding as before, we define

$$P = \left(A^2 b_1 \,\middle|\, A b_1 \,\middle|\, b_1 \,\middle|\, b_2 \right)$$

$$P^{-1} B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$P^{-1} A P =$

$$\left(\begin{array}{ccc|c} \alpha_4 & 1 & 0 & 0 \\ \alpha_3 & 0 & 1 & \beta_3 \\ \alpha_1 & 0 & 0 & \beta_1 \\ \hline \alpha_2 & 0 & 0 & \beta_2 \end{array}\right)$$

Define

$$z = P^{-1}x \quad \text{as before}$$

we obtain

$$\begin{pmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \\ \dot z_4 \end{pmatrix} = \left(\begin{array}{ccc|c} \alpha_4 & 1 & 0 & 0 \\ \alpha_3 & 0 & 1 & \beta_3 \\ \alpha_1 & 0 & 0 & \beta_1 \\ \hline \alpha_2 & 0 & 0 & \beta_2 \end{array}\right) \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

controllable canonical form

$$+ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

The above system is always controllable

Once again note that $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3$ are uniquely defined from page 36.

Remark:

Notice that the canonical forms in the two examples are different. In fact one has 8 free parameters other has 7 free parameters.

Remark:

For $n=4$, $m=2$ $\qquad P=(A^3 b_1, A^2 b_1, A b_1, b_1)$.

$\kappa_1 = 4$, $\kappa_2 = 0$

we can write

$$\boxed{\begin{aligned} b_2 &= \beta_1 b_1 \\ A^4 b_1 &= \alpha_1 b_1 + \alpha_2 A b_1 + \alpha_3 A^2 b_1 + \alpha_4 A^3 b_1 \end{aligned}}$$

$$\begin{pmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \\ \dot z_4 \end{pmatrix} = \begin{pmatrix} \alpha_4 & 1 & 0 & 0 \\ \alpha_3 & 0 & 1 & 0 \\ \alpha_2 & 0 & 0 & 1 \\ \alpha_1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & \beta_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Five free parameters

Example

Assume $n = 4$, $m = 2$

$$\kappa_1 = \kappa_2 = 2$$

as in page 32.

Find if possible a matrix P:

$$P^{-1}B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

As before, this question is motivated from the pole placement theorem.

$$P^{-1}AP = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ \times & \times & \times & \times \\ \hline 0 & 0 & 0 & 1 \\ \times & \times & \times & \times \end{array}\right)$$

To answer ~~this~~ this, assume

$$P = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \end{pmatrix}$$

$$\therefore \quad P^{-1} B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

it follows that $p_2 = b_1$, $p_4 = b_2$.

where $B = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$, hence

$$P = \begin{pmatrix} p_1 & b_1 & p_3 & b_2 \end{pmatrix}$$

writing

$$P^{-1} A P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ 0 & 0 & 0 & 1 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{pmatrix}, \text{ we have}$$

$$(Ap_1 \quad Ab_1 \quad Ap_3 \quad Ab_2) =$$

$$\left(\gamma_1 b_1 + \delta_1 b_2 \,\middle|\, p_1 + \gamma_2 b_1 + \delta_2 b_2 \,\middle|\, \gamma_3 b_1 + \delta_3 b_2 \,\middle|\, p_3 + \gamma_4 b_1 + \delta_4 b_2\right)$$

It follows that

$$p_1 = Ab_1 - \gamma_2 b_1 - \delta_2 b_2$$

$$p_3 = Ab_2 - \gamma_4 b_1 - \delta_4 b_2$$

Moreover

$$Ap_1 = \gamma_1 b_1 + \delta_1 b_2 = A^2 b_1 - \gamma_2 Ab_1 - \delta_2 Ab_2$$

$$Ap_3 = \gamma_3 b_1 + \delta_3 b_2 = A^2 b_2 - \gamma_4 Ab_1 - \delta_4 Ab_2$$

Hence

$$A^2 b_1 = \gamma_1 b_1 + \delta_1 b_2 + \gamma_2 A b_1 + \delta_2 A b_2$$

$$A^2 b_2 = \gamma_3 b_1 + \delta_3 b_2 + \gamma_4 A b_1 + \delta_4 A b_2$$

From page 32, it would follow that

$$\gamma_1 = \alpha_1 \quad \delta_1 = \alpha_2 \quad \gamma_2 = \alpha_3 \quad \delta_2 = \alpha_4$$

$$\gamma_3 = \beta_1 \quad \delta_3 = \beta_2 \quad \gamma_4 = \beta_3 \quad \delta_4 = {}_4$$

