Controllability

"Isn't that an old story??"

Lec 7

Let us consider, once again the eq^n

$$\dot{\underline{X}}(t) = A(t)\underline{X}(t) + B(t)u(t)$$
$$\underline{X}(t_0) = \underline{X}_0 .$$ *

Variation of constants formula says that if $\Phi(t,t_0)$ is the transition matrix for $\dot{\underline{X}}(t) = A(t)\underline{X}(t)$, ie if

$$\dot{\Phi}(t,t_0) = A(t)\Phi(t,t_0)$$
$$\Phi(t_0,t_0) = I$$

then the unique sol^n of * is given by

$$\underline{X}(t) = \Phi(t,t_0)\underline{X}_0 + \int_{t_0}^{t} \Phi(t,\sigma)B(\sigma)u(\sigma)\, d\sigma .$$ $\triangle$

splitting

$$\Phi(t,\sigma) = \Phi(t,t_0)\,\Phi(t_0,\sigma)$$

we obtain

$$\Phi(t,t_0)^{-1}\underline{X}(t) - \underline{X}_0 = \int_{t_0}^{t} \Phi(t_0,\sigma)B(\sigma)u(\sigma)\,d\sigma.$$

Recognizing that $\Phi(t,t_0)^{-1} = \Phi(t_0,t)$

we write

$$\int_{t_0}^{t} \Phi(t_0,\sigma)\,B(\sigma)\,u(\sigma)\,d\sigma = \Phi(t_0,t)\,\underline{X}(t) - \underline{X}_0$$

The problem of controllability is to find $u(\sigma)$, $0 \le \sigma \le T$, if possible, such that at $t = T$, $\underline{X}(T) = \underline{X}_1$, starting from $\underline{X}(t_0) = \underline{X}_0$

Thus we have

$$\int_{t_0}^{T} \Phi(t_0, \sigma) B(\sigma) u(\sigma)\, d\sigma =$$

$$\Phi(t_0, T) X_1 - X_0 \triangleq \xi .$$

The transfer between $X(t_0) = X_0$ to $X(T) = X_1$ is possible if ξ lies in the range space of

$$L(u) = \int_{t_0}^{T} \Phi(t_0, \sigma) B(\sigma) u(\sigma)\, d\sigma .$$

In fact, (*) is controllable if the range space of L is the entire $\mathbb{R}^n$; indicating that transfer between any two states is possible.

Define a matrix

$$W(t_0, T) \triangleq \int_{t_0}^{T} \Phi(t_0, \sigma) B(\sigma) B^T(\sigma) \Phi^T(t_0, \sigma) \, d\sigma .$$

The matrix $W(t_0, T)$ is called

"controllability Gramian".

This is because of the following surprising result.

- The linear operator L and the matrix $W(t_0,T)$ has the same range space. In particular if $W(t_0, T)$ is of rank n, then $(*)$ is controllable

Theorem (Controllability)

There exists a $u(t)$ which drives the state of the system.

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

from the value x_0 at $t = t_0$ to x_1 at $t = T > t_0$ iff

$$\Phi(t_0, T)\,x_1 - x_0 \triangleq \xi.$$

belongs to the range space of

$$W(t_0, T) = \int_{t_0}^{T} \Phi(t_0,\sigma)\,B(\sigma)\,B^T(\sigma)\,\Phi^T(t_0,\sigma)\,d\sigma.$$

Moreover if η_0 is one solution of

$$W(t_0, T)\,\eta_0 = \xi$$

then u as given by $u(t) = -B^T(t)\,\Phi^T(t_0,t)\,\eta_0$ is one control which accomplishes the desired transfer.

The following summarizes some of the important properties of controllability gramian matrix $W(t_0, T)$.

Thm:

(i) $W(t_0, T)$ is symmetric.

(ii) $W(t_0, T)$ is nonnegative definite for $t_0 \geq T$.

(iii) $W(t, T)$ satisfies the following matrix differential Eqn

$$\dot{W}(t,T) = A(t)W(t,T) + W(t,T)A^T(t) - B(t)B^T(t).$$

$$W(T,T) = 0.$$

Linear Time Invariant Systems.

We are looking at

$$\dot{\underline{X}}(t) = A\,\underline{X}(t) + B\,u(t)$$

$$\underline{X}(t_0) = \underline{X}_0 .$$

where A, B are constant $n \times n$ & $n \times m$ matrices. In this case $\Phi(t, t_0)$ takes a very explicit form

$$\Phi(t, t_0) = e^{A(t-t_0)} .$$

The controllability gramian matrix is given by

$$W(t_0, T) = \int_{t_0}^{T} e^{A(t_0-\sigma)} B \; B^T e^{A^T(t_0-\sigma)} \, d\sigma$$

We have the following theorem

Theorem:

The range space and null space of $W(t_0, T)$ for $T > t_0$, coincide with the range and null space of the $n \times n$ matrix

$$W_I = [B, AB, \ldots, A^{n-1}B][B, AB, \ldots, A^{n-1}B]^T.$$

Moreover, for any vector x_0 and any $T > t_0$, we have

$$\text{rank}[W(t_0, T), x_0] = \text{rank}[B, AB, \ldots, A^{n-1}B, x_0].$$

Remark: The power of the above theorem is quite evident. The property of controllability can be checked purely as a rank condition on a suitable matrix constructed out of B and A. This provides an easy checkable condition.

