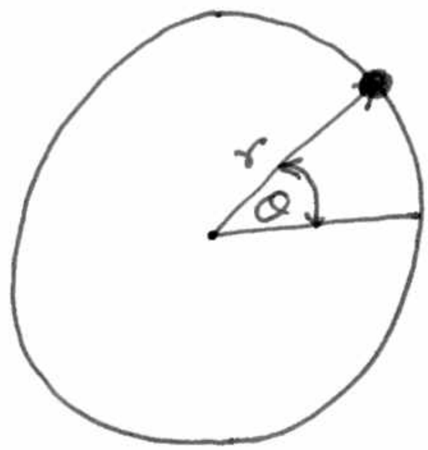
Lec 3

Example (satellite problem)



Consider a point mass in an inverse square law force field. The motion of a unit mass is governed by a pair of second order equations in the radius 'r' and angle 'θ'

If we assume that the unit mass has the capability of thrusting in the radial direction with a thrust u_1 and thrusting in the tangential direction with thrust u_2 then we have

$$\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1$$

$$\ddot{\theta} = -\frac{2\dot{\theta}\dot{r}}{r} + \frac{1}{r}u_2 \qquad (*)$$

If $u_1 = u_2 = 0$, $(*)$ admit the sol$^{\underline{n}}$

$r(t) = \sigma \qquad \sigma, \omega$ constant.

$\theta(t) = \omega t \qquad k = \sigma^3\omega^2$

i.e. circular orbits are possible.

Define $\quad x_1 = r - \sigma$

$x_2 = \dot{r}$

$x_3 = \sigma(\theta - \omega t)$

$x_4 = \sigma(\dot{\theta} - \omega)$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = r\dot{\theta}^2 - \frac{k}{r^2} + u_1$$

$$= (x_1 + \sigma)\left(\frac{x_4}{\sigma} + \omega\right)^2 - \frac{\sigma^3 \omega^2}{(x_1 + \sigma)^2} + u_1$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \sigma \ddot{\theta} = \left(-\frac{2\dot{\theta}\dot{r}}{r} + \frac{1}{r} u_2\right)\sigma$$

$$= \frac{-2\sigma\left(\frac{x_4}{\sigma} + \omega\right) x_2}{x_1 + \sigma} + \frac{\sigma u_2}{x_1 + \sigma}$$

(4)

$$f(x) = (x_1+\sigma)\left(\frac{x_4}{\sigma}+\omega\right)^2 - \frac{\sigma^3\omega^2}{(x_1+\sigma)^2}$$

$$\left.\frac{\partial f}{\partial x_1}\right|_{x=0} = \left(\frac{x_4}{\sigma}+\omega\right)^2 - \left.\frac{\sigma^3\omega^2(-2)}{(x_1+\sigma)^3}\right|_{x=0}$$

$$= \omega^2 + \frac{2\sigma^3\omega^2}{\sigma^3} = 3\omega^2$$

$$\left.\frac{\partial f}{\partial x_4}\right|_{x=0} = \left.(x_1+\sigma)\,2\left(\frac{x_4}{\sigma}+\omega\right)\frac{1}{\sigma}\right|_{x=0}$$

$$= \frac{2\sigma\omega}{\sigma} = 2\omega.$$

$$g(x) = \frac{-2\sigma\left(\frac{x_4}{\sigma} + \omega\right)x_2}{x_1 + \sigma}$$

$$\left.\frac{\partial g}{\partial x_1}\right|_{x=0} = \left.\frac{+2\sigma\left(\frac{x_4}{\sigma} + \omega\right)x_2}{(x_1 + \sigma)^2}\right|_{x=0} = 0$$

$$\left.\frac{\partial g}{\partial x_2}\right|_{x=0} = \left.\frac{-2\sigma\left(\frac{x_4}{\sigma} + \omega\right)}{x_1 + \sigma}\right|_{x=0}$$

$$= -2\omega$$

$$\left.\frac{\partial g}{\partial x_4}\right|_{x=0} = \left.\frac{-2\sigma x_2}{x_1 + \sigma}\,\frac{1}{\sigma}\right|_{x=0}$$

$$= \left.\frac{-2x_2}{x_1 + \sigma}\right|_{x=0} = 0$$

Linearized equations of motion about the circular orbit is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

— x —

We shall abbreviate the above equation as

$$\dot{\underline{X}} = A\underline{X} + Bu \qquad \underline{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Calculating the eigenvalues of the matrix A:—

check that

$$\det(\lambda \underline{I} - A) =$$

$$\det \begin{pmatrix} \lambda & -1 & 0 & 0 \\ -3\omega^2 & \lambda & 0 & -2\omega \\ 0 & 0 & \lambda & -1 \\ 0 & 2\omega & 0 & \lambda \end{pmatrix}$$

$$= \lambda^2 (\lambda^2 + \omega^2)$$

Eigenvalues are at

$0, 0, i\omega, -i\omega.$

Eigenvector calculation

"zero eigenvalue"

$$A \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = 0$$

Assume $\omega \neq 0$

$\Rightarrow v_2 = 0$

$3\omega^2 v_1 + 2\omega v_4 = 0 \quad \Rightarrow v_1 = v_2 = v_4 = 0$

$v_4 = 0$

$-2\omega v_2 = 0$

Eigenvector at $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = u_1$

Eigenvalue 0 has eigenvector

$$u_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Q: Does $\exists$ another nonzero vector u_2 such that

- u_2 is an eigenvector for eigenvalue 0. i.e $Au_2 = 0u_2 = 0$
- u_1 & u_2 are independent.

