H.W 5 (solutions)

1 Ans.

A

a) Write

$$A = \lambda I + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e^A = e^\lambda I * e^{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}$$ [show commutativity

$$= e^\lambda I \cdot \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^\lambda & e^\lambda & e^\lambda/2 \\ 0 & e^\lambda & e^\lambda \\ 0 & 0 & e^\lambda \end{pmatrix}$$

(b)

$$B = \begin{pmatrix} \mu & \mu & \mu/2 \\ 0 & \mu & \mu \\ 0 & 0 & \mu \end{pmatrix} \qquad \mu = e^{\lambda}$$

B has eigenvalues at μ, μ, μ.

with eigenvectors/gen. eigenvectors at

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/\mu \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{2\mu^2} \\ \frac{1}{\mu^2} \end{pmatrix}$$

(c) Define

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\mu & -1/2\mu^2 \\ 0 & 0 & 1/\mu^2 \end{pmatrix}$$

$$BP =$$

$$\begin{pmatrix} \mu & \mu & \mu/2 \\ 0 & \mu & \mu \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\mu & -\frac{1}{2\mu^2} \\ 0 & 0 & \frac{1}{\mu^2} \end{pmatrix}.$$

$$= \begin{pmatrix} \mu & 1 & 0 \\ 0 & 1 & \frac{1}{2\mu} \\ 0 & 0 & 1/\mu \end{pmatrix}.$$

$$PC = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\mu} & -\frac{1}{2\mu^2} \\ 0 & 0 & \frac{1}{\mu^2} \end{pmatrix} \begin{pmatrix} \mu & 1 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{pmatrix}$$

$$= \begin{pmatrix} \mu & 1 & 0 \\ 0 & 1 & \frac{1}{2\mu} \\ 0 & 0 & 1/\mu \end{pmatrix}.$$

Hence

$$P^{-1}BP = C$$

B

a

$$M = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

$$e^{M} = \begin{pmatrix} e^{\sigma} & 0 \\ 0 & e^{\sigma} \end{pmatrix} e^{\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}}$$

$$= \begin{pmatrix} e^{\sigma} & 0 \\ 0 & e^{\sigma} \end{pmatrix}\begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}$$

$$= \begin{pmatrix} e^{\sigma}\cos\omega & e^{\sigma}\sin\omega \\ -e^{\sigma}\sin\omega & e^{\sigma}\cos\omega \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} M & I & 0 \\ 0 & M & I \\ 0 & 0 & M \end{pmatrix}$$

$$= \begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{pmatrix} + \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix}$$

$$e^A = \begin{pmatrix} e^M & 0 & 0 \\ 0 & e^M & 0 \\ 0 & 0 & e^M \end{pmatrix} \begin{pmatrix} I & I & I/2 \\ 0 & I & I \\ 0 & 0 & I \end{pmatrix}$$

(show commutativity).

$$= \begin{pmatrix} e^M & e^M & e^M/2 \\ 0 & e^M & e^M \\ 0 & 0 & e^M \end{pmatrix}$$

c) We want to show that

$$B = \begin{pmatrix} e^M & e^M & e^M/2 \\ 0 & e^M & e^M \\ 0 & 0 & e^M \end{pmatrix}$$

is similar to

$$C = \begin{pmatrix} e^M & I & 0 \\ 0 & e^M & I \\ 0 & 0 & e^M \end{pmatrix}$$

Remark: This would be a hard problem if we have to find the generalized eigenvectors and construct the P matrix.

However in view of the 1st part of this exercise, choose

$$P = \begin{pmatrix} I & 0 & 0 \\ 0 & N^{-1} & -\frac{N^{-2}}{2} \\ 0 & 0 & N^{-1} \end{pmatrix}$$

where $N = e^{M}$.

$$BP = \begin{pmatrix} N & I & 0 \\ 0 & I & \frac{N^{-1}}{2} \\ 0 & 0 & N^{-1} \end{pmatrix} = PC$$

Hence B & C are similar.

