Lec 8

The story of observability and impulse response

1. Let us start with

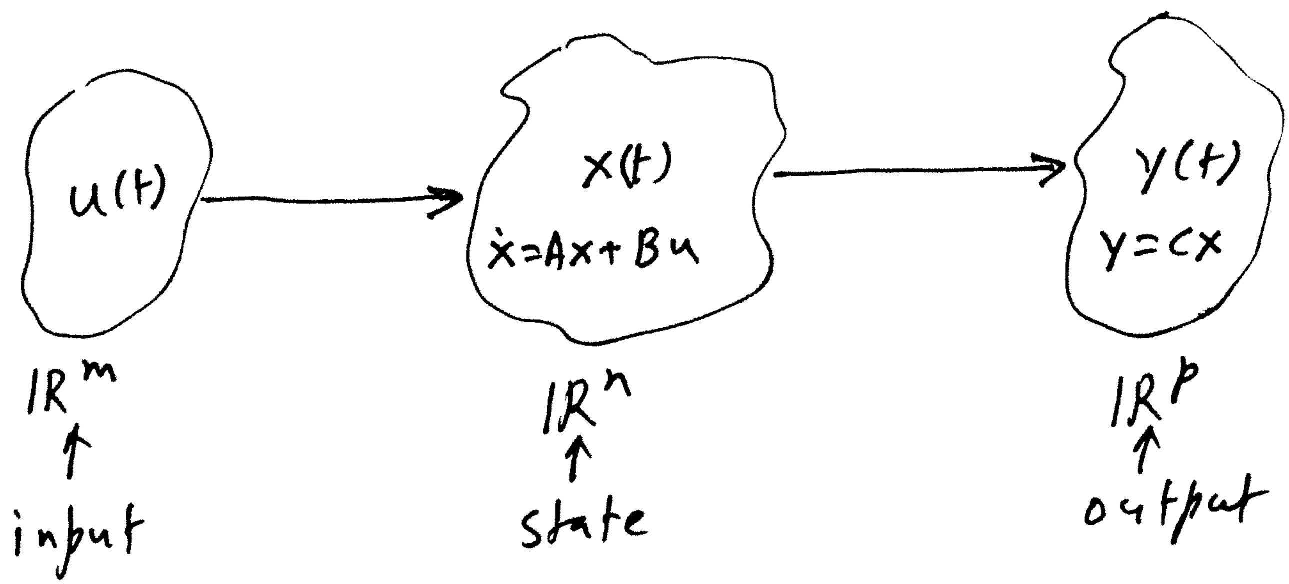
$$\dot{\underline{x}} = A(t)\underline{x}(t) + B(t)u(t), \quad \underline{x}(0) = \underline{x}_0 \qquad (8.1)$$

where we assume that $A(t)$ is $n \times n$
$B(t)$ is $n \times m$

We now add an observation equation

$$y(t) = C(t)\underline{x}(t) \qquad (8.2)$$

where $C(t)$ is $p \times n$, $y(t) \in \mathbb{R}^p$



The equations (8.1), (8.2) taken together is going to be called a

Linear Dynamical System

with m ~~p~~ inputs and p outputs, with n states.

Remark: When $m = p = 1$, the linear dynamical system would be called "single input single output." When either m or p is > 1, the linear dynamical system would be called "multi input multi output". They are ~~so~~ acronymed as SISO & MIMO for short.

2. In view of the variation of constants formula, we can explicitly write down $y(t)$ as follows:

$$\underline{X}(t) = \phi(t,0)\,\underline{X}(0) + \int_0^t \phi(t,\sigma)\,B(\sigma)\,u(\sigma)\,d\sigma \tag{8·3}$$

$$y(t) = C(t)\,\phi(t,0)\,\underline{X}(0) + \int_0^t C(t)\,\phi(t,\sigma)\,B(\sigma)\,u(\sigma)\,d\sigma. \tag{8·4}$$

watch out for 't' & 'σ'.