i.e

$$p_1 = A b_1 - \alpha_3 b_1 - \alpha_4 b_2$$

$$p_3 = A b_2 - \beta_3 b_1 - \beta_4 b_2$$

$$P = \left(A b_1 - \alpha_3 b_1 - \alpha_4 b_2 \,\middle|\, b_1 \,\middle|\, A b_2 - \beta_3 b_1 - \beta_4 b_2 \,\middle|\, b_2 \right)$$

$$P^{-1}AP =$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha_1 & \alpha_3 & \beta_1 & \beta_3 \\ 0 & 0 & 0 & 1 \\ \alpha_2 & \alpha_4 & \beta_2 & \beta_4 \end{pmatrix}$$

$$P^{-1}B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$z = P^{-1}x$$

$$\Downarrow$$

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha_1 & \alpha_3 & \beta_1 & \beta_3 \\ 0 & 0 & 0 & 1 \\ \alpha_2 & \alpha_4 & \beta_2 & \beta_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Define

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\alpha_1 & -\alpha_3 & -\beta_1+1 & -\beta_3 \\ k_1 & k_2 & k_3 & k_4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

we obtain.

$$\begin{pmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \\ \dot z_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha_2+k_1 & \alpha_4+k_2 & \beta_2+k_3 & \beta_4+k_4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Characteristic polynomial can be arbitrarily assigned

From the canonical form in page 45, we infer the following:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \times & \times & \times & \times \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} V_2$$

is controllable. We can therefore state the following lemma.

<u>Lemma (Heyman's Lemma):</u>

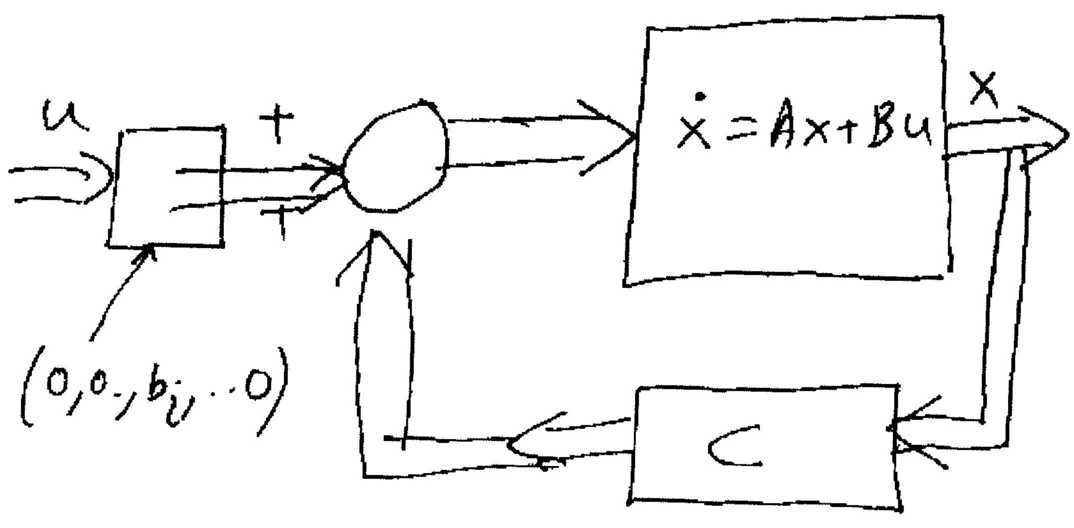
Let $\dot{x} = Ax + Bu$ $\quad x \in \mathbb{R}^n, u \in \mathbb{R}^m$ be controllable. $\exists\, C$:

$$\dot{x} = (A + BC)x + b_i u_i$$

is also controllable.

Remark:

Heymann's lemma states that if a dynamical system is controllable with multiple input then it is controllable with any given input after a possible state feedback.



Note that the lemma is not true without the state feedback.

Proof: Can be easily shown for each of the Kronecker Indices separately.

Exercise:

Consider a dynamical system

$$\dot{x} = Ax + Bu$$

where $n=4$, $m=2$, $\kappa_1=3$, $\kappa_2=1$

Find if possible a P where

$$z = P^{-1}x$$

such that

$$P^{-1}AP = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$

$$P^{-1}B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(1) Find P and also the '$\times$' in the above matrix $P^{-1}AP$ using α, β defined in page 36.

(2) Find a 2×4 matrix C:

$$\dot{x} = (A + BC)x + b_2 u_2$$

is controllable.