Ans: NO.

Q: Does $\exists$ another non-zero vector u_2 such that

- u_2 is a generalized eigenvector for eigenvalue 0

  i.e. $Au_2 = 0\,u_2 + u_1$

- u_1 and u_2 are independent.

Ans: Yes

If $u_2 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$, we write

$$Au_2 = u_1 \Rightarrow \alpha_2 = 0,\ 3\omega^2\alpha_1 + 2\omega\alpha_4 = 0$$

$$\alpha_4 = 1,\ -2\omega\alpha_2 = 0$$

It follows that

$$u_2 = \begin{pmatrix} -\frac{2}{3\omega} \\ 0 \\ \alpha_3 \\ 1 \end{pmatrix}$$

is a generalized eigenvector for eigenvalue 0.

where α_3 is arbitrary and can be chosen to be 0.

Conclusion:

$$\underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{= u_1}, \underbrace{\begin{pmatrix} -2/3\omega \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{= u_2}$$

is a linearly independent pair of generalized eigenvectors.

Review of generalized eigenvectors

Assume that a matrix A has an

eigenvalue λ repeated m times.

We define a set of m vectors

$$v_1, v_2, \dots, v_m$$

to be a chain of generalized eigenvectors of the matrix A corresponding to the eigenvalue λ if

- $\{v_1, \dots, v_m\}$ is a l.i. set.

- $$\begin{aligned} Av_1 &= \lambda v_1 \\ Av_2 &= \lambda v_2 + v_1 \\ Av_3 &= \lambda v_3 + v_2 \\ &\vdots \\ Av_m &= \lambda v_m + v_{m-1}. \end{aligned}$$

Remark:

A matrix A with eigenvalue λ repeated m times may not necessarily have a single chain of g. eigenvectors. It may have two or more subchains of g. eigenvectors.

For example the subchains could be

$$\{v_1, v_2, \ldots, v_{m_1}\} \; \& \; \{w_1, w_2, \ldots, w_{m_2}\}$$

where

- $v_1 \ldots v_{m_1}, w_1, \ldots, w_{m_2}$ are l.i.
- $m_1 + m_2 = m$
- $Av_1 = \lambda v_1$ $\qquad$ $Aw_1 = \lambda w_1$

$Av_2 = \lambda v_2 + v_1$ $\qquad$ $Aw_2 = \lambda w_2 + w_1$

$\vdots$ $\qquad$ $\vdots$

$Av_{m_1} = \lambda v_{m_1} + v_{m_1 - 1}$ $\qquad$ $Aw_{m_2} = \lambda w_{m_2} + w_{m_2 - 1}$

Theorem :—

Let A be a $n \times n$ matrix with eigenvalue λ repeated m times. There always exist a l.i. set of vectors $v_1, \dots, v_m$ in $\mathbb{R}^n$ such that

$$\{v_1, \dots, v_{m_1}\},$$
$$\{v_{m_1+1}, \dots, v_{m_1+m_2}\}$$
$$\{v_{m_1+m_2+1}, \dots, v_{m_1+m_2+m_3}\}$$

$$\{v_{m_1+m_2+\dots+m_{k-1}+1}, \dots, v_{m_1+m_2\dots+m_k}\}$$

form a set of k chains of generalized eigenvectors of A corresponding to eigenvalue λ and such that

$$m_1 + m_2 + \dots + m_k = m.$$

For the eigenvalue $i\omega$, we can show that

$$\underbrace{\begin{pmatrix} \alpha_1 \\ \frac{\omega}{2}\alpha_3 \\ \alpha_3 \\ -2\omega\alpha_1 \end{pmatrix}}_{= u_3} + i \underbrace{\begin{pmatrix} -\alpha_3/2 \\ \omega\alpha_1 \\ 2\alpha_1 \\ \omega\alpha_3 \end{pmatrix}}_{= u_4}$$

is an eigenvector. In fact it can be checked that

$$Au_3 = -\omega u_4 \, ; \; Au_4 = \omega u_3$$

Hence

$$A(u_3 + i u_4) = -\omega u_4 + i \omega u_3 .$$
$$= i\omega (u_3 + i u_4)$$

For the eigenvalue $-i\omega$, we can show likewise that

$$A(u_3 - iu_4) = -\omega u_4 - i\omega u_3.$$
$$= -i\omega(u_3 - iu_4).$$

Hence $u_3 - iu_4$ is the associated eigenvector.

Note that α_1 and α_3 are arbitrary parameters and we can choose $\alpha_1 = 0$, $\alpha_3 = 2$, without any loss of generality.