3. Example:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u(t)$$

$$y(t) = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$\phi(t,0) = \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} \qquad \phi^{-1}(t,0) = \begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix}$$

$$y(t) = C(t)\,\phi(t,0)\left[\underline{X}(0) + \int_0^t \phi(\sigma,0)^{-1}\,B(\sigma)\,u(\sigma)\,d\sigma\right]$$

$$C(t)\,\phi(t,0) =$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t \end{pmatrix}$$

$$\int_0^t \phi(\sigma,0)^{-1} B(\sigma)\, u(\sigma) \quad d\sigma$$

$$= \int_0^t \begin{pmatrix} 1 & -\sigma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(\sigma)\, d\sigma$$

$$= \int_0^t \begin{pmatrix} -\sigma \\ 1 \end{pmatrix} u(\sigma)\, d\sigma$$

$$\therefore\ y(t) = \begin{pmatrix} 1 & t \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \int_0^t (t-\sigma)\, u(\sigma)\, d\sigma$$

$$\boxed{y(t) = x_1(0) + t\, x_2(0) + \int_0^t (t-\sigma)\, u(\sigma)\, d\sigma}$$

Remark: There are two parts to $y(t)$.

$y_1(t) = x_1(0) + t\, x_2(0)$ ← Depends on $\underline{x}(0)$
Does not depend on $u(t)$.

$y_2(t) = \int_0^t (t-\sigma)\, u(\sigma)\, d\sigma$ ← Depends on $u(t)$
Does not depend on $\underline{x}(0)$.

4. In general, we can write

$$y(t) = y_1(t) + y_2(t) \qquad (8\cdot5)$$

where

$$y_1(t) = C(t)\,\phi(t,0)\,\underline{x}(0) \qquad (8\cdot6)$$

$$y_2(t) = \int_0^t C(t)\,\phi(t,\sigma)\,B(\sigma)\,u(\sigma)\,d\sigma \qquad (8\cdot7)$$

initial condition response with zero input

input response with zero initial condions.

Remark:
We would like to talk about $y_1(t)$ & $y_2(t)$ in some more details.

5. Let us start with $y_2(t)$.

Q: compute $y_2(t)$ if $u(t)$ is an unit impulse at time $t=0$, ie $u(t)=\delta(t)$.

$$y_2(t) = C(t)\phi(t,0)\int_0^t \phi(0,\sigma)B(\sigma)\delta(\sigma)\,d\sigma$$

$$= [C(t)\phi(t,0)][\underbrace{\phi(0,0)}_{= I}B(0)]$$

$$= C(t)\phi(t,0)B(0). \qquad (8.8)$$

"$C(t)\phi(t,0)B(0)$" is an impulse response of the linear dynamical system (8.1), (8.2) corresponding to unit impulse at $t=0$.

Remark: show that "$C(t)\phi(t,t_0)B(t_0)$" is an impulse response of the l.d.s. (8.1)(8.2) corresponding to unit impulse at $t=t_0$.

⑥ If we define

$$h(t,\sigma) = C(t)\,\phi(t,\sigma)\,B(\sigma)$$

impulse response due to an unit impulse at $t=\sigma$.

we obtain

$$y_2(t) = \int_0^t h(t,\sigma)\,u(\sigma)\,d\sigma. \qquad (8.9)$$

Thus unit impulse response f^n $h(t,\sigma)$ completely characterizes $y_2(t)$, response due to an arbitrary input $u(\sigma)$.

⑦ If $A(t)$, $B(t)$ & $C(t)$ are constant matrices we have

$$\dot{\underline{x}} = A\underline{x} + Bu$$

$$y(t) = C\underline{x}$$

$$\phi(t,\sigma) = e^{A(t-\sigma)}$$

$$h(t,\sigma) = C e^{A(t-\sigma)} B \triangleq h_0(t-\sigma)$$

Thus (8.9) reduces to

$$Y_2(t) = \int_0^t h_0(t-\sigma)\, u(\sigma)\, d\sigma \qquad (8.10)$$

Remark: Note that $h_0(\cdot)$ is a f^n of only one variable. If we define

$$h_0(t) = C e^{At} B \qquad (8.11)$$

then

$$Y_2(t) = \int_0^t h_0(t-\sigma) u(\sigma)\, d\sigma \triangleq \underbrace{h_0(t) ** u(t)}_{\text{convolution of } h_0(t) \,\&\, u(t).} \qquad (8.12)$$

Def (Convolution)

Let $f(t)$ & $g(t)$ be two f^{ns} of t. We define

$$s(t) = f(t) ** g(t) = \int_0^t f(t-\sigma)\, g(\sigma)\, d\sigma.$$

If $f(t)$ & $g(t)$ are scalars, we can talk about

$$g(t) ** f(t) = \int_0^t g(t-\sigma) f(\sigma)\, d\sigma$$

and it is easy to see that

$$f(t) ** g(t) = g(t) ** f(t).$$

8 Example on page 8.3.

$$h(t,\sigma) =$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t-\sigma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = t-\sigma = h_0(t-\sigma)$$

∴ The impulse response f^n $h_0(t) = t$

$$Y_2(t) = h_0(t) ** u(t)$$

$$= \int_0^t (t-\sigma)\, u(\sigma)\, d\sigma$$

we already have this on page 8.4.

