$\boxed{\text{H. W. 5}}$

1 In this problem we want to study if exponentiating a matrix preserves the Jordan structure.

A Let

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

← This is already in Jordan canonical form.

Define $B = e^{A}$

a show that

$$B = \begin{pmatrix} e^{\lambda} & e^{\lambda} & \tfrac{1}{2}e^{\lambda} \\ 0 & e^{\lambda} & e^{\lambda} \\ 0 & 0 & e^{\lambda} \end{pmatrix}$$

b Calculate eigenvalues and generalized eigenvectors of B.

(c) Verify if the following statement is true:

" B is similar to a matrix C where

$$C = \begin{pmatrix} e^{\lambda} & 1 & 0 \\ 0 & e^{\lambda} & 1 \\ 0 & 0 & e^{\lambda} \end{pmatrix}$$ "

Remark: Remember that B is similar to C if $\exists$ a nonsingular matrix P : $P^{-1} B P = C$.

B Let

$$A = \begin{pmatrix} M & I & 0 \\ 0 & M & I \\ 0 & 0 & M \end{pmatrix}$$

where $M = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Define $B = e^A$.

(a) Show that

$$e^M = \begin{pmatrix} e^\sigma \cos\omega & e^\sigma \sin\omega \\ -e^\sigma \sin\omega & e^\sigma \cos\omega \end{pmatrix}$$

(b) show that

$$B = \begin{pmatrix} e^M & e^M & \frac{1}{2}e^M \\ 0 & e^M & e^M \\ 0 & 0 & e^M \end{pmatrix}$$

(c) Verify if the following statement is true:

B is similar to a matrix C where

$$C = \begin{pmatrix} e^M & I & 0 \\ 0 & e^M & I \\ 0 & 0 & e^M \end{pmatrix}$$

2 In this problem, we want to understand natural log of matrices ie $\ln(B)$ where B is a square matrix.

Formally

$$\ln B = C \quad \text{if } e^{C} = B$$

(This is the definition of logarithm)

Recall that we can write

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} \cdots$$

where the equality is valid when the r.h.s. converges. For example at $x=-1$, the r.h.s. doesnot converge. However for $x>-1$ it does.

If A is a $n \times n$ matrix, we define

$$\ln(I+A) = A - \frac{A^2}{2!} + \frac{A^3}{3!} - \frac{A^4}{4!} \cdots$$

provided the r.h.s. converges.

(a) If

$$B = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$ where $\lambda_i > 0$, $i = 1, 2, 3$.

Show that

$$\ln B = \begin{pmatrix} \ln \lambda_1 & 0 & 0 \\ 0 & \ln \lambda_2 & 0 \\ 0 & 0 & \ln \lambda_3 \end{pmatrix}$$

(b) If

$$B = \begin{pmatrix} 1 & \mu & 0 \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{pmatrix}$$ Calculate $\ln B$.

(c) Let C & D be two square $n \times n$ matrices for which $\ln C$ & $\ln D$ are defined

show that

$$\ln CD = \ln C + \ln D$$

if C & D commutes. ie $CD = DC$.

(d) Let

$$B = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \lambda > 0$$

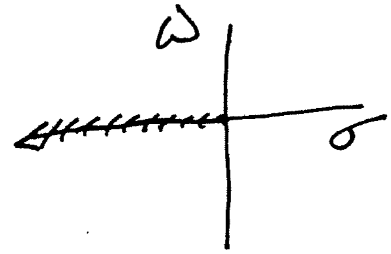
Calculate $\ln B$

Hint: Write

$$B = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 1/\lambda & 0 \\ 0 & 1 & 1/\lambda \\ 0 & 0 & 1 \end{pmatrix}$$

use the product formula on top of this page.

