H. W. 3

Answers.

1 Ans:

Define

$$\psi_1(t) = e^{B_1 \int_0^t f(\sigma)\, d\sigma}$$

$$\psi_2(t) = e^{B_2 \int_0^t g(\sigma)\, d\sigma}$$

It would follow that

$$\phi(t,0) = \psi_1(t)\, \psi_2(t).$$

Note that $\phi(0,0) = \psi_1(0)\, \psi_2(0) = I$.

On the other hand

$$\dot{\phi}(t,0) = \dot{\psi}_1 \psi_2 + \psi_1 \dot{\psi}_2$$

$$= [f(t)\, B_1\, \psi_1]\, \psi_2 + \psi_1\, [g(t)\, B_2\, \psi_2]$$

$$= f(t)\, B_1\, [\psi_1 \psi_2] + g(t)\, \psi_1\, B_2\, \psi_2.$$

We now claim that since B_1 & B_2 commutes we have

$$Y_1 B_2 = B_2 Y_1 .$$

To prove the claim, note that

$$Y_1 = I + B_1 \int_0^t f(\sigma)\,d\sigma + \frac{B_1^2 \left[\int_0^t f(\sigma)\,d\sigma\right]^2}{2!}$$

$$+ \cdots$$

$$Y_1 B_2 = B_2 + B_1 B_2 \int_0^t f(\sigma)\,d\sigma$$

$$+ \frac{B_1^2 B_2}{2!} \left[\int_0^t f(\sigma)\,d\sigma\right]^2$$

$$+ \cdots$$

$$+ \cdots$$

$$= B_2 + B_2 B_1 \int_0^t f(\sigma)\,d\sigma$$

$$+ B_2 \frac{B_1^2}{2!} \left[\int_0^t f(\sigma)\,d\sigma\right]^2 + \cdots$$

$$= B_2 Y_1$$

Thus we have

$$\dot{\phi}(t,0) =$$

$$f(t)\, B_1 Y_1 Y_2 + g(t)\, Y_1 B_2 Y_2$$

$$= f(t)\, B_1 Y_1 Y_2 + g(t)\, B_2 Y_1 Y_2$$

$$= \left[f(t) B_1 + g(t) B_2 \right] Y_1 Y_2 .$$

$$= A(t)\, \phi(t,0) .$$

Hence $\phi(t,0)$ is the transition matrix.

— x —

If $A(t) =$

$$\cos\omega t \, I + \sin\omega t \, A$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we have

$$\phi(t,0) = e^{I \int_0^t \cos\omega\sigma \, d\sigma} \; e^{A \int_0^t \sin\omega\sigma \, d\sigma}$$

$$= e^{\frac{\sin\omega t}{\omega} I} \; e^{\frac{1-\cos\omega t}{\omega} A}.$$

$$e^{\frac{\sin\omega t}{\omega} I} = \begin{pmatrix} e^{\sin\omega t/\omega} & 0 \\ 0 & e^{\frac{\sin\omega t}{\omega}} \end{pmatrix}$$

we now calculate

$$e^{\frac{1-\cos\omega t}{\omega} A} = \alpha_0 I + \alpha_1 A .$$

$$e^{\frac{1-\cos\omega t}{\omega}\lambda} = \alpha_0 + \alpha_1 \lambda$$

where $\lambda = \pm i$.

$\alpha_0 = \cos[\beta(t)]$ where

$\alpha_1 = \sin[\beta(t)]$ $\beta(t) = \dfrac{1-\cos\omega t}{\omega}$.

$$\therefore \; e^{\beta(t)A} = \begin{pmatrix} \cos\beta(t) & \sin\beta(t) \\ -\sin\beta(t) & \cos\beta(t) \end{pmatrix}$$

Thus.

$$\phi(t,0) = e^{\alpha(t)} \begin{pmatrix} \cos\beta(t) & \sin\beta(t) \\ -\sin\beta(t) & \cos\beta(t) \end{pmatrix}$$

where
$$\alpha(t) = \frac{\sin\omega t}{\omega}.$$

$$\beta(t) = \frac{1-\cos\omega t}{\omega}.$$

2 Ans:

$$A(t) = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}}_{B_1} + \cos t \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{B_2}.$$