(9) Conclusion so far.

For a linear time invariant system

$$\dot{x} = Ax + Bu, \quad y = Cx$$

the impulse response f^n $h_o(\cdot)$ is a f^n of one variable and defined as

$$h_o(t) = Ce^{At}B$$

For a linear time varying system

$$\dot{x} = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t);$$

the impulse response f^n $h(\cdot,\cdot)$ is a f^n of two variables and is defined as

$$h(t,\sigma) = C(t)\,\phi(t,\sigma)\,B(\sigma).$$

(10)

Consider the L.D.S.

$$\dot{\underline{x}}(t) = A(t)\,\underline{x}(t) + B(t)u(t)$$
$$y(t) = C(t)\,\underline{x}(t) \qquad (8.13)$$

where

$$A(t) = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \qquad (8.14)$$

$$B(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C(t) = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Calculate $h(t,\sigma)$, the impulse response f^A.

Sol^n: From H.W.3 (Answers) page 6 we know that

8.12

$\phi(t,0) =$

$$e^{\alpha(t)} \begin{bmatrix} \cos\beta(t) & \sin(\beta(t)) \\ -\sin\beta(t) & \cos\beta(t) \end{bmatrix} \qquad (8.15)$$

where

$$\alpha(t) = \frac{\sin \omega t}{\omega}, \quad \beta(t) = \frac{1-\cos\omega t}{\omega}$$

Writing $\phi(t,\sigma)$ as $\phi(t,0)\,\phi(\sigma,0)^{-1}$.
we obtain

$\phi(t,\sigma) =$

$$e^{[\alpha(t)-\alpha(\sigma)]} \begin{bmatrix} \cos[\beta(t)-\beta(\sigma)] & \sin[\beta(t)-\beta(\sigma)] \\ -\sin[\beta(t)-\beta(\sigma)] & \cos[\beta(t)-\beta(\sigma)] \end{bmatrix}$$

Hence
$$h(t,\sigma) = \sin[\beta(t)-\beta(\sigma)] + \cos[\beta(t)-\beta(\sigma)].$$

where

$$\beta(t) - \beta(\sigma) =$$

$$\frac{1-\cos\omega t}{\omega} - \frac{1-\cos\omega\sigma}{\omega}$$

$$= \frac{\cos\omega\sigma - \cos\omega t}{\omega}$$

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Consider the Linear Dynamical system

$$\dot{\underline{x}} = A(t)\,\underline{x}(t) + b\,u(t) \qquad (8.18)$$

$$y = c\,\underline{x}(t)$$

where

$$A(t) = \begin{pmatrix} -1+\cos t & 0 \\ 0 & -2+\cos t \end{pmatrix} \qquad (8.19)$$

$$b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \; ; \quad c = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Calculate $h(t,\sigma)$ the impulse response $f^{\underline{n}}$.

Sol$^{\underline{n}}$: From H.W.3 Q2 we know that

$$\phi(t,0) = e^{\begin{pmatrix}-1 & 0\\ 0 & -2\end{pmatrix}t}\, e^{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\sin t}$$

Hence

$$\phi(t,\sigma) = e^{\begin{pmatrix}-1 & 0\\ 0 & -2\end{pmatrix}t}\, e^{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\sin t}\, e^{-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\sin\sigma}\, e^{\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\sigma}$$

$$= e^{\begin{pmatrix}-1 & 0\\ 0 & -2\end{pmatrix}(t-\sigma)}\, e^{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}[\sin t-\sin\sigma]}$$

$$= \begin{pmatrix} e^{-(t-\sigma)+(\sin t-\sin\sigma)} & 0 \\ 0 & e^{-2(t-\sigma)+(\sin t-\sin\sigma)} \end{pmatrix}$$

$$\therefore \boxed{h(t,\sigma) = e^{-2(t-\sigma)}\, e^{(\sin t-\sin\sigma)}}$$

(8.16)

(11) We now proceed to talk about $Y_1(t)$, defined in (8.6). Let us consider the following problem.

Q: In (8.1), (8.2) assume that $\underline{X}_0$ is unknown and we assume that $A(t)$, $B(t)$, $C(t)$, $u(t)$, $y(t)$ are all known. Can we recover $\underline{X}_0$ from this data.

Ans: Since $y(t)$ & $Y_2(t)$ are separately known we can assume that $Y_1(t)$ is known.

