$$\boxed{\text{Lec } 9}$$

Stability of Linear Dynamical systems.

Remark:

"stability" is perhaps one of the most important subject that keeps a control scientist busy. The main point is that a dynamical system must be "stable" for it to have any practical value. It will not be an understatement if I said that "some of the best researchers in control have spent a good deal of their time on stability".

9.1 We start with the homogeneous system

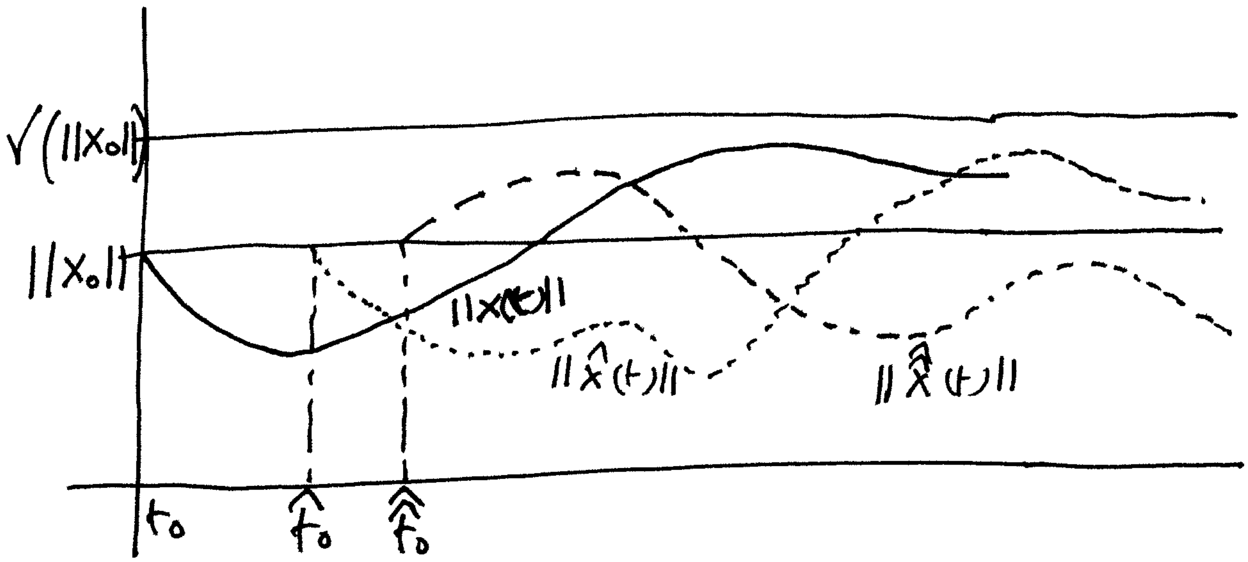
$$\dot{X} = A(t)\,X(t)$$
$$X(t_0) = X_0 \qquad (9.1)$$

Def 9.1 (Uniform stability)

The linear state eqn (9.1) is called "uniformly stable" if $\exists$ a finite positive constant γ : for any t_0 & x_0 we have

$$\|x(t)\| \leq \gamma \|x_0\|, \; t \geq t_0. \qquad (9.2)$$

Evaluation of (9.2) at $t = t_0$ shows that the constant $\gamma \geq 1$. The adjective "uniform" in the definition refers precisely to the fact that γ must not depend on t_0.



"Figure showing uniform stability where γ is independent of t_0"

Example: 9.2

$$\dot{x} = (4t\sin t - 2t)x(t), \quad x(t_0) = x_0 \qquad (9.3)$$

has the solution

$$x(t) = \exp\left[4\sin t - 4t\cos t - t^2 - 4\sin t_0 + 4t_0\cos t_0 + t_0^2\right] x_0 \qquad (9.4)$$

For a fixed t_0, ~~there~~ is a γ : (9.4) is bounded by $\gamma\|x_0\|$ $\forall\, t \geq t_0$, since $-t^2$ term dominates the exponent as t increases.