Thus we have

$$u_3 = \begin{pmatrix} 0 \\ \omega \\ 2 \\ 0 \end{pmatrix} \quad \& \quad u_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 2\omega \end{pmatrix}$$

Conclusion:

$\{u_1, u_2\}$ is a chain of g. eigenvectors for eigenvalue $\lambda = 0$

$u_3 \pm i u_4$ is a complex conjugate eigenvector for eigenvalue $\lambda = \pm i\omega$.

— x —

$$u_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -\frac{2}{3\omega} \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ \omega \\ 2 \\ 0 \end{pmatrix}, \quad u_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 2\omega \end{pmatrix}$$

If we define a matrix T as follows:

$$T = \left(u_1 \,\middle|\, u_2 \,\middle|\, u_3 \,\middle|\, u_4 \right)$$

one can easily verify that

$$AT = T \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{pmatrix}$$

This is because

$$Au_1 = 0$$

$$Au_2 = u_1$$

$$Au_3 = -\omega u_4$$

$$Au_4 = \omega u_3$$

Going back to the equation

$$\dot{\underline{x}} = A\underline{x} + Bu$$

if we define a new set of variables z: $\underline{x} = Tz$ we have

$$\dot{z} = T^{-1}\dot{\underline{x}}$$

$$= T^{-1}A\underline{x} + T^{-1}Bu$$

$$\dot{z} = T^{-1}ATz + T^{-1}Bu.$$

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \begin{pmatrix} -\frac{2}{\omega} & 0 \\ 0 & -3 \\ \frac{1}{\omega} & 0 \\ 0 & \frac{2}{\omega} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Note that in the z coordinates, the equations are block-decoupled. We have

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} -2/\omega & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} + \begin{pmatrix} \frac{1}{\omega} & 0 \\ 0 & \frac{2}{\omega} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

For $u_1 = u_2 = 0$, we can actually solve the equation.

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix}$$

$$\begin{bmatrix} z_3(t) \\ z_4(t) \end{bmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} z_3(0) \\ z_4(0) \end{pmatrix}$$

$z_2(t)$ is constant.

$z_1(t)$ changes linearly in time

$z_3(t)$, $z_4(t)$ is oscillatory.

What are these variables

$z_1 \quad z_2 \quad z_3 \quad z_4$??

Recall $\underline{X} = T Z$

$$\Rightarrow Z = T^{-1} \underline{X}$$

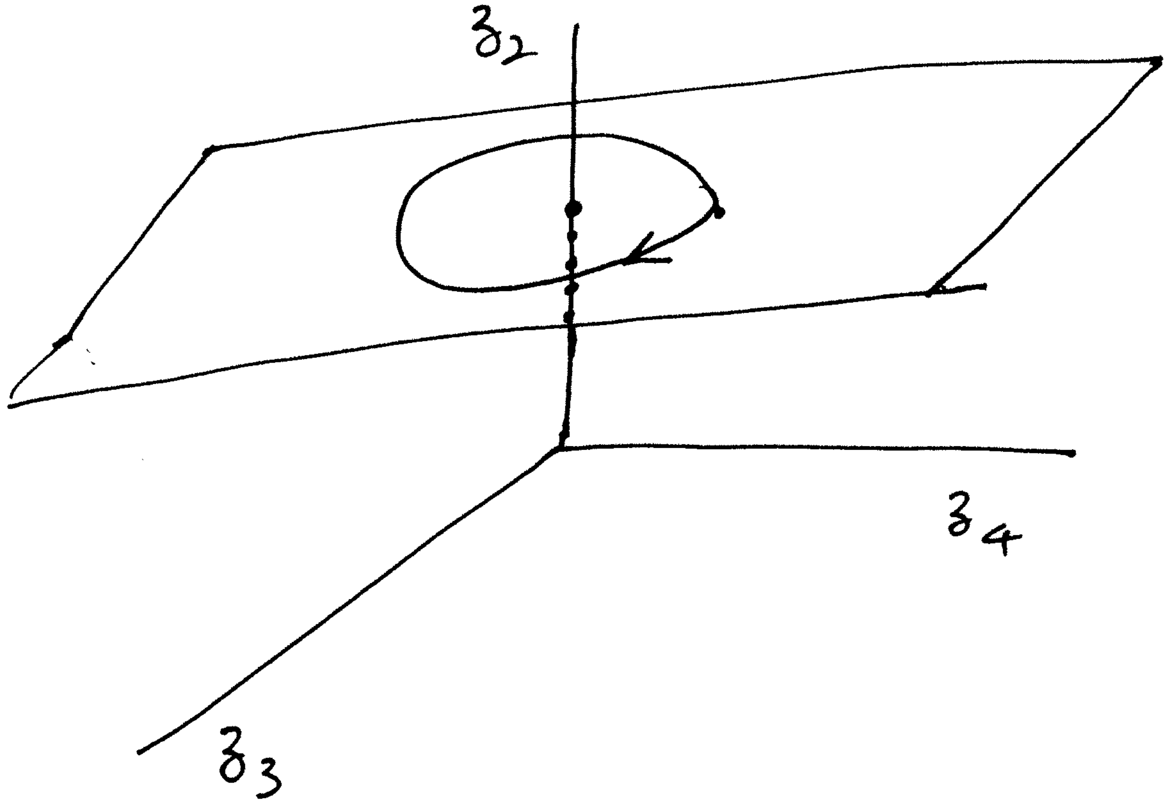
$$\Rightarrow \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{2}{\omega} & 1 & 0 \\ -6\omega & 0 & 0 & -3 \\ 0 & \frac{1}{\omega} & 0 & 0 \\ 3 & 0 & 0 & \frac{2}{\omega} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Hence $z_1 = -\frac{2}{\omega} x_2 + x_3 = -\frac{2}{\omega}(\dot{r}) + \sigma(\theta - \omega t)$

$$z_2 = -6\omega x_1 - 3x_4 = -6\omega(r - \sigma) - 3\sigma(\dot{\theta} - \omega)$$