Since $A(t)$ is known, we also assume that $\phi(t,0)$ is known.

Thus, in the eqn

$$y_1(t) = C(t)\,\phi(t,0)\,\underline{x}(0) \qquad (8.18)$$

we assume that $y_1, C, \phi(t,0)$ are known and we want to calculate (if possible) $\underline{x}(0)$ from this data

Switching t to σ in (8.18) we write

$$\phi^T(\sigma,0)\,C^T(\sigma)\,y_1(\sigma) = \qquad (8.19)$$
$$\phi^T(\sigma,0)\,C^T(\sigma)\,C(\sigma)\,\phi(\sigma,0)\,\underline{x}(0)$$

which implies

$$\int_0^T \phi^T(\sigma,0)\,C^T(\sigma)\,y_1(\sigma)\,d\sigma =$$
$$\left[\int_0^T \phi^T(\sigma,0)\,C^T(\sigma)\,C(\sigma)\,\phi(\sigma,0)\right]\underline{x}(0)$$
$$(8.20)$$

for some $T > 0$.

Remark:

(8·20) appears to be considerably more complicated compared to (8·18), which is what we had started from, except that it is not. If we define

$$\xi = \int_0^T \phi^T(\sigma, 0)\, C^T(\sigma)\, Y_1(\sigma)\, d\sigma$$

& $$M(0,T) = \int_0^T \phi^T(\sigma, 0)\, C^T(\sigma)\, C(\sigma)\, \phi(\sigma, 0)\, d\sigma \quad (8\cdot 21)$$

we can rewrite (8·20) as.

$$M(0,T)\, \underline{X}(0) = \xi \quad (8\cdot 22).$$

If $M(0,T)$ is invertible, we can write

$$\underline{X}(0) = M(0,T)^{-1}\xi \quad (8.23)$$

and solve for $\underline{X}(0)$ uniquely this way.

If $M(0,T)$ is not invertible, then (8.22) cannot be solved for $\underline{X}(0)$ uniquely and it turns out that (8.18) cannot be solved for $\underline{X}(0)$ uniquely either.

If we define a map L as follows

$$\begin{aligned} L : \mathbb{R}^n &\longrightarrow C^p[0,T] \\ x_0 &\longmapsto C(t)\phi(t,0)x_0 \end{aligned} \quad (8.24)$$

In order to solve X_0 uniquely, L must be a 1-1 map, ie Null space of L must be trivial. Here is a surprising but not totally unexpected theorem:

<u>Theorem: 8.1</u>

Null space of L conicides with the null space of $M(0,T)$.

Def: If $M(0,T)$ is invertible, we shall call the L.D.S. (8.1), (8.2) to be observable.

Remark: Note that $M(0,T)$ does not depend on $B(t), u(t)$ i.e observability does not depend on the choice of control.

Hence w.a.l.o.g we assume that $u(t) \equiv 0$ and start from the homogeneous system.

$$\dot{x}(t) = A(t)\, x(t) \qquad x(0) = x_0 \qquad (8.25)$$
$$y(t) = C(t)\, x(t)$$

(12) Theorem 8.1 is actually quite easy to prove and we might just do it now.

Proof of Theorem 8.1

Assume that the null space of L is non-trivial ie $\exists\, v : L(v) = 0 \quad v \neq 0$. It follows that

$$C(\sigma)\,\phi(\sigma,0)\, v = 0 \quad \forall \sigma \in [0,T]. \qquad (8.26)$$

From (8-26) we write

$$\left[\int_0^T \phi^T(\sigma,0)\, C^T(\sigma)\, C(\sigma)\, \phi(\sigma,0)\, d\sigma\right] v = 0 \qquad (8\cdot 27).$$

$\Rightarrow M(0,T)\, v = 0.$

$\Rightarrow$ Null space of $M(0,T)$ is non-trivial.

Conversely, assume that $\exists\, w \neq 0$:

$M(0,T)\, w = 0.$

$$\Rightarrow \left[\int_0^T \phi^T(\sigma,0)\, C^T(\sigma)\, C(\sigma)\, \phi(\sigma,0)\, d\sigma\right] w = 0$$

$$= \int_0^T \left[w^T \phi^T(\sigma,0)\, C^T(\sigma)\right]\left[C(\sigma)\, \phi(\sigma,0)\, w\right] = 0 \quad (8\cdot 28)$$

$$= \int_0^T \| C(\sigma)\, \phi(\sigma,0)\, w \|^2\, d\sigma = 0$$

Because $C(\cdot)$ & $\phi(\cdot, 0)$ are continuous fn's of σ we have

$$C(\sigma)\,\phi(\sigma, 0)\,w \equiv 0 \quad \forall \sigma \in [0, T]. \tag{8·29}$$

From (8·29) we conclude that the null space of L is nontrivial.