Canonical forms:

We start with

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0$$

and define new state variable

$$\underline{x}(t) = P\, z(t)$$

or equivalently

$$z(t) = P^{-1} \underline{x}(t)$$

where P is a constant, nonsingular matrix. It follows that

$$\dot{z} = P^{-1} \dot{\underline{x}} = P^{-1} A x + P^{-1} B u$$

i.e.

$$\dot{z} = \underbrace{P^{-1} A P}_{F} \, z(t) + \underbrace{P^{-1} B}_{G} \, u(t).$$

Define

$$F = P^{-1} A P \, ; \quad G = P^{-1} B$$

we obtain

$$\dot{z} = F z(t) + G u(t).$$

We have thus defined a transformation on the (A, B) pair given by

$$(A, B) \longmapsto (F, G)$$

where $F = P^{-1} A P$, $G = P^{-1} B$.

We would now show that there is a state variable change that displays the "controllability part" of the state eq$^{\underline{n}}$. This is summarized as follows:

Theorem:
Assume that the controllability matrix for the linear state eq$^{\underline{n}}$

$\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A is $n \times n$
B is $n \times m$

satisfies
$\operatorname{rank}(B, AB, \ldots, A^{n-1}B) = p$ where

$$0 < p < n$$

Then $\exists$ a non-singular matrix P such that

$$P^{-1}AP = \begin{pmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{pmatrix}, \quad P^{-1}B = \begin{pmatrix} G_{11} \\ 0 \end{pmatrix}$$

where

F_{11} is $p \times p$, F_{22} is $(n-p)\times(n-p)$ etc.

G_{11} is $p \times m$, and

$$\operatorname{rank}\left(G_{11}, F_{11}G_{11}, F_{11}^2 G_{11}, \ldots, F_{11}^{p-1}G_{11}\right) = p.$$

Remark: If we split the state variable z as $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ and write

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} G_{11} \\ 0 \end{pmatrix} u$$

it follows that

$$\dot{z}_2 = F_{22} z_2$$

$$\dot{z}_1 = (F_{11} z_1 + G_{11} u) + F_{12} z_2$$

The states z_1 is controllable by $u(t)$ and the states z_2 is not controllable by $u(t)$. In fact if we assume that

$$z_1(0) = \xi \qquad \text{ie } z(0) = \begin{pmatrix} \xi \\ 0 \end{pmatrix}$$
$$z_2(0) = 0$$

we obtain

$$z_2(t) = e^{F_{22}t} z_0(0) = 0 \ \forall t.$$

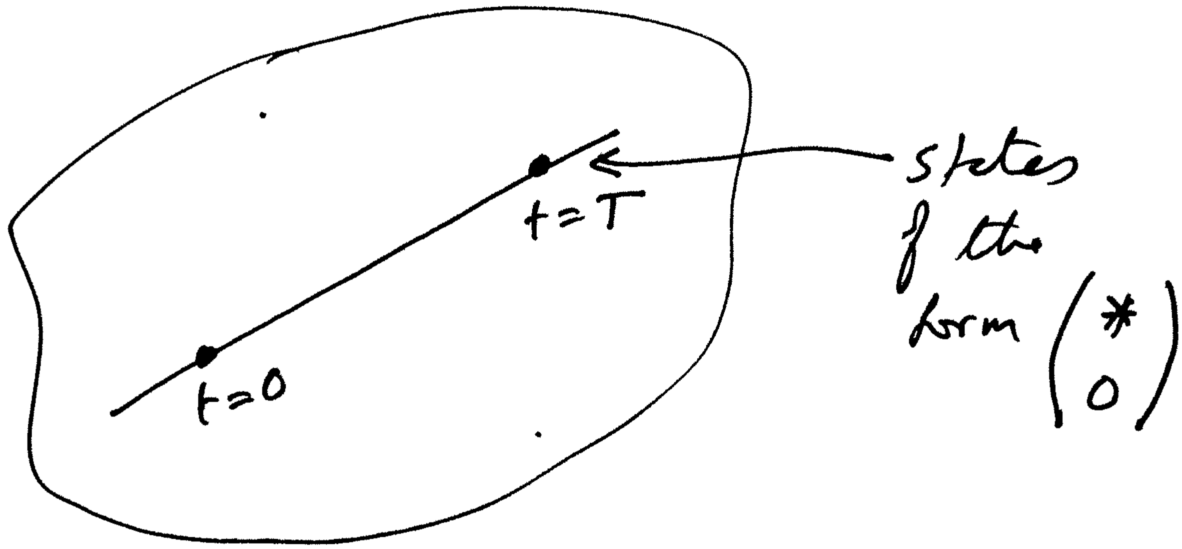
Hence

$$\dot{z}_1 = F_{11} z_1 + G_{11} u$$
$$z_1(0) = \xi.$$

which is a controllable dynamical system.

Hence we can steer between two initial conditions of the form

$$\begin{pmatrix} \xi \\ 0 \end{pmatrix} \text{ at } t=0 \quad \text{to} \quad \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} \text{ at } t=T.$$



controllable part of the state space is given by $z = \begin{pmatrix} z_1 \\ 0 \end{pmatrix}$

or equivalently

$$\underline{x} = P \begin{pmatrix} * \\ 0 \end{pmatrix}$$

Let V be a subspace of $\mathbb{R}^n$ defined as follows:

$$V = \{ v : \exists\, \xi \in \mathbb{R}^p \text{ and } v = P \begin{pmatrix} \xi \\ 0 \end{pmatrix} \}.$$

Clearly V is a p dimensional subspace of $\mathbb{R}^n$, with the following property.