$z_3 = \frac{x_2}{\omega} = \frac{1}{\omega}\dot{r}$; $z_4 = 3x_1 + \frac{2}{\omega} x_4$

$$= 3(r - \sigma) + \frac{2\sigma}{\omega}(\dot{\theta} - \omega)$$



In the (z_2, z_3, z_4) co-ordinates z_2 is constant & z_3, z_4 are sinusoidal, hence orbits are circular. It follows that given $(z_2^{(0)}, z_3^{(0)}, z_4^{(0)})$, $\exists\ M > 0$ such that

$$|z_i(t)| < M \quad \text{for } i = 2, 3, 4 \text{ and}$$

$t \geq 0$, where M depends only on the initial condition.

We can easily argue from here that the variables

$r(t)$, $\dot{r}(t)$ and $\dot{\theta}(t)$

remains bounded away from the equilibrium. $r(t) = \sigma$; $\dot{r}(t) = 0$, $\dot{\theta}(t) = \omega$.

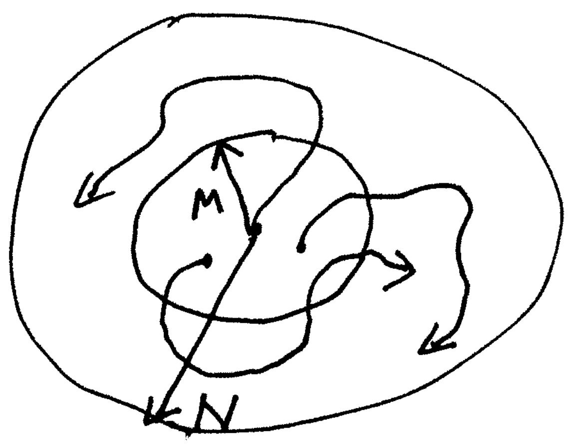
for an initial condition chosen to be sufficiently close to $(\sigma, 0, \omega)$.

The variable $\theta - \omega t$, on the other hand drifts linearly away from 0.

We will formally define the concept of stability.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 3\omega^2 & 0 & 2\omega \\ 0 & -2\omega & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

is zero input stable ie for $u_1 = u_2 = 0$, $|\underline{x}(0)| < M$ $\exists N$ depending only on M such that $|\underline{x}(t)| < N$ $\forall t \geq 0$.



Actually it can be shown, but we do not show here that if

$$u_1 = 0, \; u_2 = 0$$

we can solve for x_1, x_2, x_4 directly as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix}(t) = \begin{pmatrix} 4 - 3\cos\omega t & \dfrac{\sin\omega t}{\omega} & \dfrac{2(1-\cos\omega t)}{\omega} \\ 3\omega \sin\omega t & \cos\omega t & 2\sin\omega t \\ 6\omega(\cos\omega t - 1) & -2\sin\omega t & -3 + 4\cos\omega t \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_4(0) \end{pmatrix}$$

where

$$x_3(t) = 6(\sin\omega t - \omega t)\, x_1(0) - \frac{2(1-\cos\omega t)}{\omega} x_2(0) + x_3(0) + \frac{4\sin\omega t - 3\omega t}{\omega} x_4(0)$$

It is evident that

$x_1(t), x_2(t), x_4(t)$

has only oscillatory terms whereas $x_3(t)$ contains a linear term ωt.

Example (Spread of an epidemic disease)

The spread of an epidemic disease can be described by a set of 3 differential equations. The population under study is made up of three groups

x_1 : # susceptible to the epidemic disease

x_2 : # infected with the disease

x_3 : # removed from the initial population.

(Removal is due to immunization death or isolation.)

$$\dot{x}_1 = -\alpha x_1 - \beta x_2 + u_1(t)$$

$$\dot{x}_2 = \beta x_1 - \gamma x_2 + u_2(t)$$

$$\dot{x}_3 = \alpha x_1 + \gamma x_2$$

$u_1(t)$ is the rate at which new susceptibles are added to the population.

$u_2(t)$ is the rate at which new infectives are added to the population.