Remark: $M(0,T)$ is called the observability gramian and characterizes to what extent initial condition $x(0) = x_0$ can be observed from $y(t)$ or equivalently $y_1(t)$.

Corollary: 8.2

Define the set

$$\mathcal{S} = \{ x : x - x_0 \in \text{Null space of } M(0,T) \}.$$

it would follow that the L.D.S.

$$\dot{x} = A(t)\,x(t), \quad y(t) = C(t)\,x(t)$$

$$x(0) = \xi.$$

would have identical output $y(t)$ for any $\xi \in \mathcal{S}$, i.e. there is no way to distinguish between two initial conditions in $\mathcal{S}$.

Proof of corollary 8.2

Let ξ_1 & ξ_2 be two vectors in $\mathcal{S}$. Define

$$y_{\xi_1}(t) = C(t)\,\phi(t,0)\,\xi_1$$

$$y_{\xi_2}(t) = C(t)\,\phi(t,0)\,\xi_2$$

We conclude that

$$y_{\xi_1}(t) - y_{\xi_2}(t) =$$

$$C(t)\,\Phi(t,0)\,[\xi_1 - \xi_2] = 0$$

$\therefore\ \xi_1 - \xi_2 \in$ Null space of L defined in (8.24).

$$\therefore\ y_{\xi_1}(t) = y_{\xi_2}(t)$$

$\blacksquare$

13 Some comments on the Gramians.

For the L.D.S. we have the following two gramians:

$$W(0,T) = \int_0^T \phi(0,\sigma) B(\sigma) B^T(\sigma) \phi(0,\sigma)^T d\sigma \quad (8.30)$$

(controllability gramian)

$$M(0,T) = \int_0^T \phi^T(\sigma,0) C^T(\sigma) C(\sigma) \phi(\sigma,0) d\sigma. \quad (8.31)$$

(observability gramian).

It is an easy exercise to verify that for the LDS

$$\dot{\mathbb{X}} = -A^T(t)\,\mathbb{X}(t) + C^T(t)\,u(t) \qquad \mathbb{X}(0) = x_0 \quad (8.32).$$

$$Y(t) = B^T(t)\,\mathbb{X}(t)$$

which is a p input m output L.D.S,

$M(0,T)$ is the associated controllability gramian and $W(0,T)$ is the associated observability gramian.

To see this, note that if $\phi(t,0)$ is the fundamental matrix of (8.1), we have

$$\dot{\phi}(t,0) = A(t)\phi(t,0).$$

ie

$$\frac{d}{dt}\left[\phi^{-1}(t,0)\right] = -\phi^{-1}(t,0)\,A(t).$$

$$\Rightarrow \frac{d}{dt}\left[\phi^{-1}(t,0)\right]^T = -A^T(t)\left[\phi^{-1}(t,0)\right]^T$$

$\Rightarrow \left[\phi^{-1}(t,0)\right]^T$ is the fundamental matrix of (8.32).

The conclusion follows quite readily

$\square$

Because of the duality between the two gramians,

(8·1) & (8·32)

are called dual systems, and we have

"(8·1) is controllable iff
(8·32) is observable"

"(8·1) is observable iff
(8·32) is controllable"

This duality comes in quite handy in proving theorems which we now like to describe:

Consider the Linear time invariant system

$$\dot{\underline{x}} = A\underline{x} \qquad (8.33)$$
$$y = C\underline{x}.$$

From Duality we ascertain that (8.33) is observable iff

$$\dot{\underline{x}} = -A^T\underline{x} + C^T u \qquad (8.34)$$

is controllable. In fact, the observability gramian of (8.33) and the controllability gramian of (8.34) are identical and is given by

$$\int_0^T e^{A^T\sigma} c^T c e^{A\sigma} d\sigma \quad (8.35).$$

We now construct the controllability matrix for (8.34) as

$$\mathcal{C} = \left(c^T \,|\, -A^T c^T \,|\, (-A^T)^2 c^T \,|\, \cdots \,|\, (-A^T)^{n-1} c^T \right) \quad (8.36)$$

It is easy to see that $\mathcal{C}$ is of full rank n iff

$$\begin{pmatrix} c \\ cA \\ cA^2 \\ \vdots \\ cA^{n-1} \end{pmatrix} \quad (8.37)$$

is of rank n.