$f(t) = 1$, $g(t) = \cos t$.

$$\int_0^t f(\sigma)\, d\sigma = t.$$

$$\int_0^t g(\sigma)\, d\sigma = \sin t$$

$$\therefore \phi(t, 0) = e^{\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} t} \cdot e^{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sin t}.$$

$\phi(t, 0)$ is periodic with $T = 2\pi$ i.e.

$$\phi(t + 2\pi, 0) = \phi(t) \quad \forall t.$$

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$\phi(T,0) =$

$$\phi(2\pi,0) = e^{\begin{pmatrix}-1 & 0\\ 0 & -2\end{pmatrix} 2\cdot\pi}.$$

$$R = \begin{pmatrix}-1 & 0\\ 0 & -2\end{pmatrix}$$

$$\phi(2\pi,0) = e^{RT}, \quad T = 2\pi$$

$$p^{-1}(t) = \phi(t,0)\cdot e^{-Rt}.$$

ie.

$$p(t) = e^{Rt}\,\phi(0,t).$$

$$\phi(0,t) = \phi(t,0)^{-1} =$$

$$e^{-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\sin t}\, e^{-(Rt)}$$

$$\therefore\ p(t) = e^{Rt}\, e^{-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\sin t}\, e^{-Rt}.$$

$$= e^{-\sin t}\cdot I.$$

Define

$$z(t) = P(t)\,x(t)$$

ie
$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} e^{-\sin t} & 0 \\ 0 & e^{-\sin t} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

we claim that

$$\dot{z} = R\mathbf{z}$$

To see that calculate

$$\dot{z}_1 = \frac{d}{dt}\left[e^{-\sin t} x_1(t)\right]$$

$$= e^{-\sin t}\,\dot{x}_1 - \cos t\, e^{-\sin t} x_1(t)$$

$$= e^{-\sin t}\left[-1 + \cos t\right] \quad x_1(t)$$

$$- \cos t\, e^{-\sin t}\, x_1(t)$$

$$= -e^{-\sin t} \quad x_1(t)$$

$$= -e^{-\sin t} e^{\sin t} z_1 = -z_1$$

$\therefore\ \dot{z}_1 = -z_1$ Likewise $\dot{z}_2 = -2z_2$ $\boxed{///}$

3 Ans:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) = e^{\sigma t} \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(0).$$

It follows that

$$\phi(t,0) = e^{\sigma t} \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}$$

choose $T = \frac{2\pi}{\omega}$, we have

$$\phi(T,0) = e^{\sigma T} I = e^{\sigma I T}$$

$$\therefore R = \sigma I.$$

Define

$$P(t) = e^{Rt} \phi(t,0)^{-1}.$$

$$= \left[e^{\sigma t} I\right] e^{-\sigma t} \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}^{-1}.$$

$$= \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}$$

Define

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

It would follow that

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = R \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Remark:

Note that the $\underline{x}$ system has eigenvalues at $\sigma \pm i\omega$ whereas

Z system " "

at σ, σ.

(b) Before we generalize the result let us consider

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} \sigma & \omega & 1 & 0 \\ -\omega & \sigma & 0 & 1 \\ 0 & 0 & \sigma & \omega \\ 0 & 0 & -\omega & \sigma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$\phi(t,0)$ for the above system is given by

$$\phi(t,0) = e^{\sigma t} \begin{pmatrix} \cos\omega t & \sin\omega t & t\cos\omega t & t\sin\omega t \\ -\sin\omega t & \cos\omega t & -t\sin\omega t & t\cos\omega t \\ 0 & 0 & \cos\omega t & \sin\omega t \\ 0 & 0 & -\sin\omega t & \cos\omega t \end{pmatrix}$$

For $t = \frac{2\pi}{\omega} = T$ we have

$$\Phi(T,0) =$$

$$e^{\sigma T}\begin{pmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= e^{\begin{pmatrix} \sigma & 0 & 1 & 0 \\ 0 & \sigma & 0 & 1 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \sigma \end{pmatrix} T}$$

$$\therefore R = \begin{pmatrix} \sigma & 0 & 1 & 0 \\ 0 & \sigma & 0 & 1 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \sigma \end{pmatrix}$$