However, (9·3) is not uniformly stable. With fixed x_0, consider a sequence of $t_0 = 2k\pi$, $k = 0, 1, 2, \ldots$ and the values of respective sol$^{\underline{n}}$ at times π units later ie

$$x(2k\pi + \pi) = \exp\left[(4k+1)\pi(4-\pi)\right] x_0 . \tag{9.5}$$

Clearly there is no bound on the exponential factor that is independent of k. In other words, a candidate bound γ must be ever larger as k, and the corresponding initial time, increases

$\blacksquare$

The following theorem characterizes uniform stability in terms of the fundamental matrix.

Theorem 9.3

The linear state eqn (9.1) is uniformly stable iff $\exists$ a finite positive constant γ:

$$\|\phi(t,\tau)\| \leq \gamma \qquad (9.6)$$

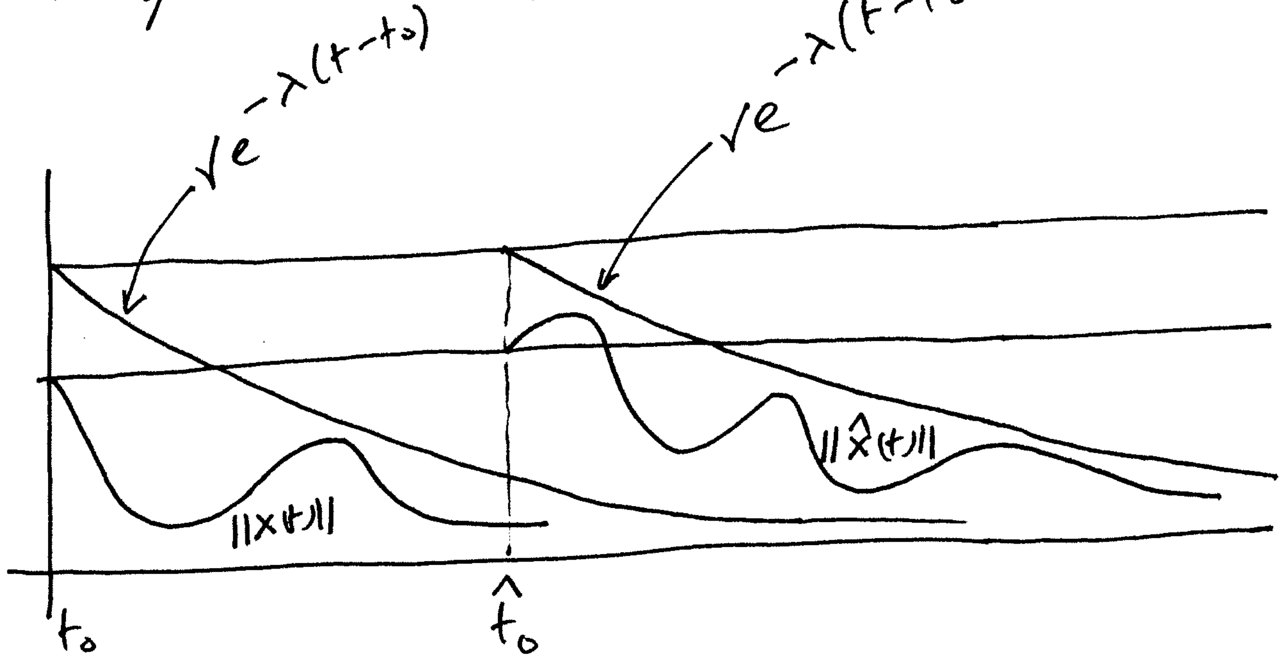
$\forall\, t,\tau : t \geq \tau.$ $\blacksquare$

Def 9.4 (Uniform Exponential Stability)

The linear state eqn (9.1) is called "uniformly exponentially stable" if $\exists$ a finite positive constant γ, λ : for any t_0 and x_0, the corresponding soln satisfies

$$\|x(t)\| \leq \gamma e^{-\lambda(t-t_0)} \|x_0\|, \quad t \geq t_0. \qquad (9.7)$$

Once again $\gamma \geq 1$ and the adjective "uniform" refers to the fact that γ and λ are independent of t_0.