We have the following

Theorem 8.3

The L.T.I. system (8.33) is observable iff

$$\text{rank}\left[\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \right] = n .$$

Theorem 8.4

The null space of the observability gramian matrix of (8.33) coincides with the null space of $\mathcal{O}^T \mathcal{O}$.

Proof: The null space of the observability gramian matrix of (8.33) is the null space of the controllability matrix of (8-34) which coincides with the null space of

$$\left[\, C^T \mid -A^T C^T \mid - - \mid (-A^T)^{n-1} C^T \,\right].$$

$$\left[\qquad \downarrow \qquad\right]^T$$

which coincides with the null space of

$$\left[\, C^T \mid A^T C^T \mid - - \mid (A^T)^{n-1} C^T \,\right]$$

$$\left[\qquad \downarrow \qquad\right]^T$$

This last matrix is $\mathcal{O}^T \mathcal{O}$.

∎

Popov Belevitch Hautus Test for observability.

Theorem:

$$\dot{x} = Ax \qquad x(0) = x_0 \qquad (*)$$
$$y = Cx$$

is observable iff

$$\text{rank} \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} = n$$

$\forall \lambda$ real or complex.

Proof:

We appeal to duality and argue that $(*)$ is observable iff

$$\dot{x} = -A^T x + C^T u$$

is controllable

iff

$$\text{rank} \left[\lambda I + A^T \,\vdots\, C^T \right] = n \qquad (\Delta)$$

$\forall \lambda$ real or complex.

ie iff

$$\operatorname{rank}\begin{pmatrix} \lambda I + A \\ C \end{pmatrix} = n$$

$\forall \lambda$ real or complex.

ie iff

$$\operatorname{rank}\begin{pmatrix} \lambda I - A \\ C \end{pmatrix} = n$$

$\forall \lambda$ real or complex.

The last assertion follows easily by writing $\lambda = -\mu$.

$\blacksquare$

Remark: This proof utilizing duality is not very direct. We sketch an alternative proof:

8.35

Alternative Proof:

Assume that

$$\text{rank}\begin{pmatrix} \lambda I - A \\ C \end{pmatrix} < n$$

ie $\exists\, v \in \mathbb{R}^n,\ v \neq 0$:

$$(\lambda I - A)v = 0$$

& $Cv = 0$

It follows that $Av = \lambda v$ & $Cv = 0$

$$\Rightarrow Ce^{At}v \equiv 0 \quad \forall t.$$

To see this, write

$$e^{At} = \alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1}$$

$$\Rightarrow Ce^{At}v = \alpha_0 Cv + \alpha_1 CAv + \cdots + \alpha_{n-1} CA^{n-1} v$$

$$= \alpha_0 Cv + \alpha_1 \lambda Cv + \cdots + \alpha_n \lambda^{n-1} Cv$$

$$= \left[\alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \cdots + \alpha_{n-1}\lambda^{n-1}\right] Cv$$

$$= 0$$

Hence v is in the null space of L (given by 8·24)

and hence ⊛ is not observable.

Conversely, assume that ⊛ is not observable. It follows that $\exists$ P and

$$x = Pz$$

such that

$$\dot{z} = P^{-1}APz$$

$$y = CPz$$

$$\& \quad P^{-1}AP = \begin{pmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{pmatrix}$$

$$CP = \begin{pmatrix} B_{11} & 0 \end{pmatrix}$$

We did not exactly prove this yet.

Let λ be an eigenvalue of A_{22} with v_2 be the corresponding eigenvector.

8.37

Define $v = \begin{pmatrix} 0 \\ v_2 \end{pmatrix}$

It is easy to see that

$$\begin{pmatrix} \lambda - A_{11} & 0 \\ -A_{12} & \lambda - A_{22} \\ B_{11} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

ie

$$\text{rank} \begin{pmatrix} \lambda I - P^{-1}AP \\ CP \end{pmatrix} < n$$

$$\Rightarrow \text{rank} \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} < n .$$

(8.3

To see the last step, note that

$$\begin{pmatrix} \lambda I - P^{-1}AP \\ CP \end{pmatrix} =$$

$$\begin{pmatrix} P^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} P$$

$\therefore$ rank $P = n$, rank $\begin{pmatrix} P^{-1} & 0 \\ 0 & I \end{pmatrix} = n + p$.

ie they are full rank matrices.

we have the last step.

$\blacksquare$

Appendix

Some Remarks on controllable and observable linear time invariant state space dynamical systems.