2 Ans.

a $B = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \ \lambda_i > 0$

To show that $C = \ln B = \begin{pmatrix} \ln\lambda_1 & 0 & 0 \\ 0 & \ln\lambda_2 & 0 \\ 0 & 0 & \ln\lambda_3 \end{pmatrix}$

$$e^C = e^{(\ )} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix} = B.$$

b $B = \begin{pmatrix} 1 & \mu & 0 \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{pmatrix} = I + \underbrace{\begin{pmatrix} 0 & \mu & 0 \\ 0 & 0 & \mu \\ 0 & 0 & 0 \end{pmatrix}}_{A}$

$$\ln B = \ln(I + A)$$

$$= A - \frac{A^2}{2} = \begin{pmatrix} 0 & \mu & 0 \\ 0 & 0 & \mu \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & 0 & \mu^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore \ln B =$

$$\begin{pmatrix} 0 & \mu & -\mu^2/2 \\ 0 & 0 & \mu \\ 0 & 0 & 0 \end{pmatrix}$$

(c) Ans:

Since $\ln C$ & $\ln D$ are defined it follows that

$$\ln C = (C - I) - \frac{(C-I)^2}{2!} + \frac{(C-I)^3}{3!} - \cdots$$

$$\ln D = (D - I) - \frac{(D-I)^2}{2!} + \frac{(D-I)^3}{3!} - \cdots$$

The series in the r.h.s converges.

Thus. $\ln C$ & $\ln D$ commutes since C&D commutes.

$$\therefore e^{\ln C + \ln D} = e^{\ln C} \cdot e^{\ln D} = CD.$$

Thus.

$$\ln C + \ln D = \ln(CD)$$

d $\quad B = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$

$$= \underbrace{\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} 1 & 1/\lambda & 0 \\ 0 & 1 & 1/\lambda \\ 0 & 0 & 1 \end{pmatrix}}_{D}$$

[C & D commute.]

Hence

$$\ln B = \ln C + \ln D$$

$$\ln C = \begin{pmatrix} \ln\lambda & 0 & 0 \\ 0 & \ln\lambda & 0 \\ 0 & 0 & \ln\lambda \end{pmatrix}$$

$$\ln D = \begin{pmatrix} 0 & 1/\lambda & -\frac{1}{2\lambda^2} \\ 0 & 0 & 1/\lambda \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \ln B = \begin{pmatrix} \ln\lambda & 1/\lambda & -\frac{1}{2\lambda^2} \\ 0 & \ln\lambda & 1/\lambda \\ 0 & 0 & \ln\lambda \end{pmatrix}$$

(e) There is nothing to do here. It was already done in the question paper.

(f) $\sigma = 2 \quad \omega = 3$

$$\sigma^2 + \omega^2 = 4 + 9 = 13$$

$$\ln \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\ln 13 & \theta \\ -\theta & \frac{1}{2}\ln 13 \end{pmatrix}$$

$$\cos\theta = \frac{2}{\sqrt{13}} \;;\; \sin\theta = \frac{3}{\sqrt{13}}.$$

write

$$A = \begin{pmatrix} \begin{matrix} 2 & 3 \\ -3 & 2 \end{matrix} & O \\ O & \begin{matrix} 2 & 3 \\ -3 & 2 \end{matrix} \end{pmatrix} \begin{pmatrix} I & \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}^{-1} \\ 0 & I \end{pmatrix}$$

(13)

$$\ln \begin{pmatrix} I & \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}^{-1} \\ 0 & I \end{pmatrix} = \begin{bmatrix} 0 & \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}^{-1} \\ 0 & 0 \end{bmatrix}$$

$\therefore \ \ln A =$

$$\left(\begin{array}{cccc} \frac{1}{2}\ln 13 & \theta & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}^{-1}$}} \\ -\theta & \frac{1}{2}\ln 13 & & \\ 0 & 0 & \frac{1}{2}\ln 13 & \theta \\ 0 & 0 & -\theta & \frac{1}{2}\ln 13 \end{array} \right)$$

3 Ans:

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = \cos t \; x_1 + 2x_2$$

Solving $\dot{x}_1 = x_1$ we obtain

$$x_1(t) = e^{t} x_1(0)$$

So we have

$$\dot{x}_2 = 2x_2 + e^{t} \cos t \; x_1(0)$$

Solving the above eqn we have

$$x_2(t) = e^{2t} x_2(0) + \int_0^t e^{2(t-\tau)} e^{\tau} \cos\tau \, d\tau \cdot x_1(0).$$

$$= e^{2t}\left[x_2(0) + \int_0^t e^{-\tau} \cos\tau \, d\tau \cdot x_1(0) \right]$$

$\int_0^t e^{-\tau} \cos\tau \, d\tau$ has to be calculated.

We now proceed to do just that.