(e) Let



$$A = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}, \quad (\sigma, \omega) \notin \text{negative real axis including } (0,0).$$

calculate $\ln A$

Hint: write A as.

$$A = \sqrt{\sigma^2 + \omega^2} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

where $\cos\theta = \dfrac{\sigma}{\sqrt{\sigma^2+\omega^2}}$, $\sin\theta = \dfrac{\omega}{\sqrt{\sigma^2+\omega^2}}$

write

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = e^{\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}}$$

$$A = \sqrt{\sigma^2+\omega^2}\, e^{\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}}$$

$$\ln A = \begin{pmatrix} \ln(\sigma^2+\omega^2)^{1/2} & \theta \\ -\theta & \ln(\sigma^2+\omega^2)^{1/2} \end{pmatrix}$$

(f) Let

$$A = \begin{pmatrix} 2 & 3 & 1 & 0 \\ -3 & 2 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

Calculate $\ln A$.

3 Consider the 2 dimensional system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \cos t & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

a Write down the fundamental matrix $\phi(t, 0)$.

b Compute $\phi(2\pi, 0)$ for $T = 2\pi$

c compute

$$R = \frac{1}{T} \ln[\phi(2\pi, 0)]$$

d compute

$$P(t) = e^{Rt} \phi(t, 0)^{-1}$$

and show that

$$P(t + 2\pi) = P(t).$$

e Define

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P(t) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and verify that $\dot{z} = Rz$.

Remark:

you cannot write

$$\begin{pmatrix} 1 & 0 \\ \cos t & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \cos t \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and claim that

$$\phi(t,0) = e^{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} t} \; e^{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \int_0^\sigma \cos\sigma \, d\sigma}$$

because ...... (can you say why?).

Think of other ideas.

4 Consider the satellite problem

$$\dot{\underline{X}} = A\underline{X} + bu$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{pmatrix} ; \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Here we choose $u_1 = 0$ and actuate the satellite using only u_2.

(i) Show that the above pair is controllable.

(ii) Choose

$$P = \left(A^3 b \,|\, A^2 b \,|\, Ab \,|\, b \right)$$

and show that

$$P^{-1} A P = \underbrace{\begin{pmatrix} \times & 1 & 0 & 0 \\ \times & 0 & 1 & 0 \\ \times & 0 & 0 & 1 \\ \times & 0 & 0 & 0 \end{pmatrix}}_{F} ; \quad P^{-1} b = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{b}$$

(iii) Find F if you have not already done so.

(iv) Write

$$\underline{X} = PZ$$

and obtain

$$\dot{z} = Fz + bu. \quad (\star)$$

$(\star)$ is called the controllable canonical form

(v) Write down the controllability matrix

$$\mathcal{C} \triangleq (b \mid Fb \mid F^2b \mid F^3b)$$

associated with $(\star)$.

5 Consider the satellite dynamics

$$\dot{\underline{X}} = A\underline{X}$$
$$y = C\underline{X}$$

where

A is defined in problem 4.

$C = (0 \ 0 \ 1 \ 0)$

Here we assume that we observe only the 3rd state variable "the angle".

(i) show that the above pair is observable.

(ii) Construct a nonsingular matrix P such that

$$P^{-1}AP = \underbrace{\begin{pmatrix} x & x & x & x \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{F}, \quad CP^{-1} = \underbrace{(0 \ 0 \ 0 \ 1)}_{g}$$

(iii) Find F if you have not already done so.

(iv) Write

$$\underset{\sim}{x} = P z$$

and obtain

$$\left.\begin{aligned} \dot{z} &= F z \\ y &= g z \end{aligned}\right\}$$

This is ~~the~~ observable canonical ~~form~~.

(v) Write down ~~the~~ observability matrix

$$\mathcal{O} \triangleq \begin{pmatrix} g \\ gF \\ gF^2 \\ gF^3 \end{pmatrix}.$$

6 Consider the satellite dynamics

$$\dot{\underline{x}} = A\underline{x} \qquad x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$
$$y = C\underline{x}$$

where

A is defined in problem 4

$C = (1 \quad 0 \quad 0 \quad 0)$

Here we assume that we observe only the 1st state variable "the radial distance"

(i) Show that the above pair is not observable.

(ii) Find the set of all initial conditions that cannot be distinguished from $x(0) = (1 \quad 1 \quad 1 \quad 1)^T$.

Remark: Try not to use the observability gramian