Define

$$P(t) = e^{Rt}\,\Phi(t,0)^{-1}.$$

$$\phi(t,0) = e^{\sigma t}\begin{pmatrix} \Omega & t\,\Omega \\ 0 & \Omega \end{pmatrix}$$

where

$$\Omega = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}$$

$$\phi(t,0)^{-1} = e^{-\sigma t}\begin{pmatrix} \Omega^{-1} & -t\,\Omega^{-1} \\ 0 & \Omega^{-1} \end{pmatrix}$$

$$= e^{-\sigma t}\underbrace{\begin{pmatrix} \cos\omega t & -\sin\omega t & -t\cos\omega t & t\sin\omega t \\ \sin\omega t & \cos\omega t & -t\sin\omega t & -t\cos\omega t \\ 0 & 0 & \cos\omega t & -\sin\omega t \\ 0 & 0 & \sin\omega t & \cos\omega t \end{pmatrix}}_{L}$$

$$\triangleq e^{-\sigma t} L$$

$$e^{Rt} =$$

$$e^{\sigma t}\begin{pmatrix} I & tI \\ 0 & I \end{pmatrix}$$

$$\therefore\ P(t) = e^{Rt}\,\phi(t,0)^{-1}$$

$$= e^{\sigma t}\begin{pmatrix} I & tI \\ 0 & I \end{pmatrix} e^{-\sigma t} L$$

$$= \begin{pmatrix} I & tI \\ 0 & I \end{pmatrix} L$$

$$= \begin{pmatrix} \Omega^{-1} & 0 \\ 0 & \Omega^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 & 0 \\ \sin\omega t & \cos\omega t & 0 & 0 \\ 0 & 0 & \cos\omega t & -\sin\omega t \\ 0 & 0 & \sin\omega t & \cos\omega t \end{pmatrix}$$

Define

$$Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = P(t) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

we have

$$\dot{Z} = RZ$$

i.e.

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} \sigma & 0 & 1 & 0 \\ 0 & \sigma & 0 & 1 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \sigma \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

But this is not in Jordan canonical form.

But if we reorder

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_3 \\ \dot{z}_2 \\ \dot{z}_4 \end{pmatrix} = \left(\begin{array}{cc|cc} \sigma & 1 & 0 & 0 \\ 0 & \sigma & 0 & 0 \\ \hline 0 & 0 & \sigma & 1 \\ 0 & 0 & 0 & \sigma \end{array}\right) \begin{pmatrix} z_1 \\ z_3 \\ z_2 \\ z_4 \end{pmatrix}$$

This is in the J. Canonical form.

Thus if we define

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_3 \\ z_2 \\ z_4 \end{pmatrix}$$

we have

$$\omega_1 = z_1 = \cos\omega t\ x_1 - \sin\omega t\ x_2$$

$$\omega_2 = z_3 = \cos\omega t\ x_3 - \sin\omega t\ x_4 .$$

$$\omega_3 = z_2 = \sin\omega t\ x_1 + \cos\omega t\ x_2 .$$

$$\omega_4 = z_4 = \sin\omega t\ x_3 + \cos\omega t\ x_4$$

we have

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\omega t & -\sin\omega t & 0 & 0 \\ 0 & 0 & \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t & 0 & 0 \\ 0 & 0 & \sin\omega t & \cos\omega t \end{pmatrix}}_{\bar{P}(t)} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

If $\omega = \bar{P}(t) x$

$$\dot{\omega} = \bar{R}\,\omega \qquad \bar{R} = \begin{pmatrix} \sigma & 1 & 0 & 0 \\ 0 & \sigma & 0 & 0 \\ 0 & 0 & \sigma & 1 \\ 0 & 0 & 0 & \sigma \end{pmatrix}$$

<u>Conclusion:</u>

The x-system with complex conjugate Eigenvalues in the Jordan form can be transformed into the constant system.

$$\dot{w} = \bar{R} w$$

where $\bar{R}$ has only real eigenvalues at σ that are repeated and has 2-chain of generalized eigenvectors.