The state equation is given by.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -\alpha & -\beta & 0 \\ \beta & -\gamma & 0 \\ \alpha & \gamma & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

For a closed population $u_1 = u_2 = 0$

Solve for $x_1(t), x_2(t), x_3(t)$.

$$A = \begin{pmatrix} -\alpha & -\beta & 0 \\ \beta & -\gamma & 0 \\ \alpha & \gamma & 0 \end{pmatrix}$$

characteristic polynomial of A is given by

$$\lambda\left(\lambda^2+(\alpha+\gamma)\lambda+\alpha\gamma+\beta^2\right)$$

Assume that

$$(\alpha+\gamma)^2 < 4\left(\alpha\gamma+\beta^2\right)$$

ie $(\alpha-\gamma)^2 < 4\beta^2$

ie $\quad |\alpha-\gamma| < |2\beta|$

Eigenvalues of A are at

$$\lambda = 0$$

$$\lambda = -\frac{\alpha+\gamma}{2} \pm \sqrt{\frac{(\alpha+\gamma)^2 - 4(\gamma\alpha+\beta^2)}{4}}$$

$$= \sigma \pm i\omega$$

where

$$\sigma = -\frac{\alpha+\gamma}{2}$$

$$\omega = \sqrt{(\gamma\alpha+\beta^2) - \frac{(\alpha+\gamma)^2}{4}}$$

$$= \sqrt{\frac{4\gamma\alpha + 4\beta^2 - \alpha^2 - \gamma^2 - 2\alpha\gamma}{4}}$$

$$= \sqrt{\frac{4\beta^2 - (\alpha-\gamma)^2}{4}} = \sqrt{\beta^2 - \left(\frac{\alpha-\gamma}{2}\right)^2}$$

Brute force sol^n

$$x_1(t) = e^{\sigma t}(a\cos\omega t + b\sin\omega t)$$
$$x_2(t) = e^{\sigma t}(c\cos\omega t + d\sin\omega t)$$

$$x_1(0) = a \; ; \; x_2(0) = c$$

$$\therefore \; x_1(t) = e^{\sigma t}(x_1(0)\cos\omega t + b\sin\omega t)$$
$$x_2(t) = e^{\sigma t}(x_2(0)\cos\omega t + d\sin\omega t)$$

$$\dot{x}_1 = e^{\sigma t}(b\omega\cos\omega t - x_1(0)\,\omega\sin\omega t)$$
$$+ \sigma e^{\sigma t}(x_1(0)\cos\omega t + b\sin\omega t).$$

$$= e^{\sigma t}\Big[(\sigma x_1(0) + b\omega)\cos\omega t$$
$$+ (\sigma b - x_1(0)\,\omega)\sin\omega t\Big]$$

$$-\alpha x_1 - \beta x_2 =$$
$$e^{\sigma t}\Big[(-\alpha x_1(0) - \beta x_2(0))\cos\omega t$$
$$(-\alpha b - \beta d)\sin\omega t\Big]$$

$\because \dot{x}_1 = -\alpha x_1 - \beta x_2$

we have

$$\sigma x_1(0) + b\omega = -\alpha x_1(0) - \beta x_2(0)$$

$$\Rightarrow b\omega = -(\sigma+\alpha) x_1(0) - \beta x_2(0)$$

$$\Rightarrow b = -\frac{(\sigma+\alpha)}{\omega} x_1(0) - \frac{\beta}{\omega} x_2(0).$$

$$= -\frac{1}{\omega}\frac{\alpha-\gamma}{2} x_1(0) - \frac{\beta}{\omega} x_2(0)$$

$$b = -\frac{\alpha-\gamma}{2\omega} x_1(0) - \frac{\beta}{\omega} x_2(0)$$

$$x_1(t) = e^{\sigma t}\left[x_1(0)\cos\omega t - \left(\frac{\alpha-\gamma}{2\omega} x_1(0) + \frac{\beta}{\omega} x_2(0)\right)\sin\omega t\right]$$

$$\dot{x}_2 =$$

$$e^{\sigma t}\left[d\omega \cos\omega t - c\omega \sin\omega t\right]$$

$$+ \sigma e^{\sigma t}\left[c \cos\omega t + d \sin\omega t\right]$$

$$= e^{\sigma t}\left[(\sigma x_2(0) + d\omega) \cos\omega t + (d\sigma - x_2(0)\omega) \sin\omega t\right]$$

$$\beta x_1 - \gamma x_2 =$$

$$e^{\sigma t}\left[\left[\beta x_1(0) - \gamma x_2(0)\right] \cos\omega t + (\beta b - \gamma d) \sin\omega t\right]$$

$\because$ $\dot{x}_2 = \beta x_1 - \gamma x_2$, we have

$$d\omega = \beta x_1(0) - (\gamma + \sigma) x_2(0)$$

$$= \beta x_1(0) - \left(\frac{\gamma - \alpha}{2}\right) x_2(0)$$

$$d = \frac{\beta}{\omega} x_1(0) - \frac{\gamma - \alpha}{2\omega} x_2(0)$$

$$x_2(t) = e^{\sigma t}\left[x_2(0)\cos\omega t + \left(\frac{\beta}{\omega} x_1(0) - \frac{\gamma-\alpha}{2\omega} x_2(0)\right)\sin\omega t\right]$$

<u>Conclusion:</u>

The brute force sol^{n} is quite messy. $x_1(t), x_2(t)$ is sinusoidal with exponentially decaying amplitude.

Where are the eigenvectors of A??