"Figure showing uniformly exponential stability where γ & λ are independent of t_0."

Theorem 9.5

The linear state eqn (9.1) is uniformly exponentially stable iff $\exists$ finite positive constants γ and λ:

$$\|\Phi(t,\tau)\| \leq \gamma e^{-\lambda(t-\tau)} \qquad (9.8)$$

for all $t, \tau : t \geq \tau$.

The following theorem is a rather surprising characterization of uniform exponential stability:

<u>Theorem 9.6</u>

Let $A(t)$ be bounded on $(-\infty, \infty)$, then any of the following four statements is equivalent to uniform exponential stability of (9.1):

(i) $\exists\ M_1 > 0 : \int_{t_0}^{t_1} \|\phi(t, t_0)\|^2 \, dt \leq M_1 \quad \forall\ t_1 \geq t_0$ (9.9)

(ii) $\exists\ M_2 > 0 : \int_{t_0}^{t_1} \|\phi(t, t_0)\| \, dt \leq M_2 \quad \forall\ t_1 \geq t_0$ (9.10)

(iii) $\exists\ M_3 > 0 : \int_{t_0}^{t_1} \|\phi(t_1, \sigma)\|^2 \, d\sigma \leq M_3 \quad \forall\ t_1 \geq t_0$ (9.11)

(iv) $\exists\ M_4 > 0 : \int_{t_0}^{t_1} \|\phi(t_1, \sigma)\| \, d\sigma \leq M_4 \quad \forall\ t_1 \geq t_0.$ (9.12)

Remark:

These four conditions, at first sight, seem less demanding for uniform, exponential stability.

When $A(t)$ is independent of t, we have

$$\phi(t,\sigma) = e^{A(t-\sigma)} \qquad (9.13)$$

we have the following corollary.

<u>Corollary 9.7</u>

The homogeneous linear time invariant system

$$\dot{\underline{x}} = A\underline{x}, \quad \underline{x}(t_0) = \underline{x}_0 \qquad (9.14)$$

is exponentially stable iff

$$\exists M : \int_0^{\infty} \| e^{At} \| \, dt < M. \qquad (9.15)$$

For time invariant systems, the adjective uniform is superfluous.

For L.T.I. systems, the characterization (9.15) can be improved considerably as the following theorem states:

Theorem 9.8

The LTI system (9.14) is exponentially stable iff all eigenvalues of A have negative real parts.

$\blacksquare$

Remark: The LTI system (9.14) is exponentially stable iff $\lim_{t\to\infty} e^{At} = 0$.

The corresponding statement is not true about (9.1) in general. For example even when

$$\lim_{t\to\infty} \phi(t,\tau) = 0 \qquad (9.16)$$

for every τ, (9.1) is not necessarily uniformly exponentially stable.

Example 9-9

Consider

$$\dot{x} = -\frac{2t}{t^2+1}\, x(t) \qquad (9.17)$$

$$\phi(t,\tau) = \frac{\tau^2+1}{t^2+1} \qquad (9.18)$$

For any τ, we have $\underset{t\to\infty}{Lt}\ \phi(t,\tau) = 0$.

However (9.17) is not uniformly exponentially stable ie $\nexists\ \gamma, \lambda$:

$$\frac{\tau^2+1}{t^2+1} \leq \gamma e^{-\lambda(t-\tau)} \quad \forall\, t,\tau : t \geq \tau, \qquad (9.19)$$

for otherwise at $\tau = 0$ we have

$$1 \leq (t^2+1)\,\gamma e^{-\lambda t}, \quad t \geq 0. \qquad (9.20)$$

The inequality (9.20) cannot be true because the r.h.s $\to 0$ as $t \to \infty$.

∎

Problems

① For what ranges of real parameter α are the following scalar linear state equations uniformly stable? Uniformly exponentially stable?

ⓐ $\dot{x} = \alpha t \, x(t)$.

ⓑ $\dot{x}(t) = \dfrac{\alpha e^{-t}}{e^{-t}+1} x(t)$.