Theorem:
Consider the L.D.S.

$$\dot{x} = Ax + Bu$$

where

$x(0) \in V$.

It follows that $x(t) \in V$ for any $u(t)$, $t > 0$. Moreover if $x_0, x_T \in V$ then $\exists\, u(\sigma)$, $0 \le \sigma \le T$ such that

$$x(0) = x_0 \;\&\; x(T) = x_T.$$

Remark: We have not said anything about how to compute this P.

How to compute P:

write

$$P = (v_1 \mid v_2 \mid \cdots \mid v_p \mid v_{p+1} \mid \cdots \mid v_n)$$

where $\{v_1, \ldots v_p\}$ is a basis of the range of the controllability matrix

$$(B \mid AB \mid \cdots \; A^{n-1}B)$$

i.e.

$\{v_1 \ldots v_p\}$

is a basis of $\operatorname{span}\{B, AB, \ldots, A^{n-1}B\}$.

choose $v_{p+1}, \ldots, v_n$ arbitrarily so that P is non-singular.

$\blacksquare$

Back to our satellite example

$$\dot{x} = Ax + bu$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

We choose $u_2 = 0$ and actuate the satellite by only one control u_1.

Bad news is that the above (A, b) is not controllable. The controllability matrix is

$$(b, Ab, A^2b, A^3b)$$

is of rank 3. In fact a set of l.i. basis vectors of the column span of the controllability matrix is given by

$$\{A^2 b, Ab, b\}$$

Note that

$$b = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad Ab = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2\omega \end{pmatrix}, \quad A^2 b = \begin{pmatrix} 0 \\ -\omega^2 \\ -2\omega \\ 0 \end{pmatrix}$$

are indeed linearly independent if $\omega \neq 0$.

Let us augment the above set of 3 basis vectors by adding the vector

$$\xi = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

It turns out that

$$\{A^2 b, Ab, b, \xi\}$$

are independent vectors if $\omega \neq 0$.

We define a matrix P by stacking the basis vectors as follows.

$$P = \left(A^2 b \,\middle|\, Ab \,\middle|\, b, \xi \right)$$

$$= \begin{pmatrix} 0 & 1 & 0 & 1 \\ -\omega^2 & 0 & 1 & 1 \\ -2\omega & 0 & 0 & 0 \\ 0 & -2\omega & 0 & 0 \end{pmatrix}$$

Claim:

$$P^{-1} b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = g$$

This is because $P \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = b$ ← the 3rd column of the P matrix.

Q: what is AP

Well,

$$AP = \left(A^3 b \mid A^2 b \mid Ab \mid A\xi\right)$$

check that

$$A^3 b = \begin{pmatrix} -\omega^2 \\ 0 \\ 0 \\ 2\omega^3 \end{pmatrix} = -\omega^2 A b$$

$$A\xi = \begin{pmatrix} 1 \\ 3\omega^2 \\ 0 \\ -2\omega \end{pmatrix} = Ab + 3\omega^2 b$$

Claim: $$P^{-1} A P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega^2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3\omega^2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = F$$

This is because

$$AP =$$
$$\left(-\omega^2 Ab,\ A^2 b,\ Ab,\ Ab + 3\omega^2 b\right)$$
$$= \left(A^2 b \,\middle|\, Ab \,\middle|\, b \,\middle|\, \xi\right) F$$
$$= PF$$

Hence $P^{-1} A P = F$.

— × —

Thus if we define

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 1 \\ -\omega^2 & 0 & 1 & 1 \\ -2\omega & 0 & 0 & 0 \\ 0 & -2\omega & 0 & 0 \end{pmatrix}}_{P} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

i.e
$$
\begin{aligned}
x_1 &= z_2 + z_4 \\
x_2 &= -\omega^2 z_1 + z_3 + z_4 \\
x_3 &= -2\omega z_1 \\
x_4 &= -2\omega z_2
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
z_1 &= -\frac{1}{2\omega} x_3 \\
z_2 &= -\frac{1}{2\omega} x_4 \\
z_4 &= x_1 + \frac{1}{2\omega} x_4 \\
z_3 &= x_2 + \omega^2 z_1 - z_4
\end{aligned}
$$

$$
\begin{aligned}
z_3 &= x_2 - \frac{\omega^{\cancel{2}}}{2\cancel{\omega}} x_3 - x_1 - \frac{1}{2\omega} x_4 \\
&= -x_1 + x_2 - \frac{\omega}{2} x_3 - \frac{1}{2\omega} x_4
\end{aligned}
$$

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we have

$$
\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix}
=
\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega^2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3\omega^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}
+
\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u .
$$

z_4 is the state that cannot be controlled. It satisfies the eqn

$$\dot{z}_4 = 0$$

ie $z_4(t) = z_4(0) \leftarrow$ a constant.