(This would validate Heyman's Lemma)

REALIZATION PROBLEM AGAIN

Realization problem again

Example

Let us start with the transfer fn

$$g(s) = \frac{3s^2+4s+5}{s^3+8s^2+2s+10}$$

$g(s)$ is of degree 3 provided

$$\det \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 4 & 8 & 3 & 1 & 0 \\ 5 & 2 & 4 & 8 & 3 \\ 0 & 10 & 5 & 2 & 4 \\ 0 & 0 & 0 & 10 & 5 \end{pmatrix} \neq 0$$

$= -2985$

(from matlab).

We would therefore like to realize $g(s)$ as a transfer $f^{\underline{n}}$ of a 3^{rd} order system

$$\dot{x} = Ax + bu \qquad x \in \mathbb{R}^3$$
$$y = cx$$

- characteristic polynomial of A must be
$$\lambda^3 + 8\lambda^2 + 2\lambda + 10$$

- The system must be controllable and observable.

Choose

$$A = \begin{pmatrix} \alpha_1 & 1 & 0 \\ \alpha_2 & 0 & 1 \\ \alpha_3 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad c = (c_1 \ \ c_2 \ \ c_3)$$

characteristic polynomial of A is given by

$$\lambda^3 - \alpha_1 \lambda^2 - \alpha_2 \lambda - \alpha_3$$

It follows that

$$\boxed{\alpha_1 = -8,\ \alpha_2 = -2,\ \alpha_3 = -10}$$

To compute c we expand

$$g(s) = \frac{3}{s} - \frac{20}{s^2} + \frac{159}{s^3} + \cdots$$

Moreover

$$g(s) = c\,(sI - A)^{-1} b$$

$$= \frac{cb}{s} + \frac{cAb}{s^2} + \frac{cA^2b}{s^3} + \cdots$$

$$\left.\begin{array}{l} cb = c_3 = 3 \\ cAb = c_2 = -20 \\ cA^2b = c_1 = 159 \end{array}\right. \Rightarrow c = \begin{pmatrix} 159 & -20 & 3 \end{pmatrix}$$

We have a realization given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -8 & 1 & 0 \\ -2 & 0 & 1 \\ -10 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 159 & -20 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The above realization is in the controllable canonical form and hence controllable.

The observability matrix is

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 159 & -20 & 3 \\ -1262 & 159 & -20 \\ 9978 & -1262 & 159 \end{pmatrix}$$

$$\det(\mathcal{O}) = 2985 \neq 0$$

Hence the realization is also observable.

We can construct the controllability matrix

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and the Hankel matrix

$$H = \mathcal{O}\mathcal{C} =$$

$$\begin{pmatrix} 3 & -20 & 159 \\ -20 & 159 & -1262 \\ 159 & -1262 & 9978 \end{pmatrix}$$

H has a negative eigenvalue at $-\cdot 617$

two positive eigenvalues at

$10140\cdot 14$ & $\cdot 477$

Signature is $2 - 1 = 1$

Example

Let $G(s)$ be a 2×2 transfer function given by

$$G(s) = \frac{H_1}{s} + \frac{H_2}{s^2} + \frac{H_3}{s^3} + \frac{H_4}{s^4} + \cdots$$

where

$$H_1 = \begin{pmatrix} 1 & -2 \\ 3 & 7 \end{pmatrix} \qquad H_2 = \begin{pmatrix} 3 & -2 \\ -1 & 5 \end{pmatrix}$$

and where H_j, $j = 3, 4, \ldots$ are defined as follows:

1st column of H_j =

$$\begin{pmatrix} H_{j-2} & H_{j-1} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -3 \\ 1 \end{pmatrix}$$

2nd column of H_j =

$$\begin{pmatrix} H_{j-2} & H_{j-1} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 3 \end{pmatrix}$$

$j = 3, 4, \ldots$

It follows that

$$H_j = \begin{pmatrix} H_{j-2} & H_{j-1} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 1 \\ 1 & 3 \end{pmatrix}$$

```
>> H1=[1 -2;3 7]

H1 =
```

$$\begin{pmatrix} 1 & -2 \\ 3 & 7 \end{pmatrix}$$

```
>> H2=[3 -2;-1 5]

H2 =
```

$$\begin{pmatrix} 3 & -2 \\ -1 & 5 \end{pmatrix}$$

```
>> P=[2 -1;1 0;-3 1;1 3]

P =
```