(A) $\lambda = 0$ Eigenvector is at $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(B) $\lambda = \sigma + i\omega$

Eigenvector $\Pi_1 + i\Pi_2$

we have

$$A(\Pi_1 + i\Pi_2) = (\sigma + i\omega)(\Pi_1 + i\Pi_2)$$

$$= \sigma\Pi_1 - \omega\Pi_2$$

$$+ i(\omega\Pi_1 + \sigma\Pi_2)$$

$$A\Pi_1 = \sigma\Pi_1 - \omega\Pi_2$$

$$A\Pi_2 = \omega\Pi_1 + \sigma\Pi_2$$

$$A\begin{pmatrix}\Pi_1 & \Pi_2\end{pmatrix} = \begin{pmatrix}\Pi_1 & \Pi_2\end{pmatrix}\begin{pmatrix}\sigma & \omega \\ -\omega & \sigma\end{pmatrix}$$

$$\begin{pmatrix}-\alpha & -\beta & 0 \\ \beta & -\gamma & 0 \\ \alpha & \gamma & 0\end{pmatrix}\begin{pmatrix}\Pi_{11} & \Pi_{21} \\ \Pi_{12} & \Pi_{22} \\ \Pi_{13} & \Pi_{23}\end{pmatrix} = \begin{pmatrix}\Pi_{11} & \Pi_{21} \\ \Pi_{12} & \Pi_{22} \\ \Pi_{13} & \Pi_{23}\end{pmatrix}\begin{pmatrix}\sigma & \omega \\ -\omega & \sigma\end{pmatrix}$$

$$\begin{pmatrix}-\alpha\Pi_{11} - \beta\Pi_{12} & -\alpha\Pi_{21} - \beta\Pi_{22} \\ \beta\Pi_{11} - \gamma\Pi_{12} & \beta\Pi_{21} - \gamma\Pi_{22} \\ \alpha\Pi_{11} + \gamma\Pi_{12} & \alpha\Pi_{21} + \gamma\Pi_{22}\end{pmatrix}$$

$$= \begin{pmatrix}\sigma\Pi_{11} - \omega\Pi_{21} & \omega\Pi_{11} + \sigma\Pi_{21} \\ \sigma\Pi_{12} - \omega\Pi_{22} & \omega\Pi_{12} + \sigma\Pi_{22} \\ \sigma\Pi_{13} - \omega\Pi_{23} & \omega\Pi_{13} + \sigma\Pi_{23}\end{pmatrix}$$

$$
\begin{pmatrix}
\sigma+\alpha & \beta & 0 & -\omega & 0 & 0 \\
\omega & 0 & 0 & \sigma+\alpha & \beta & 0 \\
-\beta & \sigma+\gamma & 0 & 0 & -\omega & 0 \\
0 & \omega & 0 & -\beta & \sigma+\gamma & 0 \\
-\alpha & -\gamma & \sigma & 0 & 0 & -\omega \\
0 & 0 & \omega & -\alpha & -\gamma & \sigma
\end{pmatrix}
\begin{pmatrix}
\Pi_{11} \\ \Pi_{12} \\ \Pi_{13} \\ \Pi_{21} \\ \Pi_{22} \\ \Pi_{23}
\end{pmatrix}
$$

$$\begin{pmatrix} \sigma+\alpha & \beta & -\omega & 0 \\ \omega & 0 & \sigma+\alpha & \beta \\ -\beta & \sigma+\gamma & 0 & -\omega \\ 0 & \omega & -\beta & \sigma+\gamma \end{pmatrix} \begin{pmatrix} \pi_{11} \\ \pi_{12} \\ \pi_{21} \\ \pi_{22} \end{pmatrix}$$

$$\begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix} \begin{pmatrix} \pi_{13} \\ \pi_{23} \end{pmatrix} = \begin{pmatrix} \alpha\pi_{11} + \gamma\pi_{12} \\ \alpha\pi_{21} + \gamma\pi_{22} \end{pmatrix}$$

$$\begin{pmatrix} \sigma+\alpha & -\omega \\ \omega & \sigma+\alpha \end{pmatrix} \begin{pmatrix} \pi_{11} \\ \pi_{21} \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \pi_{12} \\ \pi_{22} \end{pmatrix}$$

$$\begin{pmatrix} -\beta & 0 \\ 0 & -\beta \end{pmatrix} \begin{pmatrix} \pi_{11} \\ \pi_{21} \end{pmatrix} = \begin{pmatrix} \sigma+\gamma & -\omega \\ \omega & \sigma+\gamma \end{pmatrix} \begin{pmatrix} \pi_{12} \\ \pi_{22} \end{pmatrix}$$

$$\begin{pmatrix} \pi_{11} \\ \pi_{21} \end{pmatrix} = -\frac{1}{\beta} \begin{pmatrix} \sigma+\gamma & -\omega \\ \omega & \sigma+\gamma \end{pmatrix} \begin{pmatrix} \pi_{12} \\ \pi_{22} \end{pmatrix}$$