② Determine if the Linear state Eqn

$$\dot{\underline{x}} = \begin{pmatrix} a(t) & 1 \\ 0 & -1 \end{pmatrix} \underline{x}(t)$$

is uniformly exponentially stable for $a(t) =$

(i) 0 (ii) -1 (iii) $-t$ (iv) $-e^{-t}$

(v) $\begin{cases} -1 & t < 0 \\ -e^{-t}, & t \geq 0. \end{cases}$

3. Is the linear state equation

$$\dot{\underline{x}}(t) = \begin{pmatrix} 1 & e^{-t} \\ -e^{-t} & 1 \end{pmatrix} \underline{x}(t)$$

uniformly stable.

4. Show that

$$\dot{\underline{x}}(t) = \begin{pmatrix} -1 & 0 \\ -e^{-3t} & -1 \end{pmatrix} \underline{x}(t)$$

is not uniformly exponentially stable.

5. If $A(t) = -A^T(t)$,
show that the L. State eq^n (9.1) is uniformly stable. Show also that $P(t) = \phi(t,0)$ is a Lyapunov Transformation.

Remark:

There is a concept of stability that we did not quite emphasize. It is called "Uniform Asymptotic Stability"

Def:

The linear state equation (9.1) is called uniformly asymptotically stable if it is uniformly stable and if given any $\delta > 0$ $\exists$ a positive time T : for any t_0 & x_0 we have

$$\|x(t)\| \leq \delta \|x_0\|, \quad t \geq t_0 + T.$$

Once again, T is independent of the choice of t_0 and depends only on δ.

The following is a rather remarkable result:

Theorem: 9.10

The linear state equation (9.1) is uniformly asymptotically stable iff it is uniformly exponentially stable.

$\blacksquare$

Thus for linear homogeneous systems, uniform asymptotic & uniform exponential stability are equivalent.

So far we have described

uniform stability

and

uniform exponential stability

for linear state space systems of the form (9.1). The theorems describe how to check these stability conditions using the fundamental matrix.

We would like to derive alternative conditions to test for stability. For linear Time Invariant system, one such condition is provided by Theorem 9.8.

In view of Theorem 9.8, one can ask the following question.

Q 9.10

This is a constant A

Under what condition is it true that the eigenvalues of a matrix A have negative real parts?

The answer to this question is provided by Routh Hurwitz condition which has been described in the Appendix.

Let us do the following exercise:

Exercise 9.11

Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix}$$

characteristic polynomial of A is given by

$$p(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0.$$

The Hurwitz matrix is given by

$$\begin{pmatrix} a_2 & a_0 & 0 \\ 1 & a_1 & 0 \\ 0 & a_2 & a_0 \end{pmatrix}$$

Condition under which $p(\lambda)$ has roots with negative real parts is given by

$$a_2 > 0, \quad \begin{vmatrix} a_2 & a_0 \\ 1 & a_1 \end{vmatrix} > 0, \quad \begin{vmatrix} a_2 & a_0 & 0 \\ 1 & a_1 & 0 \\ 0 & a_2 & a_0 \end{vmatrix} > 0$$

ie

$$a_2 > 0 \quad a_2 a_1 > a_0, \quad a_0 \begin{vmatrix} a_2 & a_0 \\ 1 & a_1 \end{vmatrix} > 0$$

ie $a_2 > 0$, $a_0 > 0$, $a_2 a_1 > a_0$.

ie $\boxed{a_2 > 0, \ a_1 > a_0/a_2, \ a_0 > 0}$

Exercise:

Under what condition does the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{pmatrix}$$

(a) has all roots in the left half of the complex plane

(b) has all roots with real parts < -5.

$\square$

Routh Hurwitz condition settles the story about Linear Time Invariant systems in a very elegent way as far as checking uniform asymptotic/exponential stability is concerned.