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\omega^2 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u .$$

$$+ \begin{pmatrix} 0 \\ 1 \\ 3\omega^2 \end{pmatrix} z_4(0)$$

The pair

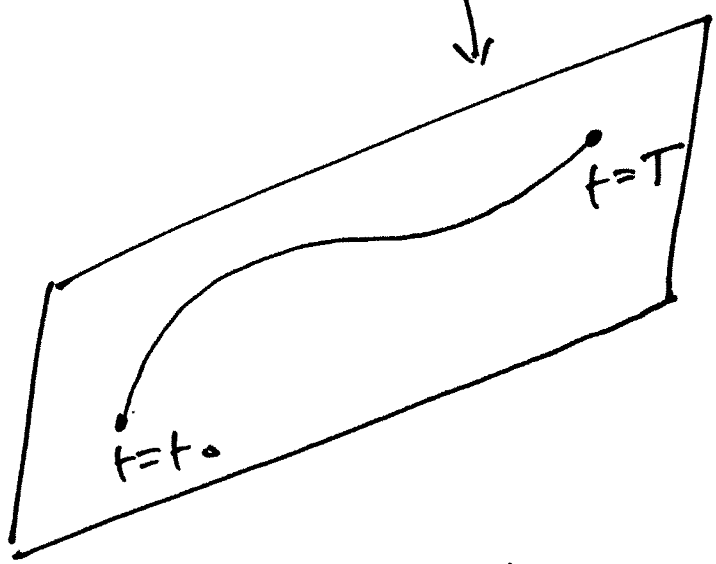
$$\begin{pmatrix} 0 & 1 & 0 \\ -\omega^2 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$ is always controllable

In the $\underline{z}$ - coordinate

$$z_4 = \text{const.}$$

is the hyperplane

$$x_1 + \frac{1}{2\omega} x_4 = \text{const.}$$



If I take two pts on this hyperplane, then I can choose a $u(\tau)$, $t_0 \leq \tau \leq T$ and can steer between the two points.

I CANNOT LEAVE THE HYPER Plane however.

Problem:

Let us now consider the problem of steering the state $\underline{X}(t)$ for the L.D.S. on page 17 from.

$$\begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ at } t=0 \qquad \underline{X}(0) = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

to

$$\begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ at } t=1. \qquad \underline{X}(1) = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Sol^{n}: We solve this problem by writing the initial and final condition $\underline{X}(0)$, $\underline{X}(1)$ in the Z co-ordinates.

From page 22 we have

$$Z(0) = \begin{pmatrix} -\frac{1}{2\omega} \cdot 0 \\ -\frac{1}{2\omega} \cdot 0 \\ -3+1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 3 \end{pmatrix}$$

$$Z(1) = \begin{pmatrix} -\frac{1}{2\omega} \\ 0 \\ -3 - \frac{\omega}{2} \cdot 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1/2\omega \\ 0 \\ -3 - \frac{\omega}{2} \\ 3 \end{pmatrix}$$

Note that $z_4(0) = z_4(1) = 3$

Thus we are steering between 2 pts on the hyperplane $z_4 = 3$.

From page 23 we have

$$\begin{pmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{pmatrix} = \underbrace{\begin{pmatrix}0&1&0\\-\omega^2&0&1\\0&0&0\end{pmatrix}}_{F_1}\begin{pmatrix}z_1\\ z_2\\ z_3\end{pmatrix} + \underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{g_1} u + \underbrace{\begin{pmatrix}0\\1\\3\omega^2\end{pmatrix}}_{g_2}\cdot 3$$

Denote

$Z_1 = \begin{pmatrix}z_1\\ z_2\\ z_3\end{pmatrix}$, we have

$$\dot{Z}_1(t) = F_1 Z_1(t) + g_1 u + 3g_2$$

$$\Rightarrow \quad Z_1(t) = e^{F_1 t} Z_1(0) + \int_0^t e^{F_1(t-\tau)} g_1 u(\tau)\,d\tau.$$

$$+ \int_0^t 3e^{F_1(t-\tau)} g_2\,d\tau.$$

We have

$$Z_1(0) = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = \eta_0 \quad \leftarrow \text{page 26}$$

and we would like to find $u(\tau)$:

$$Z_1(1) = \begin{pmatrix} -\frac{1}{2\omega} \\ 0 \\ -3-\frac{\omega}{2} \end{pmatrix} = \eta_1 \quad \leftarrow \text{page 26}$$

Thus we have

$$e^{-F_1 T} Z_1(T) - Z_1(0) - \int_0^T 3e^{-F_1 \tau} g_2 \, d\tau .$$

$$= \int_0^T e^{-F_1 \tau} g_1 u(\tau) \, d\tau .$$

Here $T = 1$.

