Home Work IV

1. In this problem, we shall study the connection between a matrix A and its eigenvalues.

(a) consider a homogeneous system

$$\dot{X} = AX, \quad X(0) = X_0$$

where we assume that all eigenvalues of A have negative real parts. Show that $\|X(t)\|$ remains bounded for all $t \geq 0$, ie $\exists M > 0$:

$$\|X(t)\| < M \quad \forall t \geq 0.$$

(b) Let

$$A = \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$X(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Show that $\|x(t)\|$ is not bounded.

$$\left[\text{Take } \|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\right]$$

for all $t \geq 0$.

(c) Define

$$B = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = e^{Bt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(i) Show that $\|z\|$ is bounded iff $\|x\|$ is bounded.

(ii) We know from (b) that $\|x(t)\|$ is not bounded. What can we say about $\|z(t)\|$.

(d) Show that

$$\dot{z} = R(t)\, z$$

$$z(0) = x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

where

$$R(t) = e^{Bt}(A+B)e^{-Bt}$$

(e) Show that the eigenvalues of $R(t)$ are independent of t and are at the are eigenvalues of $A+B$.

(f) Show that the eigenvalues of $A+B$ are at the zeros of the polynomial.

$$\lambda^2 + 3\lambda + 5$$

Hence conclude that the eigenvalues have negative real parts.

In fact the eigenvalues are at

$$\lambda_1, \lambda_2 = -\frac{3}{2} \pm i\sqrt{\frac{11}{4}}$$

(g) Conclusion:—

We have a dynamical system

$$\dot{z} = R(t)\, z \,, \quad z(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

where

$$R(t) = e^{Bt} \begin{pmatrix} -4 & 3 \\ -3 & 1 \end{pmatrix} e^{-Bt}$$

The eigenvalues of $R(t)$ have negative real parts $\forall\, t \geq 0$. However $\|z\|$ is not bounded.

Remark: There isn't anything to show for part (g). This problem is constructed by Vinogradov, Markus-Yamabe, Rosenbrock

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A Consider

$$\dot{x} = A(t)\,x(t)$$

$$x(0) = x_0.$$

Let $\phi(t,0)$ be the fundamental matrix

ie

$$\dot{\phi}(t,0) = A(t)\,\phi(t,0)$$

(i) show that:

$$\frac{d}{dt}\left[\phi^{-1}(t,0)\right] = -\phi^{-1}(t,0)\,A(t).$$

(ii) Let $P(t)$ be a $n \times n$ square matrix defined as

$$P(t) = e^{Rt}\,\phi(t,0)^{-1}$$

for some square $n \times n$ constant matrix R.

Argue why $P(t)$ is nonsingular for all $t \geq 0$.

(iii) Show that

$$\dot{P}(t) = R\,P(t) - P(t)\,A(t).$$

(iv) Define a new variable $z(t)$ as follows —

$$z(t) = P(t)\,x(t),$$

show that

$$\dot{z}(t) = (\dot{P} + PA)\,P^{-1} z(t).$$

(v) From (iii) & (iv) conclude that

$$\dot{z}(t) = R\,z.$$

Conclusion: Any time varying homogeneous system can be converted to a time invariant homogeneous system in z co-ordinates.

(vi) Argue why in general the transformation

$$z(t) = P(t)\,x(t)$$

is not a Lyapunov Transformation.

(vii) Assume that $\exists T > 0 : A(t+T) = A(t)$ $\forall t \geq 0$ ie $A(t)$ is periodic and assume that $\exists R : \phi(T,0) = e^{RT}$,

show that

$$P^{-1}(t+T) = P^{-1}(t), \quad \forall t \geq 0.$$

i.e P^{-1} and hence P is periodic.

Hint:

$$P^{-1}(t+T) = \phi(t+T, 0)\, e^{-R(t+T)}$$

$$= \phi(t+T, T)\, \phi(T, 0)\, e^{-R(t+T)}$$

$$= \phi(t+T, T)\, e^{-Rt}$$

(viii) Argue why "periodicity of $P(t)$" implies that

$$z(t) = P(t)\,x(t)$$

is a Lyapunov Transformation.

B Consider

$$\dot{x} = A(t)\,x(t)$$

where

$$A(t) = \alpha(t)\,A, \quad A \text{ is a constant matrix.}$$

and where $\alpha(t+T) = \alpha(t)$, $t \geq 0$

i.e. $\alpha(t)$ is periodic. Find a $P(t)$ such that when we define

$$z(t) = P(t)\,x(t)$$

we obtain

$$\dot{z} = \left[\frac{1}{T}\int_0^T \alpha(\sigma)\,d\sigma\right] A\,z.$$

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$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{B} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \quad (*)$$

(a) computing the rank of the controllability matrix

$$(B \mid AB \mid A^2B) = \mathcal{C}$$

show that $(*)$ is not controllable

(b) Calculate the range of the 3×3 matrix $\mathcal{C}\mathcal{C}^T$.