$\boxed{\text{Lec 9}}$

Input Output Stability

We consider a linear dynamical system (possibly time varying) of the form

$$\dot{\underline{X}}(t) = A(t)\,\underline{X}(t) + B(t)\,u(t).$$

$$y(t) = C(t)\,\underline{X}(t). \qquad (9.21)$$

where we assume that $\underline{X}(t_0) = 0$.

The input-output behavior of (9.21) is completely specified by the impulse response function.

$$h(t,\sigma) = C(t)\,\phi(t,\sigma)\,B(\sigma). \quad (9.22)$$

for $t \geq \sigma$.

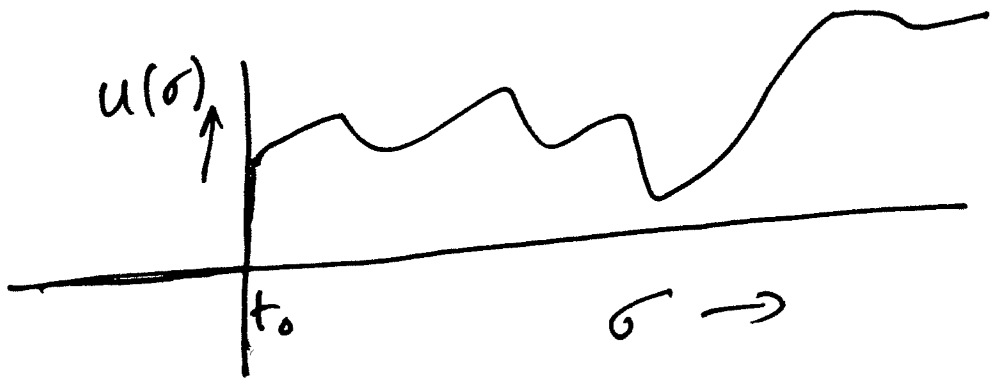
and we have

$$y(t) = \int_{t_0}^{t} h(t,\sigma)\,u(\sigma)\,d\sigma \qquad (9.23)$$

In (9.23) we assume

$$u(\sigma) = 0 \quad \sigma < t_0$$

$y(t)$ is the response to an input which is non-zero in the interval $[t_0, \infty)$



Def (BIBO)

A linear state eqn (9.21) is called uniformly B.I.B.O. stable if $\exists$ a finite constant η: for any 't_0' and any input signal $u(t)$ the corresponding zero state response $y(t)$ satisfies.

$$\sup_{t \geq t_0} \|y(t)\| \leq \eta \sup_{t \geq t_0} \|u(t)\| . \quad (9.24)$$

Remark:

The adjective "uniform" refers to the fact that η is independent of t_0 and choice of $u(t)$.

Example

Consider a single input single output (SISO) linear time invariant system.

$$\dot{x} = Ax + bu,\ y = cx,\ x(0) = 0, \qquad (9.25)$$

we get

$$y(t) = \int_0^t c e^{A(t-\sigma)} b\, u(\sigma)\, d\sigma \qquad (9.26)$$

where

$$h(t,\sigma) = \bar{h}(t-\sigma) = c e^{A(t-\sigma)} b \quad (9.27)$$

is the impulse response function.

Let us construct a f^n $u(\cdot)$ as follows:

$$\begin{aligned} u(t) &= 0 \qquad t < 0 \\ &= 1 \qquad \text{when } t > 0 \ \& \ \bar{h}(t) > 0 \qquad (9.28) \\ &= -1 \qquad \text{when } t > 0 \ \& \ \bar{h}(t) < 0. \end{aligned}$$

with the choice of $u(\cdot)$ defined in (9.28), we have

$$y(t) = \int_0^t |c e^{A(t-\sigma)} b| \, d\sigma \quad (9.29)$$

where

$|\cdot|$ refers to absolute value.

Note that $u(\cdot)$ defined in (9.28) is bounded. Hence for (9.25) to be BIBO stable $y(t)$ defined in (9.29) must also be bounded. Thus a necessary condition for BIBO stability is that

$$\sup_{t \geq 0} \int_0^t |c e^{A(t-\sigma)} b| \, d\sigma < \infty \quad (9.30)$$

ie the impulse response f^n is absolutely summable.