Our first job is to compute

$$e^{F_1 t}.$$

check that

$$e^{F_1 t} = \begin{pmatrix} \cos\omega t & \frac{\sin\omega t}{\omega} & \frac{1-\cos\omega t}{\omega^2} \\ -\omega\sin\omega t & \cos\omega t & \frac{\sin\omega t}{\omega} \\ 0 & 0 & 1 \end{pmatrix}$$

&

$$e^{-F_1 t} = \begin{pmatrix} \cos\omega t & -\frac{\sin\omega t}{\omega} & \frac{\cos\omega t - 1}{\omega^2} \\ \omega\sin\omega t & \cos\omega t & -\frac{\sin\omega t}{\omega} \\ 0 & 0 & 1 \end{pmatrix}$$

It would follow that

$$e^{-F_1 \tau} g_1 = \begin{pmatrix} \frac{\cos\omega\tau - 1}{\omega^2} \\ -\frac{\sin\omega\tau}{\omega} \\ 1 \end{pmatrix}$$

The controllability gramian matrix is given by

$$\int_0^1 \begin{pmatrix} \frac{\cos\omega\tau - 1}{\omega^2} \\ -\frac{\sin\omega\tau}{\omega} \\ 1 \end{pmatrix} \begin{pmatrix} \frac{\cos\omega\tau - 1}{\omega^2} & -\frac{\sin\omega\tau}{\omega} & 1 \end{pmatrix} d\tau \triangleq W(0,1).$$

The control $u(\tau)$ would be of the form

$$u(\tau) = \Pi_1 \frac{\cos\omega\tau - 1}{\omega^2} - \Pi_2 \frac{\sin\omega\tau}{\omega} + \Pi_3$$

where

$$\begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \end{pmatrix} = W(0,1)^{-1}\left[e^{-F_1 T} Z_1(T) - Z_1(0) - \int_0^T 3e^{-F_1 \tau} g_2 \, d\tau \right]$$

It would help to use a symbolic toolbox at this pt. Good Luck

$\blacksquare$

Exercise:

consider the satellite problem

$$\dot{\underline{x}} = A\underline{x} + bu$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Here we choose $u_1 = 0$ and actuate the satellite using only u_2.

① show that the above pair is controllable

② choose $P = (A^3 b, A^2 b, Ab, b)$ and show that

$$P^{-1}AP = \underbrace{\begin{pmatrix} \times & 1 & 0 & 0 \\ \times & 0 & 1 & 0 \\ \times & 0 & 0 & 0 \\ \times & 0 & 0 & 1 \end{pmatrix}}_{F}, \quad P^{-1}b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

③ Find F if you have not already done so.

④ Write $\underline{x} = PZ$ and obtain.

$$\dot{Z} = FZ + bu$$

⑤ compute e^{Ft}.

⑥ compute $W(0, 1)$

Proof of theorems

$$\dot{x} = B(t)\,u(t) \tag{7.1}$$

Let

$$L : C[0,T] \longrightarrow \mathbb{R}^n \tag{7.2}$$
$$u(t) \longmapsto \int_0^T B(\sigma)\,u(\sigma)\,d\sigma$$

Define

$$W(0,T) = \int_0^T B(\sigma)\,B^T(\sigma)\,d\sigma \tag{7.3}$$

Theorem 7.1

An n-tuple x_1 lies in the range space of L iff it lies in the range space of the matrix $W(0,T)$.

Proof of theorem 7·1

If $x_1 \in$ Range space of $W(0,T)$ it follows that $\exists$ a vector η:

$$W(0,T)\eta = x_1. \quad (7\cdot4)$$

From (7·3) & (7·4) we obtain.

$$\int_0^T B(\sigma)B^T(\sigma)\eta \, d\sigma = x_1 \quad (7\cdot5)$$

Define

$$u(\sigma) = B^T(\sigma)\eta \quad (7\cdot6)$$

From (7·5) & (7·6) we have

$$\int_0^T B(\sigma)u(\sigma)d\sigma = x_1 \quad (7\cdot7)$$

Thus x_1 lies in the range space of L.

Conversely:

If $x_1 \notin$ Range space of $W(0,T)$ it would follow that $\exists$ a vector ξ in the null space of $W^T(0,T)$ such that $\xi \not\perp x_1$

i.e $\exists\ \xi \in \mathbb{R}^n$:

$$W^T(0,T)\,\xi = 0 \qquad (7.8)$$

&

$$\xi \cdot x_1 \neq 0 \qquad (7.9)$$

$\because$ $W(0,T)$ is symmetric, it would follow that

$$W(0,T)\,\xi = 0. \quad (7.10)$$

From (7.3) & (7.10) we have

$$\int_0^T B(\sigma)\,B^T(\sigma)\,\xi\, d\sigma = 0 \quad (7.11)$$

which implies that

$$\int_0^T \xi^T B(\sigma)\,B^T(\sigma)\,\xi\, d\sigma = 0$$

$$\Rightarrow \int_0^T \|B^T(\sigma)\,\xi\|^2\, d\sigma = 0$$

$\because$ $B(\sigma)$ is a continuous f^n, it follows that $B^T(\sigma)\,\xi \equiv 0 \ \forall\ 0 \le \sigma \le T$. (7.12)

It would follow that $x_1 \notin$ Range space of L for otherwise

$\exists\, u(\sigma)$:

$$\int_0^T B(\sigma)\, u(\sigma)\, d\sigma = x_1$$

$\Rightarrow$

$$\int_0^T \xi^T B(\sigma)\, u(\sigma)\, d\sigma = \xi^T x_1 \qquad (7.13)$$

$$= \xi \cdot x_1 \neq 0$$

from (7.9).