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 1 \\ 1 & 3 \end{pmatrix}$$

```
>> H3=[H1 H2]*P

H3 =
```

$$\begin{pmatrix} -11 & -4 \\ 21 & 11 \end{pmatrix}$$

```
>> H4=[H2 H3]*P

H4 =
```

$$\begin{pmatrix} 33 & -26 \\ -49 & 55 \end{pmatrix}$$

```
>> H5=[H3 H4]*P

H5 =
```

$$\begin{pmatrix} -151 & -34 \\ 255 & 95 \end{pmatrix}$$

```
>> H=[H1 H2 H3;H2 H3 H4;H3 H4 H5]

H =
```

$$\left(\begin{array}{cc|cc|cc} 1 & -2 & 3 & -2 & -11 & -4 \\ 3 & 7 & -1 & 5 & 21 & 11 \\ \hline 3 & -2 & -11 & -4 & 33 & -26 \\ -1 & 5 & 21 & 11 & -49 & 55 \\ \hline -11 & -4 & 33 & -26 & -151 & -34 \\ 21 & 11 & -49 & 55 & 255 & 95 \end{array}\right)$$

```
>> rank(H)

ans =

     4
```

We have a block Hankel matrix of rank 4

$G(s)$ can be realized as a state space system of order 4

$$\dot{x} = Ax + \overbrace{(b_1 \;\; b_2)}^{=B}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$y = \underbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}_{=C} x$$

It would follow that

$$CB = H_1$$

$$CAB = H_2$$

$$CA^2B = H_3$$

$$\vdots$$

and so on.

From page 32 we obtain

$$CA^2B = (CB \quad CAB)\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \\ \alpha_4 & \beta_4 \end{pmatrix}$$

It would follow that

$$\alpha_1 = 2 \qquad \beta_1 = -1$$
$$\alpha_2 = 1 \qquad \beta_2 = 0$$
$$\alpha_3 = -3 \qquad \beta_3 = 1$$
$$\alpha_4 = 1 \qquad \beta_4 = 3$$

Using the controllable canonical form on page 34 we have

$$A = \begin{pmatrix} -3 & 1 & 1 & 0 \\ 2 & 0 & -1 & 0 \\ 1 & 0 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{pmatrix}$$

We compute

$$CB = \begin{pmatrix} c_{12} & c_{14} \\ c_{22} & c_{24} \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 3 & 7 \end{pmatrix} = H_1$$

$$CAB = \begin{pmatrix} c_{11} & c_{13} \\ c_{21} & c_{23} \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 5 \end{pmatrix} = H_2$$

It follows that

$$C = \begin{pmatrix} 3 & 1 & -2 & -2 \\ -1 & 3 & 5 & 7 \end{pmatrix}$$

The state space realization is clearly controllable. To check observability we compute the observability matrix as follows:

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```
To get started, select MATLAB Help or Demos from the Help menu.

>> A=[-3 1 1 0;2 0 -1 0;1 0 3 1;1 0 0 0]

A =

    -3     1     1     0
     2     0    -1     0
     1     0     3     1
     1     0     0     0

>> C=[3 1 -2 -2;-1 3 5 7]

C =

     3     1    -2    -2
    -1     3     5     7

>> O=[C;C*A;C*A*A;C*A*A*A]

O =

     3     1    -2    -2
    -1     3     5     7
   -11     3    -4    -2
    21    -1    11     5
    33   -11   -26    -4
   -49    21    55    11
  -151    33   -34   -26
   255   -49    95    55

>> rank(O)

ans =

     4

>>
```

observability matrix has rank 4

Hence we have a controllable and observable realization.

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$$G(s) = C(sI - A)^{-1}B$$

$$\begin{pmatrix} \dfrac{s^3+3s^2-23s+3}{s^4-12s^2+6s+1} & \dfrac{-2s^3-2s^2+20s-14}{s^4-12s^2+6s+1} \\[2ex] \dfrac{3s^3-s^2-15s-19}{s^4-12s^2+6s+1} & \dfrac{7s^3+5s^2-73s+37}{s^4-12s^2+6s+1} \end{pmatrix}$$

Exercise:

Consider the controllable system

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix}}_{} = \underbrace{\begin{pmatrix} -3 & 1 & 1 & 0 \\ 2 & 0 & -1 & 0 \\ 1 & 0 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}_{u}$$

$$\dot{x} = Ax + Bu$$

Find

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{pmatrix}$$

such that

$$\dot{x} = (A + BK)x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_2$$

is controllable and $A+BK$ has eigenvalues at $-1, -2, -3, -4$.