(c) Find the set of all points in $\mathbb{R}^3$ one can reach at $t = T = 10$ starting from $x_1(0) = 2, x_2(0) = x_3(0) = 1$ at $t = 0$, for a suitable choice of control vector (u_1, u_2).

Hint: Use the range calculation in (b). Dont use the controllability gramian.

(d) Describe the set of all points in $\mathbb{R}^3$ that can be reached for some final time $T > 0$ starting from $x_1(0) = x_2(0) = x_3(0) = 1$ at $t = 0$, for a suitable choice of control vector (u_1, u_2).

(e) Let P be a nonsingular 3×3 matrix such that

$$P^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

define

$$z = Px$$

show that

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (\star)$$

(f) Argue that the states $z_1(t)$, $z_2(t)$ are controllable but $z_3(t)$ is not.

(g) From (A) it follows that

$$z_3(t) = e^t z_3(0),$$

why??

show that $z_3 = x_1 - x_3$, hence

$$z_3(0) = x_1(0) - x_3(0) = 1.$$

Hence conclude that

$$x_1(t) - x_3(t) = e^t \cdot 1 = e^t \quad \forall\, t \geq 0$$

(h) Argue that the set of points in $\mathbb{R}^3$ that can be reached at $t = T = 10$ must be contained in the plane

$$x_1 - x_3 = e^{10} \quad (**)$$

Argue that the set of all points in $\mathbb{R}^3$ that can be reached at $t = T = 10$ is given by the plane

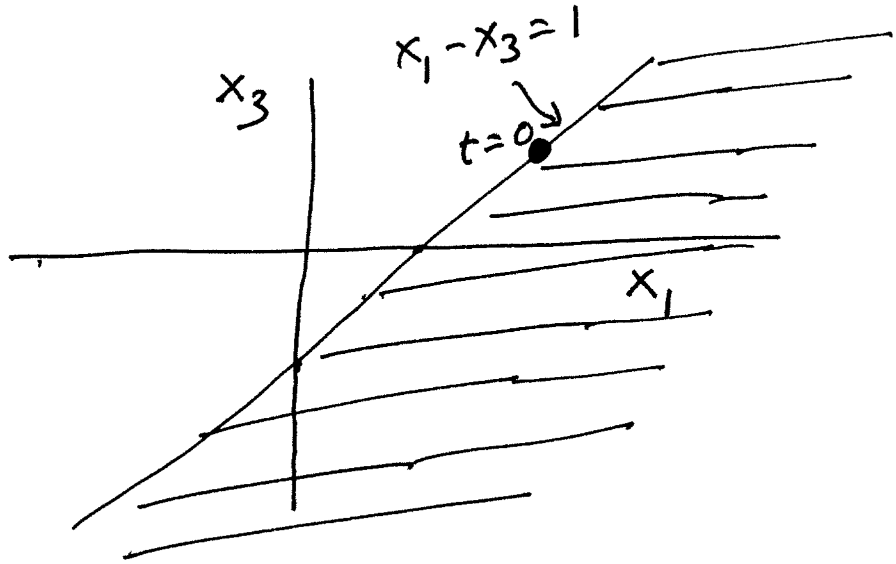
$$x_1 - x_3 = e^{10}.$$

(i) Finally show that the set of all points in $\mathbb{R}^3$ that can be reached at $t=T$ is given by the plane

$$x_1 - x_3 = e^T$$

For any arbitrary $T>0$, the set of points in $\mathbb{R}^3$ that can be reached is given by

$$x_1 - x_3 > 1$$



(j) Find $u_1(t), u_2(t)$ which will drive the state from

$$\underline{x}(0) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

to the state

$$\underline{x}(T) = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$$

Compute T and then find $u_1(t), u_2(t)$ $0 \le t \le T$ using controllability gramian.

Hint: You may like to find the control in the z space. Will the same control work in the x space??

(k) For the pair of matrices A, B on page 9, where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

show that

$$\operatorname{rank}\left(\lambda I - A \mid B \right) < 3$$

for some λ.

(you need to find the λ and show that the rank is < 3).

Remark: This is the Popov, Belevitch, Hautus test of controllability.