$$\int_0^t \phi(0,\sigma)\begin{pmatrix}0\\1\end{pmatrix}\cos\sigma\, d\sigma$$

$$=\int_0^t \begin{pmatrix}-\sin\sigma\cos\sigma\\ \cos^2\sigma\end{pmatrix} d\sigma$$

$$=\frac{1}{2}\int_0^t \begin{pmatrix}-\sin 2\sigma\\ 1+\cos 2\sigma\end{pmatrix} d\sigma$$

$$=\frac{1}{2}\begin{pmatrix}\frac{\cos 2\sigma}{2}\\ \sigma+\frac{\sin 2\sigma}{2}\end{pmatrix}_0^t$$

$$=\frac{1}{4}\begin{pmatrix}\cos 2\sigma\\ 2\sigma+\sin 2\sigma\end{pmatrix}_0^t = \frac{1}{4}\left[\begin{pmatrix}\cos 2t\\ 2t+\sin 2t\end{pmatrix}-\begin{pmatrix}1\\0\end{pmatrix}\right]$$

$$=\frac{1}{4}\begin{pmatrix}\cos 2t-1\\ 2t+\sin 2t\end{pmatrix}$$

$$\therefore\ \mathbf{x}(t) =$$

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} +$$

$$\frac{1}{4} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \cos 2t - 1 \\ 2t + \sin 2t \end{pmatrix}$$

$$x_1(t) = x_1(0) \cos t + x_2(0) \sin 2t$$
$$+ \frac{1}{4} \cos t \left(\cos 2t - 1\right)$$
$$+ \frac{1}{4} \sin t \left(2t + \sin 2t\right)$$

$$x_2(t) = - x_1(0) \sin t + x_2(0) \cos t$$
$$+ \frac{1}{4} \sin t \left(1 - \cos 2t\right)$$
$$+ \frac{1}{4} \cos t \left(2t + \sin 2t\right).$$

Remark:

In this problem, the natural frequency is given by $\omega = 1$ which is the frequency of e^{λ}s in $\phi(t, 0)$.

The frequency of $u(t)$ is also $\omega = 1$

Because, the input frequency matches with the natural frequency, we have what is called "resonance", as a results both

$x_1(t)$ & $x_2(t)$ has terms like

$t \sin t$ & $t \cos t$

These terms are unbounded, making $x_1(t)$ & $x_2(t)$ unbounded.