This contradicts (7.12) that

$$\xi^T B(\sigma) \equiv 0$$

$\blacksquare$

$$\dot{x} = A(t)x(t) + B(t)u(t). \qquad (7.14)$$

Variation of constants formula says that

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t,\sigma)B(\sigma)u(\sigma)\,d\sigma \qquad (7.15)$$

$$\Rightarrow \phi(t_0,t)\,x(t) = x_0 + \int_{t_0}^{t} \phi(t_0,\sigma)B(\sigma)u(\sigma)\,d\sigma \qquad (7.16)$$

Let

$$L_1 : C[t_0, T] \longrightarrow \mathbb{R}^n$$

$$u(t) \longmapsto \int_{t_0}^{T} \phi(t_0,\sigma)B(\sigma)u(\sigma)\,d\sigma \qquad (7.17)$$

Define

$$W_1(t_0, T) = \int_{t_0}^{T} \Phi(t_0,\sigma)B(\sigma)B^T(\sigma)\Phi^T(t_0,\sigma)\,d\sigma \qquad (7.18)$$

Theorem 7.2

An n tuple x_1 lies in the range space of L_1 iff it lies in the range space of the matrix $W_1(t_0, T)$.

Proof of Theorem 7.2

We shall prove this theorem by reducing (7.14) to the case (7.1). This is done by defining a new state variable

$$z(t) = \phi(t_0, t)\, x(t) \qquad (7.19)$$

We obtain

$$\dot{z}(t) = \left[\frac{d}{dt}\phi(t_0, t)\right] x(t) + \phi(t_0, t)\,\dot{x}(t) \qquad (7.20)$$

Ofcourse

$$\dot{x}(t) = Ax + Bu \qquad (7.21)$$

$$\phi(t_0, t)\,\phi(t, t_0) = I$$

$$\Rightarrow \left[\frac{d}{dt}\phi(t_0, t)\right]\phi(t, t_0) + \phi(t_0, t)\,\dot{\phi}(t, t_0) = 0$$

$$\Rightarrow \left[\frac{d}{dt}\phi(t_0, t) + \phi(t_0, t)\,A(t)\right]\phi(t, t_0) = 0$$

$\because$ $\phi(t, t_0)$ is a nonsingular matrix

it follows that

$$\frac{d}{dt}\phi(t_0, t) = -\phi(t_0, t)\,A(t). \qquad (7.22)$$

combining (7.20), (7.21) and (7.22) we obtain

$$\dot{z}(t) = -\phi(t_0, t)\,A(t)\,x(t) + \phi(t_0, t)\,A(t)\,x(t) + \phi(t_0, t)\,B(t)\,u(t).$$

Thus.

$$\dot{z}(t) = \Phi(t_0, t) B(t)\, u(t) \quad (7.23)$$

Note that (7.23) is of the form (7.1) and Theorem 7.2 easily follows from Theorem 7.1. $\blacksquare$

Although redundant, it may be instructive to write out the linear time invariant case.

$$\dot{x} = Ax + Bu. \qquad (7.24)$$

A & B are constant matrices.

$$\phi(t, t_0) = e^{A(t-t_0)} \qquad (7.25)$$

$$L_1 : C[t_0, T] \to \mathbb{R}^n$$

$$u(t) \mapsto \int_{t_0}^{T} e^{A(t_0-\sigma)} B \, u(\sigma)\, d\sigma \qquad (7.26)$$

Define

$$W_1(t_0, T) = \int_{t_0}^{T} e^{A(t_0-\sigma)} B B^T e^{A^T(t_0-\sigma)} d\sigma \qquad (7.27)$$

From the variation of constants formula (43)

we have

$$e^{A(t_0-t)} x(t) - x(t_0) = \int_{t_0}^{t} e^{A(t_0-\sigma)} B \; u(\sigma)\, d\sigma \qquad (7.28)$$

If we substitute

$$u(\sigma) = B^T e^{A^T(t_0-\sigma)} \eta \qquad (7.29)$$

we have

$$e^{A(t_0-T)} x(T) - x(t_0) = W_1(t_0, T)\,\eta \qquad (7.30)$$

From Theorem 7.2 we would conclude that iff $W_1(t_0,T)$ is invertible, one could solve (7.30) for an arbitrary left hand side. In this case we would write

$$\eta = W_1(t_0, T)^{-1}\left[e^{A(t_0 - T)}x(T) - x(t_0)\right] \tag{7.31}$$

and construct a control signal using (7.29) and (7.31) given by

$$u(\sigma) = B^T e^{A^T(t_0 - \sigma)} W_1(t_0, T)^{-1} \left[e^{A(t_0 - T)}x(T) - x(t_0)\right] \tag{7.32}$$

Substituting (7.32) into (7.24) we have the linear dynamical system

$$\dot{x} = Ax + BB^T e^{A^T(t_0-\sigma)} W_1(t_0,T)^{-1} \left[e^{A(t_0-T)} x(T) - x(t_0) \right]$$

which when initialized at $x(t_0)$, $t=t_0$ will hit $x(T)$ at $t=T$.