Define

$$I_1 = \int_0^t e^{-\tau} \cos\tau \, d\tau$$

$$I_2 = \int_0^t e^{-\tau} \sin\tau \, d\tau$$

It follows that

$$I_1 = e^{-\tau} \sin\tau \Big|_0^t - \int_0^t -e^{-\tau} \sin\tau \, d\tau .$$

$$= e^{-t} \sin t + I_2 \qquad (*)$$

$$I_2 = -e^{-\tau} \cos\tau \Big|_0^t + \int_0^t -e^{-\tau} \cos\tau \, d\tau$$

$$= \left[-e^{-t} \cos t + 1\right] - I_1 \qquad (**)$$

From $(*)$ & $(**)$ we obtain.

$$I_1 = e^{-t} \sin t + \left[1 - e^{-t} \cos t\right] - I_1 .$$

$$\therefore I_1 = \frac{1}{2} \left[1 - e^{-t} \cos t + e^{-t} \sin t\right]$$

$$\therefore\ x_2(t) = e^{2t}\, x_2(0) + \frac{1}{2}\left[e^{2t} - e^{t}\cos t + e^{t}\sin t\right] x_1(0)$$

$$\therefore \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{pmatrix} e^{t} & 0 \\ \frac{1}{2}\left[e^{2t} - e^{t}\cos t + e^{t}\sin t\right] & e^{2t} \end{pmatrix}}_{\phi(t,0)} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.$$

(b)

$$\Phi(2\pi, 0) = \begin{pmatrix} e^{2\pi} & 0 \\ \frac{1}{2}\left[e^{4\pi} - e^{2\pi}\right] & e^{4\pi} \end{pmatrix}$$

(c) Denote $\alpha = e^{2\pi}$, $\beta = e^{4\pi}$

$$A = \begin{pmatrix} \alpha & 0 \\ \frac{1}{2}(\beta - \alpha) & \beta \end{pmatrix} \qquad \begin{aligned} \lambda_1 &= \alpha \\ \lambda_2 &= \beta \end{aligned}$$

$$\ln A = \alpha_0 I + \alpha_1 A$$

$$\ln \lambda_1 = \alpha_0 + \alpha_1 \lambda_1 = 2\pi$$

$$\ln \lambda_2 = \alpha_0 + \alpha_1 \lambda_2 = 4\pi$$

$$\alpha_1 = \frac{2\pi}{\beta - \alpha}, \quad \alpha_0 = 2\pi - \frac{2\pi}{\beta - \alpha}\alpha$$

$$= 2\pi\left(\frac{\beta - \alpha - \alpha}{\beta - \alpha}\right) = 2\pi\left(\frac{\beta - 2\alpha}{\beta - \alpha}\right)$$

$$\ln A =$$

$$\begin{pmatrix} 2\pi \dfrac{\beta-2\alpha}{\beta-\alpha} + 2\pi \dfrac{\alpha}{\beta-\alpha} & 0 \\ \pi & 2\pi \dfrac{\beta-2\alpha}{\beta-\alpha} + \dfrac{2\pi}{\beta-\alpha}\beta \end{pmatrix}$$

$$= \begin{pmatrix} 2\pi & 0 \\ \pi & 4\pi \end{pmatrix}$$

$$R = \begin{pmatrix} \dfrac{2\pi}{2\pi} & 0 \\ \dfrac{\pi}{2\pi} & \dfrac{4\pi}{2\pi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 2 \end{pmatrix}$$

$$\phi(t,0)^{-1} = \frac{\begin{pmatrix} e^{2t} & 0 \\ -\frac{1}{2}\left[e^{2t} - e^{t}\cos t + e^{t}\sin t\right] & e^{t} \end{pmatrix}}{e^{3t}}$$

$$= \begin{pmatrix} e^{-t} & 0 \\ -\frac{1}{2}\left[e^{-t} - e^{-2t}\cos t + e^{-2t}\sin t\right] & e^{-2t} \end{pmatrix}$$

We now compute e^{Rt}

$$e^{Rt} = \alpha_0 I + \alpha_1 R$$

$$e^{t} = \alpha_0 + \alpha_1$$
$$e^{2t} = \alpha_0 + 2\alpha_1$$

$$\alpha_1 = e^{2t} - e^{t}$$
$$\alpha_0 = e^{t} - e^{2t} + e^{t} = 2e^{t} - e^{2t}$$

$$\therefore e^{Rt} = \begin{pmatrix} 2e^{t} - \cancel{e^{2t}} + \cancel{e^{2t}} - e^{t} & 0 \\ \frac{1}{2}e^{2t} - \frac{1}{2}e^{t} & \cancel{2}e^{t} - e^{2t} + 2e^{2t} - \cancel{2}e^{t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{t} & 0 \\ \frac{1}{2}e^{2t} - \frac{1}{2}e^{t} & e^{2t} \end{pmatrix}$$