Conversely, assume that (9.30) is satisfied and that the input is bounded i.e. $\exists\, M$:

$$\|u(\sigma)\| \leq M, \quad 0 \leq \sigma < \infty \qquad (9.31)$$

it follows that $y(t)$ is also bounded for otherwise if we let $y(t)$ to be unbounded we have from (9.26) the following

$$\|y(t)\| \leq \int_0^t |c e^{A(t-\sigma)} b| \, \|u(\sigma)\| \, d\sigma$$

$$\leq \int_0^t |c e^{A(t-\sigma)} b| \, d\sigma \cdot M .$$

$$(9.32)$$

$\therefore$ the l.h.s is unbounded, it follows that the r.h.s. is also unbounded ie

$\underset{t \to \infty}{Lt} \int_0^t |c e^{A(t-\sigma)} b| \, d\sigma = \infty$, which contradicts (9.30).

$\blacksquare$

Theorem (BIBO stability of linear time invariant system)

$\boxed{\text{Theorem 1}}$

The L.T.I. system (9.25) is BIBO stable

iff

$$\int_0^\infty \|ce^{At}b\| \, d\sigma < \infty \quad (9.33).$$

Remark: For multiple input multiple outputs systems, $ce^{At}B$ is a matrix and one needs to take a suitable norm.

The above theorem can be generalized to linear time varying systems as follows:

Theorem 2 (BIBO stability of linear time varying system)

The linear time varying system (9.21) is BIBO stable iff $\exists \rho > 0$:

$$\int_{-\infty}^{t} \|h(t,\sigma)\| \, d\sigma \le \rho, \text{ for all } t.$$

It turns out that if (9.25) is controllable and observable, then theorem 1 can be refined as follows:

Theorem 3: Assume that (9.25) is C&O.

I The LTI system (9.25) is BIBO stable iff

II $\int_0^\infty \| c e^{At} b \| d\sigma < \infty$ (9.34)

iff

III $\underset{t \to \infty}{Lt}\, e^{At} = 0$ (9.35)

iff

IV All eigenvalues of A have negative real parts.

iff

V The homogeneous LTI system (9.14) is exponentially stable iff

VI $\int_0^\infty \|e^{At}\| \, dt < \infty$ (9.36)

Proof of Theorem 3

I $\Longleftrightarrow$ II follows from Theorem 1 on page 9.24

III $\Rightarrow$ IV $\Rightarrow$ V $\Rightarrow$ VI
follows from our discussion on exponential stability of linear time invariant system.

To show VI $\Rightarrow$ II :

Note that

$$\int_0^\infty \|c e^{A\sigma} b\| \, d\sigma \le \|c\| \, \|b\| \int_0^\infty \|e^{A\sigma}\| \, d\sigma$$

(9.34) clearly follows from (9.36).

(9.27)

To show

$\underline{\text{II}} \Rightarrow \underline{\text{III}}$

$$\int_0^\infty \| c e^{At} b \| d\sigma < \infty$$

$$\Rightarrow \operatorname*{Lt}_{t \to \infty} c e^{At} b = 0.$$

Define $\phi(t) = c e^{At} b$ where

$$\operatorname*{Lt}_{t \to \infty} \phi(t) = 0.$$

It follows that

$$\operatorname*{Lt}_{t \to \infty} \phi^{(n)}(t) = 0$$

where $\phi^{(n)}(t)$ is the n^{th} derivative of $\phi(t)$.