Remark:

If $W_1(t_0,T)$ is not invertible, then the transfer of state from $x(t_0)$ at $t=t_0$ to $x(T)$ at $t=T$ would still be possible if $\exists\, \eta^*$:

$$e^{A(t_0-T)} x(T) - x(t_0) = W_1(t_0,T)\eta^*.$$

In this case one of the control signal that will perform the desired transfer is given by

$$u(\sigma) = B^T e^{A^T(t_0-\sigma)} \eta^* .$$

For the time invariant case, invertibility of $W_1(t_0, T)$ (see (7.27)) has further interpretations. In fact invertibility of $W_1(t_0, T)$ does not even depend on t_0, T as long as $t_0 < T$. This is described by the following theorem:

Theorem 7.3 :-

If A is $n \times n$ and A, B are constant matrices then the range space and null space of $W(t_0, T)$, $T > t_0$ coincide with the range and null space of the $n \times n$ matrix

$$W_I = [B | AB | \cdots | A^{n-1}B][B | AB | \cdots | A^{n-1}B]^T . \quad (7.33)$$

Proof:

Let us assume that $x_1 \in$ Null space of $W(t_0, T)$.

$$\Rightarrow W(t_0, T)\, x_1 = 0$$

$$\Rightarrow x_1^T\, W(t_0, T)\, x_1 = 0$$

$$\Rightarrow \int_{t_0}^{T} x_1^T e^{A(t_0-\sigma)} B\, B^T e^{A^T(t_0-\sigma)} x_1\, d\sigma = 0$$

$$\Rightarrow \int_{t_0}^{T} \| B^T e^{A^T(t_0-\sigma)} x_1 \|^2\, d\sigma = 0$$

$$\Rightarrow B^T e^{A^T(t_0-\sigma)} x_1 \equiv 0$$

$$t_0 \le \sigma \le T$$

$$\Rightarrow B^T x_1 = 0 \qquad (7.34)$$

$$B^T A^T x_1 = 0$$

$$\vdots$$

$$B^T (A^T)^{n-1} x_1 = 0$$

$\Rightarrow$

$$x_1^T \left[B \,|\, AB \,|\, A^2B \,|\, \cdots \,|\, A^{n-1}B \right] = 0$$

Thus

x_1 is in the null space of

$\left[B \,|\, AB \,|\, \cdots \,|\, A^{n-1}B \right]^T$

and hence in the null space of W_I.

Conversely, let us assume that

$x_1 \in$ Null space of W_I

ie $\quad W_I x_1 = 0$

$\Rightarrow x_1^T W_I x_1 = 0$

$\Rightarrow \left\| \left(B \,|\, AB \,|\, \cdots \,|\, A^{n-1}B \right)^T x_1 \right\|^2 = 0$

$\Rightarrow x_1 \in$ Null space of $\left[B \,|\, AB \,|\, \cdots \,|\, A^{n-1}B \right]^T$

ie $B^T x_1 = 0,\ B^T A^T x_1 = 0 \cdots$

$\cdots\ B^T (A^T)^{n-1} x_1 = 0 \quad (7.35)$

Expanding

$$e^{A(t_0-\sigma)} =$$

$$I + A(t_0-\sigma) + \frac{A^2(t_0-\sigma)^2}{2!} + \cdots$$

$$= \alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1}$$

(by Cayley Hamilton's Th$\underline{m}$)

It follows that

$$x_1^T e^{A(t_0-\sigma)} B =$$

$$\alpha_0 x_1^T B + \alpha_1 x_1^T AB + \alpha_2 x_1^T A^2 B + \cdots$$
$$\cdots + \alpha_{n-1} x_1^T A^{n-1} B.$$

$= 0$ from (7.30).

Hence $W_1(t_0, T)\, x_1 = 0$ ie

$x_1 \in$ Null space of $W_1(t_0, T)$.

We have shown so far that

$$W(t_0, T) \ \& \ W_I$$

have the same null spaces.

But since these matrices are symmetric matrices, their range and null spaces are perpendicular to each other. Hence

$$W(t_0, T) \ \& \ W_I$$

must also have the same range spaces.

$\blacksquare$

Let $\mathcal{C}$ be the controllability matrix

$$\mathcal{C} = \left[B \,|\, AB \,|\, A^2B \,|\, \text{---} \,|\, A^{n-1}B \right] \tag{7.36}$$

If A is $n \times n$, B is $n \times m$ then $\mathcal{C}$ is $n \times mn$ matrix.

We now define two linear transformations.

$$L_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^{mn}$$
$$\xi^T \longmapsto \xi^T \mathcal{C}$$

$$L_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\xi^T \longmapsto \xi^T W_I \quad (\text{see } 7.33)$$

From the proof of theorem 7.3,

we have already seen that

Null space of L_1

coincides with the

Null space of L_2.

Furthermore, for any linear transformation L, over finite dimensional vector spaces we have

dim(Null space of L) +

dim(Range space of L)

= dim(Base space of L).

It would follow that

dim(Range space of L_1) =

dim(Range space of L_2)

Thus

$$\operatorname{rank} \mathcal{C} = \operatorname{rank} W_I .$$

We state this as a theorem:

Theorem 7.4

The rank of the following two matrices are the same

- $[B \mid AB \mid \cdots \mid A^{n-1}B][B \mid AB \cdots \mid A^{n-1}B]^T$
- $[B \mid AB \mid \cdots \mid A^{n-1}B]$

In particular

W_I is invertible iff

$\mathcal{C}$ is of rank n.

Remark:

① W_I & $\mathcal{C}$ are not matrices of the same order. One is $n\times n$ the other is $n\times mn$.

② $\mathcal{C}$ is useful only in checking the invertibility of W_I which is equivalent to invertibility of $W(t_0,T)$, $T>t_0$.

We state this as follows.

Theorem 7.5

The following statements are equivalent. Here A is $n\times n$, B is $n\times m$

ⓐ $\dot{x}=Ax+Bu$ is controllable
A,B constant matrices.