(a)

$$P(t) = e^{Rt}\,\phi(t,0)^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{1}{2}e^{t} - \frac{1}{2} - \frac{1}{2}\left(e^{t} - \cos t + \sin t\right) & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{1}{2}\left[\cos t - \sin t - 1\right] & 1 \end{pmatrix}$$

Clearly $P(t+2\pi) = P(t)$

(e)

$$z_1 = x_1$$

$$z_2 = x_2 + \frac{1}{2}[\cos t - \sin t - 1]\, x_1$$

$$\dot{z}_1 = \dot{x}_1 = x_1 = z_1 .$$

$$\dot{z}_2 = \dot{x}_2 + \frac{1}{2}[\cos t - \sin t - 1]\, \dot{x}_1$$
$$+ \frac{1}{2}[-\sin t - \cos t]\, x_1$$

$$= \cos t . x_1 + 2x_2$$
$$+ \frac{1}{2}[\cancel{\cos t} - \sin t - 1 - \sin t - \cancel{\cos t}]\, x_1$$

$$= 2x_2 + \left(\cos t - \sin t - \frac{1}{2}\right) x_1 .$$

$$= 2\left[z_2 - \frac{1}{2}(\cos t - \sin t - 1) z_1\right]$$
$$+ \left(\cos t - \sin t - \frac{1}{2}\right) z_1$$

$$= \frac{1}{2} z_1 + 2 z_2$$

Thus

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/2 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

④ Ans:

(i) The matrix

$$(b \mid Ab \mid A^2 b \mid A^3 b)$$

is given by

See Lec 4 page 13

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & 2\omega & 0 \\ 0 & 2\omega & 0 & -2\omega^3 \\ 0 & 1 & 0 & -4\omega^2 \\ 1 & 0 & -4\omega^2 & 0 \end{pmatrix}$$

$$\det \mathcal{C} = 2\omega \begin{vmatrix} 2\omega & -2\omega^3 \\ 1 & -4\omega^2 \end{vmatrix}$$

$$= 2\omega\left[-8\omega^3 + 2\omega^3\right]$$

$$= 2\omega\left[-6\omega^3\right] = -12\omega^4 \neq 0$$

if $\omega \neq 0$.

Hence rank $\mathcal{C} = 4$

The pair is controllable.

(ii) To show

$$P^{-1}AP = F$$

we verify that

$$PF = AP$$

$$AP = \left(A^4 b \mid A^3 b \mid A^2 b \mid Ab \right)$$

$$PF = \left(\pi \mid A^3 b \mid A^2 b \mid Ab \right)$$

writing

$$F = \begin{pmatrix} c_1 & 1 & 0 & 0 \\ c_2 & 0 & 1 & 0 \\ c_3 & 0 & 0 & 1 \\ c_4 & 0 & 0 & 0 \end{pmatrix}$$

we get

$$\pi = c_1 A^3 b + c_2 A^2 b + c_3 Ab + c_4 b.$$

From Cayley Hamilton Theorem we know that $p(A) = 0$ where $p(\lambda) = \lambda^4 + \omega^2 \lambda^2$

$$\therefore A^4 + \omega^2 A^2 = 0$$

$$\Rightarrow A^4 = -\omega^2 A^2$$

$$\Rightarrow A^4 b = -\omega^2 A^2 b$$

$$\therefore c_1 = 0,\ c_2 = -\omega^2,\ c_3 = 0,\ c_4 = 0.$$

$$\& \quad A^4 b = \pi$$

Finally note that

$$P\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = b \quad \text{Hence } P^{-1}b = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

(iii) $$F = \begin{pmatrix}0 & 1 & 0 & 0\\ -\omega^2 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

(iv) $$\dot{z} = P^{-1}\dot{x} = P^{-1}(A\underline{x} + bu)$$
$$= P^{-1}APz + P^{-1}bu$$
$$= Fz + bu.$$

(v)

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$\mathcal{C}$ is of rank 4

Remark: The point is that if

$$F = \begin{pmatrix} c_1 & 1 & 0 & 0 \\ c_2 & 0 & 1 & 0 \\ c_3 & 0 & 0 & 1 \\ c_4 & 0 & 0 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

the dynamical system

$$\dot{z} = Fz + gu$$

is always controllable irrespective of the values of c_1, c_2, c_3, c_4. Hence this is called the controllable canonical form.