$$\left[\begin{pmatrix} \sigma+\alpha & -\omega \\ \omega & \sigma+\alpha \end{pmatrix} \begin{pmatrix} \sigma+\gamma & -\omega \\ \omega & \sigma+\gamma \end{pmatrix} + \begin{pmatrix} \beta^2 & 0 \\ 0 & \beta^2 \end{pmatrix} \right] \begin{bmatrix} \pi_{12} \\ \pi_{22} \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} (\sigma+\alpha)(\sigma+\gamma)-\omega^2 + \beta^2 & -\omega(2\sigma+\alpha+\gamma) \\ \omega(2\sigma+\alpha+\gamma) & (\sigma+\alpha)(\sigma+\gamma)-\omega^2 \end{pmatrix} \begin{pmatrix} \Pi_{12} \\ \Pi_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is a zero matix

$$\sigma^2 + \alpha\sigma + \gamma\sigma + \cancel{\alpha\gamma} - \cancel{\gamma\alpha} - \cancel{\beta^2} + \frac{(\alpha+\gamma)^2}{4} + \cancel{\beta^2}$$

$$2\sigma^2 + \sigma(-2\sigma) = 0$$

$$\begin{pmatrix} \pi_{12} \\ \pi_{22} \end{pmatrix} \text{ is arbitrary}$$

choose $\pi_{12} = 1 \quad \pi_{22} = 0$

$$\begin{pmatrix} \pi_{11} \\ \pi_{21} \end{pmatrix} = -\frac{1}{\beta} \begin{pmatrix} \sigma + r \\ \omega \end{pmatrix} = \begin{pmatrix} -\frac{\sigma + r}{\beta} \\ -\frac{\omega}{\beta} \end{pmatrix}$$

$$\alpha \pi_{11} + r \pi_{12} = r - \frac{\alpha}{\beta}(\sigma + r)$$

$$\alpha \pi_{21} + r \pi_{22} = -\frac{\alpha}{\beta} \omega$$

$$\begin{pmatrix} \pi_{13} \\ \pi_{23} \end{pmatrix} = \frac{\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} \begin{pmatrix} r - \frac{\alpha}{\beta}(\sigma + r) \\ -\frac{\alpha}{\beta}\omega \end{pmatrix}}{\sigma^2 + \omega^2}$$

$$(\Pi_1 \quad \Pi_2) =$$

$$\begin{pmatrix} -\frac{\sigma+r}{\beta} & -\frac{\omega}{\beta} \\ 1 & 0 \end{pmatrix}$$

$$\frac{\sigma r - \frac{\alpha}{\beta}\sigma(\sigma+r) - \frac{\alpha}{\beta}\omega^2}{\sigma^2+\omega^2} \qquad \frac{-\omega r + \frac{\alpha}{\beta}\omega(\sigma+r) - \frac{\alpha}{\beta}\omega\sigma}{\sigma^2+\omega^2}$$

$$= \left(\frac{\alpha}{\beta} - 1\right)\omega r$$

$$\frac{\left(1-\frac{\alpha}{\beta}\right)\sigma r - \frac{\alpha}{\beta}(\sigma^2+\omega^2)}{\sigma^2+\omega^2}$$

$$(\Pi_1, \ \Pi_2) =$$

$$\begin{pmatrix} -\dfrac{\sigma + r}{\beta} & -\dfrac{\omega}{\beta} \\ 1 & 0 \\ \left(1 - \dfrac{\alpha}{\beta}\right)\dfrac{\sigma r}{\sigma^2 + \omega^2} - \dfrac{\alpha}{\beta} & \left(\dfrac{\alpha}{\beta} - 1\right)\dfrac{\omega r}{\sigma^2 + \omega^2} \end{pmatrix}$$

Eigenvalue	Eigenvector
0	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = e_3$
$\sigma + i\omega$	$\Pi_1 + i\Pi_2$

Π_1 & Π_2 given on page 44

Define

$$T = \begin{pmatrix} \Pi_1 & \Pi_2 & e_3 \end{pmatrix}$$

Define new state variables

$$\underline{X} = T Z$$

we Obtain

$$\dot{Z} = T^{-1} A T Z + T^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$T^{-1} A T =$$

$$\begin{pmatrix} \sigma & \omega & 0 \\ -\omega & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In the new co-ordinates we have