ⓑ $W(t_0,T)$ is invertible $T>t_0$.

ⓒ W_I is invertible.

ⓓ $\operatorname{rank}\mathcal{C}=n$.

Theorem 7.6

The following statements are equivalent.

(a) For the dynamical system

$$\dot{x} = Ax + Bu, \; A \text{ is } n\times n, \; B \text{ is } n\times m$$

there exist a control $u(\sigma)$, $t_0 \le \sigma \le T$ which transfers the state from $x(t_0)$ at $t=t_0$ to $x(T)$ at $t=T$.

(b) $\exists \, \eta \in \mathbb{R}^n$:

$$e^{A(t_0-T)} x(T) - x(t_0) = W_1(t_0,T)\,\eta. \tag{7.37}$$

(c) $\exists \, \eta' \in \mathbb{R}^n$:

$$e^{A(t_0-T)} x(T) - x(t_0) = W_I \, \eta' \tag{7.38}$$

The proof of theorem 7·6 follows quite easily from the fact that $W_1(t_0,T)$ & W_I have the same range space. Note that the controllability matrix $\mathcal{C}$ does not have a role in Theorem 7·6. Finally note that η & η' are not necessarily the same vector.

Finally note that in order to test if a transfer between $x(t_0)$ & $x(T)$ is possible, it is not necessary to compute the controllability gramian $W_1(t_0,T)$. Instead computing W_I and its range space would suffice.

The controllability gramian matrix $W_1(t_0, T)$ is required only when the goal is to compute $u(\sigma)$ given by (7.32).

— × —

Rank Tests for Controllability

For reasons that are not very hard to guess, a lot more effort has gone into analyzing linear time invariant dynamical systems (LTI), to assess controllability. In the following pages I would describe two theorems:

Theorem 7.7

The L.T.I. dynamical system

$$\dot{x} = Ax + Bu \qquad (7.39)$$

where $x \in \mathbb{R}^n$, A is $n \times n$, B is $n \times m$
$u \in \mathbb{R}^m$, A, B constant matrices,

is controllable iff

(a) rank $[B \,;\, AB | \cdots | A^{n-1}B] = n$ (7.40)

(b) rank $[sI - A \mid B] = n$ (7.41)

$\forall$ s real or complex.

Remark: The matrix in (7.40) is the controllability matrix that we have already seen in (7.36).

The rank test (7.41) is due to "Popov, Belevitch and Hautus" and is called P.B.H rank test.

Let us first show that if

rank$[sI - A \mid B] < n$ (7.42)

for some $s = \lambda$, then (7.39) is not controllable.

$(7.42) \Rightarrow \exists \lambda:$

$$\text{rank}\,[\lambda I - A \,|\, B] < n$$

ie $\exists\, \xi^T \in \mathbb{R}^n$:

$$\xi^T [\lambda I - A \,\vdots\, B] = 0$$

$$\Rightarrow \xi^T A = \lambda \xi^T \;\&\; \xi^T B = 0$$

Thus $\xi^T B = 0$

$$\xi^T A B = (\xi^T A) B = \lambda \xi^T B = 0$$

$$\xi^T (A^{n-1} B) = (\xi^T A) A^{n-2} B = \lambda \xi^T A^{n-2} B$$

$$= \lambda^2 \xi^T A^{n-3} B \;---$$

$$--- = \lambda^{n-1} \xi^T B = 0$$

We conclude

$$\xi^T [B \,|\, AB \,|\, --- \,|\, A^{n-1} B] = 0$$

$$\Rightarrow [B \,|\, AB \,|\, --- \,|\, A^{n-1} B] < n$$

Hence $(\;)_{(7.39)}$ is not controllable.

Conversely, we would like to show that if (7·39) is not controllable then the PBH rank condition (7·41) is not satisfied.

If (7·39) is not controllable, it follows that

$$\operatorname{rank}\left[B \mid AB \mid \text{---} \mid A^{n-1}B\right] = q < n$$

for some q. It follows that upto change of variable, (7·39) takes the form

$$\dot{x} = Ax + Bu$$

See theorem on p 11, Lec 7

Where

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} \\ 0 \end{pmatrix} \qquad (7·43).$$

Where A_{11} is $q \times q$, A_{22} is $(n-q) \times (n-q)$, B_{11} is $q \times m$, 0 is the zero matrix.

Let $\rho \in \mathbb{R}^{n-q}$ be a left eigenvector of A_{22} ie

$$\rho^T A_{22} = \lambda^* \rho^T, \qquad (7.44)$$

for some eigenvalue λ^*. We define

$$\xi^T = \begin{pmatrix} 0 & \rho^T \end{pmatrix} \qquad (7.45)$$

and conclude that

$$\xi^T B = 0\, B_{11} + \rho^T 0 = 0$$

$$\begin{aligned} \xi^T A &= \begin{pmatrix} 0 & \rho^T A_{22} \end{pmatrix} \\ &= \lambda^* \begin{pmatrix} 0 & \rho^T \end{pmatrix} \\ &= \lambda^* \xi^T \end{aligned}$$

Thus

$$\xi^T \left[\lambda^* I - A \,|\, B \right] = 0$$

$$\Rightarrow \operatorname{rank} \left[\lambda^* I - A \,|\, B \right] < n \qquad \blacksquare$$