5 Ans

(i) Construct the observability matrix

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \\ -6\omega^3 & 0 & 0 & -4\omega^2 \end{pmatrix}$$

$$\det \mathcal{O} = 12\omega^4 \leftarrow \text{upto sign.}$$

$$\neq 0 \text{ if } \omega \neq 0.$$

Hence $\operatorname{rank} \mathcal{O} = 4$.

ii

Define

$$A_1 = -A^T$$
$$b_1 = c^T$$

$$\therefore\ A_1 = \begin{pmatrix} 0 & -3\omega^2 & 0 & 0 \\ -1 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 0 \\ 0 & -2\omega & -1 & 0 \end{pmatrix}$$

$$b_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Define

$$P_1 = \left(-A_1^3 b_1 \mid A_1^2 b_1 \mid -A_1 b_1 \mid b_1 \right)$$

we know from problem 4 that

$$P_1^{-1} A_1 P_1 = \begin{pmatrix} x & -1 & 0 & 0 \\ x & 0 & -1 & 0 \\ x & 0 & 0 & -1 \\ x & 0 & 0 & 0 \end{pmatrix}$$

$$P_1^{-1} b_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

which implies that

$$P_1^T A_1^T (P_1^T)^{-1} = \begin{pmatrix} * & x & x & x \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$b_1^T (P_1^T)^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$$

Define $P = (P_1^T)^{-1}$ we obtain.

$$P^{-1} A P = \begin{pmatrix} x & x & x & x \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$cP = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}.$$

Computing P_1 we get

$$P_1 = \begin{pmatrix} 6\omega^3 & 0 & 0 & 0 \\ 0 & 2\omega & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 4\omega^2 & 0 & 1 & 0 \end{pmatrix}$$

$$P_1^T = \begin{pmatrix} 6\omega^3 & 0 & 0 & 4\omega^2 \\ 0 & 2\omega & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$P = (P_1^T)^{-1} =$$

$$\begin{pmatrix} \frac{1}{6\omega^3} & 0 & -\frac{2}{3\omega} & 0 \\ 0 & \frac{1}{2\omega} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(iii) writing

$$P_1^{-1} A_1 P_1 = \begin{pmatrix} c_1 & -1 & 0 & 0 \\ c_2 & 0 & -1 & 0 \\ c_3 & 0 & 0 & -1 \\ c_4 & 0 & 0 & 0 \end{pmatrix}$$

we have

$$-A_1^4 b_1 = -c_1 A_1^3 b_1 + c_2 A_1^2 b_1 - c_3 A_1 b_1 + c_4 b_1 .$$

$$\Rightarrow A_1^4 b_1 = c_1 A_1^3 b_1 - c_2 A_1^2 b_1 + c_3 A_1 b_1 - c_4 b_1 .$$

Of course we can check that

$$A_1^4 = -\omega^2 A_1^2$$

$$\therefore c_1 = c_3 = c_4 = 0 \quad c_2 = \omega^2$$

It would follow that

$$P^{-1} A P = \begin{pmatrix} -c_1 & -c_2 & -c_3 & -c_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = F$$

$$F = \begin{pmatrix} 0 & -\omega^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(iv) $\dot{z} = P^{-1}\dot{\underline{x}} = P^{-1}A\underline{x}$

$= P^{-1}APz = Fz.$

$y = C\underline{x} = CPz.$

$= gz.$

(v) $\mathcal{O} = \begin{pmatrix} g \\ gF \\ gF^2 \\ gF^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$\operatorname{rank} \mathcal{O} = 4.$

6 Ans:

(i) $$\mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & -\omega^2 & 0 & 0 \end{pmatrix}$$

rank $\mathcal{O} = 3$

Hence the pair is not observable.

(ii) Null space of $\mathcal{O}$ is given by vectors of

the form $\begin{pmatrix} 0 \\ 0 \\ \alpha \\ 0 \end{pmatrix}$.

If $X(0) = \begin{pmatrix} 1 \\ 1 \\ \alpha \\ 1 \end{pmatrix}$, all these initial conditions would produce the same output at $X(0) = (1 \ 1 \ 1 \ 1)^T$.

This is because

$$y(t) = C e^{At} x(0)$$

$$= C e^{At} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C e^{At} \begin{pmatrix} 0 \\ 0 \\ \alpha - 1 \\ 0 \end{pmatrix}$$

writing $C e^{At} =$

$$\left(C + CAt + C\frac{A^2 t^2}{2!} + \cdots\cdots \right)$$

we conclude that

$$C e^{At} \begin{pmatrix} 0 \\ 0 \\ \alpha - 1 \\ 0 \end{pmatrix} = 0 \quad \forall\, \alpha \in \mathbb{R}$$