Assume that we have

$$\dot{x} = Ax + Bu \qquad (1)$$
$$y = Cx$$

where (1) is both controllable and observable.

Here $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and

A is $n \times n$, B is $n \times m$, C is $p \times n$.

The controllability and Observability matrices are respectively

$$\mathcal{C} = \left(B \,|\, AB \,|\, A^2B \,|\, \cdots \,|\, A^{n-1}B\right) \qquad (2)$$

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} \qquad (3)$$

where $\mathcal{C}$ is $n \times nm$ matrix of rank n

$\mathcal{O}$ is $pn \times n$ " " " n.

The product $\mathcal{O}\mathcal{C}$ is given by

$$\begin{pmatrix} CB & CAB & CA^2B & \cdots & CA^{n-1}B \\ CAB & CA^2B & CA^3B & \cdots & CA^nB \\ CA^2B & CA^3B & CA^4B & \cdots & CA^{n+1}B \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ CA^{n-1}B & CA^nB & CA^{n+1}B & \cdots & CA^{2n-2}B \end{pmatrix} \quad (4)$$

The matrix $\mathcal{O}\mathcal{C}$ is $pn \times mn$ with a block Hankel structure.

$$\begin{pmatrix} H_0 & H_1 & \cdots & H_{n-1} \\ H_1 & H_2 & \cdots & H_n \\ \vdots & \vdots & \cdots & \vdots \\ H_{n-1} & H_n & \cdots & H_{2n-2} \end{pmatrix}$$

We denote the matrix (4) by $\mathcal{H}$ and

verify that

$$\operatorname{rank} \mathcal{H} = n \qquad (5)$$

<u>Exercise:</u>

Show that if A is a $q \times n$ and B is a $n \times l$ matrix, each of rank n where $q \geq n, l \geq n$ then

$$\operatorname{rank} AB = n.$$

<u>Theorem (1)</u>

The dynamical system is controllable and observable iff $\operatorname{rank} \mathcal{H} = n$.

Proof: The necessary part has already been shown. To show sufficiency if

$$\operatorname{rank} \mathcal{C} < n \text{ or } \operatorname{rank} \mathcal{O} < n$$

it follows that

$$\operatorname{rank} \mathcal{H} \leq \min[\operatorname{rank} \mathcal{O}, \operatorname{rank} \mathcal{C}] < n$$

$\blacksquare$

In closing, we would like to comment that the blocks $CB, CAB, \dots, CA^{n-1}B$ etc actually occurs in the impulse response of (1). Recall that the impulse response of (1) is given by

$$H(t) = Ce^{At}B \quad (6)$$

which can be written as

$$H(t) = CB + CABt + CA^2B\frac{t^2}{2!} + \cdots$$

$$= \sum_{j=0}^{\infty} CA^jB\frac{t^j}{j!} \quad (7)$$

$\blacksquare$

Using the matrices A, B, C we can design a matrix rational f^n as follows:

$$G(s) = C(sI - A)^{-1}B. \quad (8).$$

In (8), 's' is to be viewed simply as a variable.

<u>Remark</u>: Later, we would view $G(s)$ as a transfer function of (1), obtained by taking the Laplace Transform of $H(t)$.

For now, note that $G(s)$ can be expanded as

$$G(s) = \frac{CB}{s} + \frac{CAB}{s^2} + \frac{CA^2B}{s^3} + \cdots \quad (9)$$

compare (9) with (7).

Definition: 2

We shall say that the degree of a strictly proper* rational function in s given by

$$G(s) = \sum_{j=0}^{\infty} \frac{H_j}{s^{j+1}} \qquad (10)$$

is q if

$$\operatorname{rank} \begin{pmatrix} H_0 & H_1 & H_2 & \cdots \\ H_1 & H_2 & H_3 & \cdots \\ H_2 & H_3 & H_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} = q.$$

* $G(s)$ is strictly proper if $\underset{s \to \infty}{\mathrm{Lt}}\, G(s) = 0$

Exercise:

(i) Calculate degree of

$$g(s) = \frac{3s+5}{s^2+17s+19}$$

(ii) calculate degree of

$$G(s) = \begin{pmatrix} \frac{1}{s+7} & \frac{2}{s+7} \\ \frac{3}{s+7} & \frac{4}{s+7} \end{pmatrix}$$

Realization Theorem :-

Let $G(s)$ be a $p \times m$ strictly proper rational function of degree q, then $\exists$

a $q \times q$ matrix A

$q \times m$ " B :

$p \times q$ " C

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^q$$

$$y = Cx \qquad u \in \mathbb{R}^m \qquad (11)$$

$$y = \mathbb{R}^p$$

is controllable and observable and

$$C(sI - A)^{-1}B = G(s).$$

Moreove the choice of A, B, C is not unique but is given upto a nonsingular $q \times q$ matrix P as follows:

If A_0, B_0, C_0 is one choice then every other choice is given by $(C_0 P, P^{-1} A_0 P, P^{-1} B_0)$.