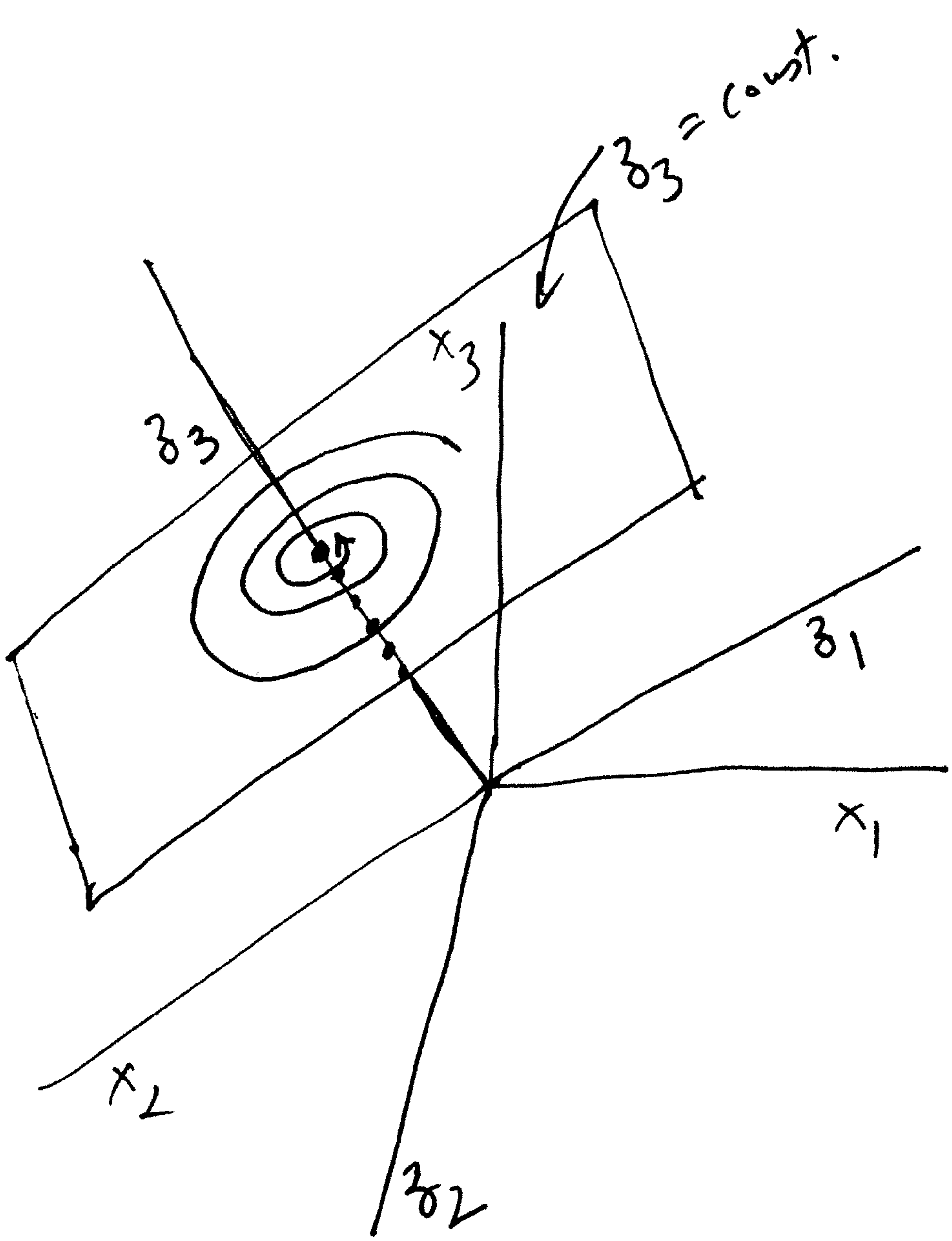
$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} \sigma & \omega & 0 \\ -\omega & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} * & * \\ * & * \\ * & * \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

we have not computed the '$*$'.

For $u_1 = u_2 = 0$ we obtain

$$z_3(t) = z_3(0).$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}(t) = e^{\sigma t} \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix}$$



$z_3(t)$ can be computed from.

$$Z = T^{-1} \underline{X}$$

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ a & b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\therefore z_3 = a x_1 + b x_2 + c x_3$

and since $z_3(t) = z_3(0) \ \forall t$.

it would follow that the plane

$$a x_1 + b x_2 + c x_3 = d$$

is invariant under the $\underline{X}$ dynamics

$$\dot{\underline{X}} = A \underline{X}$$

$$\Rightarrow (a \quad b \quad c) A \underline{X} = 0$$

$$\Rightarrow (a \quad b \quad c) \begin{pmatrix} -\alpha & -\beta \\ \beta & -\gamma \\ \alpha & \gamma \end{pmatrix} = (0 \quad 0)$$

$$\begin{pmatrix} -\alpha & \beta & \alpha \\ -\beta & -\gamma & \gamma \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Choose $c = 1$ we have

$$\begin{pmatrix} -\alpha & \beta \\ -\beta & -\gamma \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\gamma \end{pmatrix}$$

$$a = \frac{\begin{vmatrix} -\alpha & \beta \\ -\gamma & -\gamma \end{vmatrix}}{\begin{vmatrix} -\alpha & \beta \\ -\beta & -\gamma \end{vmatrix}} = \frac{\alpha\gamma + \beta\gamma}{\alpha\gamma + \beta^2} = \frac{(\alpha+\beta)\gamma}{\alpha\gamma+\beta^2}$$

$$b = \frac{\begin{vmatrix} -\alpha & -\alpha \\ -\beta & -\gamma \end{vmatrix}}{\alpha\gamma + \beta^2} = \frac{\alpha\gamma - \alpha\beta}{\alpha\gamma + \beta^2} = \frac{\alpha(\gamma-\beta)}{\alpha\gamma+\beta^2}$$

Scaling by $\alpha\gamma + \beta^2$ we obtain.

$$z_3 = (\alpha+\beta)\gamma\, x_1 + \alpha(\gamma-\beta)\, x_2 + (\gamma\alpha+\beta^2)\, x_3$$

Eqn of the invariant plane

$$(\alpha+\beta)\gamma\, x_1 + \alpha(\gamma-\beta)x_2 + (\gamma\alpha+\beta^2)x_3 = \text{const}.$$