Thus

$$\operatorname*{Lt}_{t \to \infty} \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} e^{At} \left(B \ AB \ \cdots \ A^{n-1}B \right) = 0. \qquad (9.37)$$

(9.28)

$\therefore$ (9.25) is C&O, it follows

from (9.37) that.

$$\underset{t\to\infty}{Lt}\; N e^{At} M = 0$$

where N is a $n\times n$ invertible submatrix of

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

where M is a $n\times n$ invertible submatrix of

$$(B \;\; AB \;\; \cdots \;\; A^{n-1}B).$$

Hence

$$\underset{t\to\infty}{Lt}\; e^{At} = 0.$$

$\blacksquare$

$$\boxed{\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx, \quad x(0) = 0\end{aligned}} \quad (*)$$

$$\boxed{\begin{aligned}\dot{x} &= Ax \\ x(0) &= x_0\end{aligned}} \quad (**)$$

Assume that (*) & (**) are LTI and that (*) is C&O. An important corollary of Theorem 3 is that

$$\boxed{\begin{gathered}(*) \text{ is BIBO stable} \\ \text{iff} \\ (**) \text{ is exponentially stable}\end{gathered}}$$

"important corollary"

A corollary similar to page 9·29 can also be written for Linear time varying system. However it is not enough to simply assume controllability and observability.

$$\boxed{\begin{aligned} \dot{x} &= A(t)\,x(t) + B(t)\,u(t) \\ y(t) &= C(t)\,x(t)\,, \quad x(t_0) = \cancel{x_0}\ 0 \end{aligned}} \quad (*)$$

$$\boxed{\begin{aligned} \dot{x} &= A(t)\,x(t) \\ x(t_0) &= x_0 \end{aligned}} \quad (**)$$

Assume that $\exists\ \alpha, \beta, \mu$:

$$\|A(t)\| < \alpha, \quad \|B(t)\| < \beta, \quad \|C(t)\| < \mu.$$

$$\forall t.$$

i.e. the matrices are all assumed to be uniformly bounded.

Recall that

$$W(t_0, t_f) = \int_{t_0}^{t_f} \phi(t_0, t) B(t) B^T(t) \phi^T(t_0, t) dt$$

"controllability gramian".

$$M(t_0, t_f) = \int_{t_0}^{t_f} \phi^T(t, t_0) C^T(t) C(t) \phi(t, t_0) dt$$

"observability gramian"

Assume that $\exists\, \varepsilon_1, \delta_1, \varepsilon_2, \delta_2$:

$$\left.\begin{aligned} \varepsilon_1 I &\leq W(t-\delta_1, t) \\ \varepsilon_2 I &\leq M(t, t+\delta_2) \end{aligned}\right\} \forall t.$$

Def: If A & B are two symmetric matrices. then we say $A \geq B$ if $A - B$ is positive definite

<u>Corollary</u>:

If $A(t), B(t), C(t)$ satisfy the conditions on pages 9.30 & 9.31 we have the following:

(*) is uniformly BIBO stable

iff

(**) is uniformly exponentially stable.

Remark: I am not saying that checking the positive definiteness condition is going to be easy and I am not telling you how to do that either.

Appendix

HO2·5

The Routh Hurwitz stability criterion:

$$P_n(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n$$

Consider $P_n(\lambda)$ a monic polynomial with real co-efficients.

Form the determinants

$$\Delta_1 = a_1$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}$$

$$\Delta_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ 1 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 1 & a_2 & a_4 \end{vmatrix}$$

$$\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2n-1} \\ 1 & a_2 & a_4 & \cdots & a_{2n-2} \\ 0 & a_1 & a_3 & \cdots & a_{2n-3} \\ 0 & 1 & a_2 & \cdots & a_{2n-4} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix}$$

where $a_k = 0 \quad k > n$.

Theorem (Routh - Hurwitz)

Every zero of $P_n(\lambda)$ has a negative real part if and only if $\Delta_r > 0$ for $r = 1, 2, \ldots, n$.

H02·7

<u>Ex 1</u> $n=2$

$$P_2(\lambda) = \lambda^2 + a_1\lambda + a_2$$

$$\Delta_1 = a_1$$

$$\Delta_2 = \begin{vmatrix} a_1 & 0 \\ 1 & a_2 \end{vmatrix}$$

Every zero of $P_2(\lambda)$ has a negative real part iff

$$\Delta_1 > 0 \ \& \ \Delta_2 > 0$$

$$\Rightarrow a_1 > 0 \ \& \ a_1 a_2 > 0$$

$$\Rightarrow a_1 > 0, \ a_2 > 0$$