Remark:

If we write

$$\dot{x} = A_0 x + B_0 u$$
$$y = C_0 x$$

Define $z = P^{-1} x$, we have.

$$\dot{z} = P^{-1}\dot{x} = P^{-1} A_0 x + P^{-1} B_0 u$$
$$= P^{-1} A_0 P z + P^{-1} B_0 u.$$

$$y = C_0 x = C_0 P z$$

Thus

$$\dot{z} = P^{-1} A_0 P z + P^{-1} B_0 u$$
$$y = C_0 P$$

Thus the realization theorem tells us that a rational f^n of degree q is realizable by a q dimensional state space system uniquely upto choice of basis in the state space. Every such realization is controllable and observable.

Proof of the realization theorem

We start by expanding $G(s)$ in accordance with (10) and write down $H_0, H_1, H_2 \cdots$ as a block Henkel matrix, which by assumption is of rank q. It is an easy exercise to assert that

$$\begin{pmatrix} H_0 & H_1 & H_2 & \cdots \\ H_1 & H_2 & H_3 & \cdots \\ H_2 & H_3 & H_4 & \cdots \\ - & & & \\ - & & & \end{pmatrix} = \begin{pmatrix} \Pi_0 \\ \Pi_1 \\ \Pi_2 \\ \vdots \end{pmatrix} \begin{pmatrix} \Delta_0 & \Delta_1 & \Delta_2 & \cdots \end{pmatrix} \qquad (12)$$

where Π_j are $p \times q$ matrices

Δ_j are $q \times m$ matrices.

and that the expansion (12) is unique up to choice of a $q\times q$ non-singular matrix ie

$$\begin{pmatrix} H_0 & H_1 & \cdots \\ H_1 & H_2 & \cdots \\ \hline & & \\ \hline \end{pmatrix} = \begin{pmatrix} \Pi_0 \\ \Pi_1 \\ \Pi_2 \\ \vdots \end{pmatrix} P \cdot P^{-1} \left(\Delta_0 \; \Delta_1 \; \Delta_2 \cdots \right) \qquad (13)$$

We define

$$C = \Pi_0 P , \quad B = P^{-1} \Delta_0$$

Matrix A is defined as follows:

write

$$\begin{pmatrix} \Pi_0 \\ \Pi_1 \\ \Pi_2 \\ \vdots \\ \Pi_{n-1} \end{pmatrix} A = \begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \vdots \\ \Pi_n \end{pmatrix} \qquad (14)$$

Solve A uniquely from (14), note that $\begin{pmatrix} \Pi_0 \\ \Pi_1 \\ \vdots \\ \Pi_{n-1} \end{pmatrix}$ is of rank q. Denote the sol^n to be A_0

We write

$$\begin{pmatrix} \Pi_0 \\ \Pi_1 \\ \\ \\ \Pi_{n-1} \end{pmatrix} P \quad P^{-1} A_0 P = \begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \\ \\ \Pi_n \end{pmatrix} P \qquad (15)$$

Thus for any choice of P in (13), the corresponding unique sol$^{\underline{n}}$ for A is given by

$$A = P^{-1} A_0 P.$$

where A_0 is the unique sol$^{\underline{n}}$ for (14).

Thus we conclude that the Hankel matrix

$$\begin{pmatrix} H_0 & H_1 & \cdots & H_{n-1} \\ \hline \\ \hline \\ H_{n-1} & H_n & \cdots & H_{2n-2} \end{pmatrix}$$

can be factored as:

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \left(B \mid AB \mid \cdots \mid A^{n-1}B \right)$$

where

$C = C_0 P$

$B = P^{-1} B_0$. $\quad$ P is any $q \times q$ nonsingular matrix.

$A = P^{-1} A_0 P$.

Here we have

$C_0 = \Pi_0$, $B_0 = \Delta_0$

and A_0 is obtained by solving (14).

Finally note that for any nonsingular P we have

$$C(sI - A)^{-1} B = C_0 (sI - A_0)^{-1} B_0 = G(s)$$

∎