If you have more time and patience you can choose

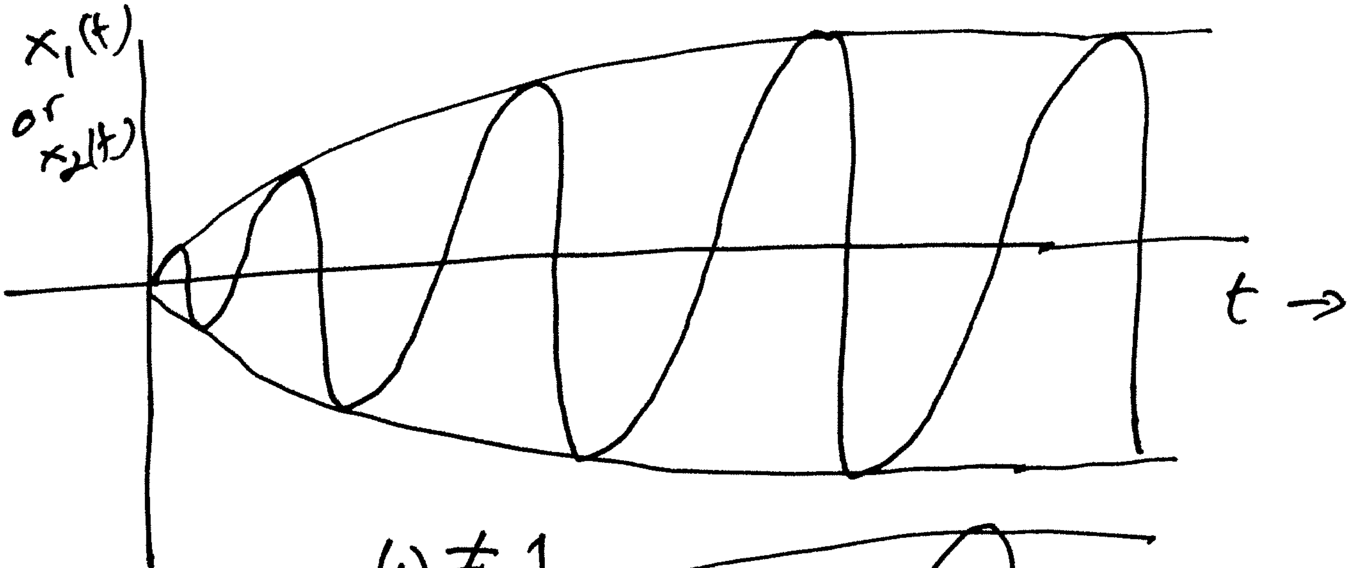
$u(t) = \cos\omega t$

ie

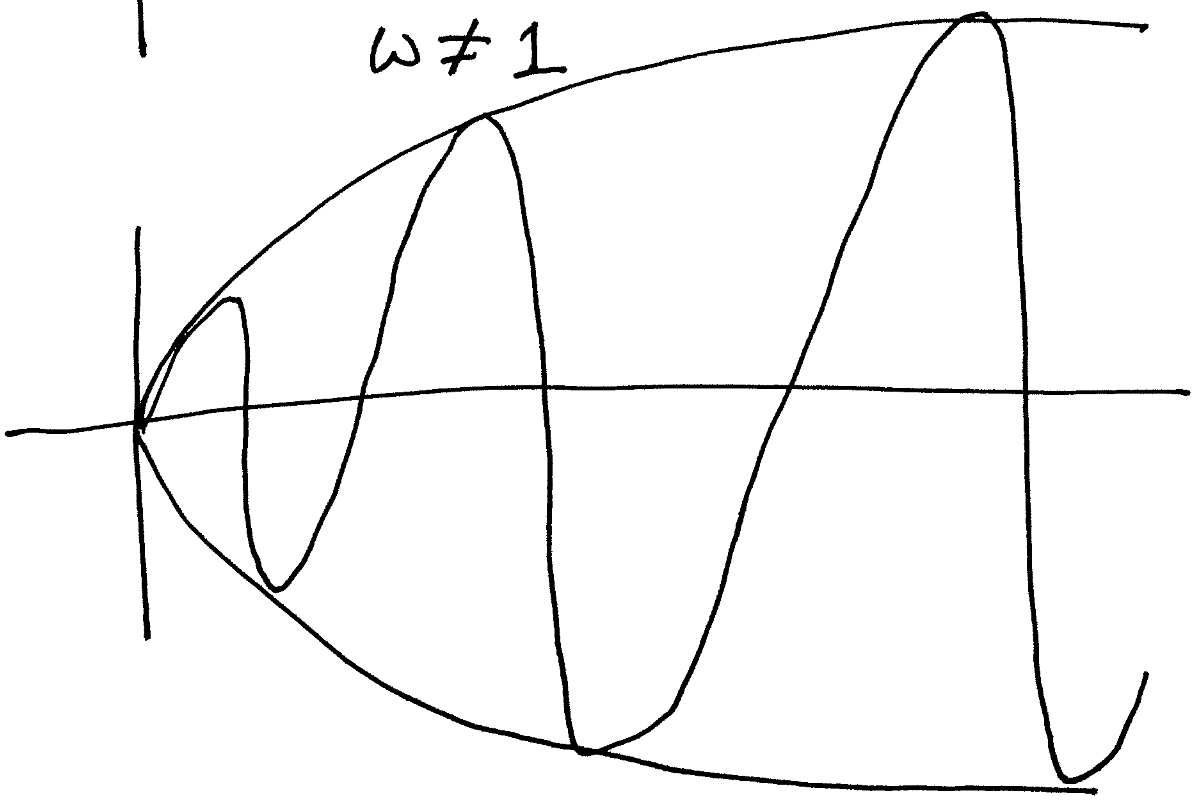
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos\omega t$$

and solve the problem for $\omega \neq 1$.

- when $\omega \neq 1$, both $x_1(t)$ & $x_2(t)$ would be oscillatory but remain bounded.
- when $\omega \longrightarrow 1$, both $x_1(t)$ & $x_2(t)$ would be bounded with increasingly large amplitude.

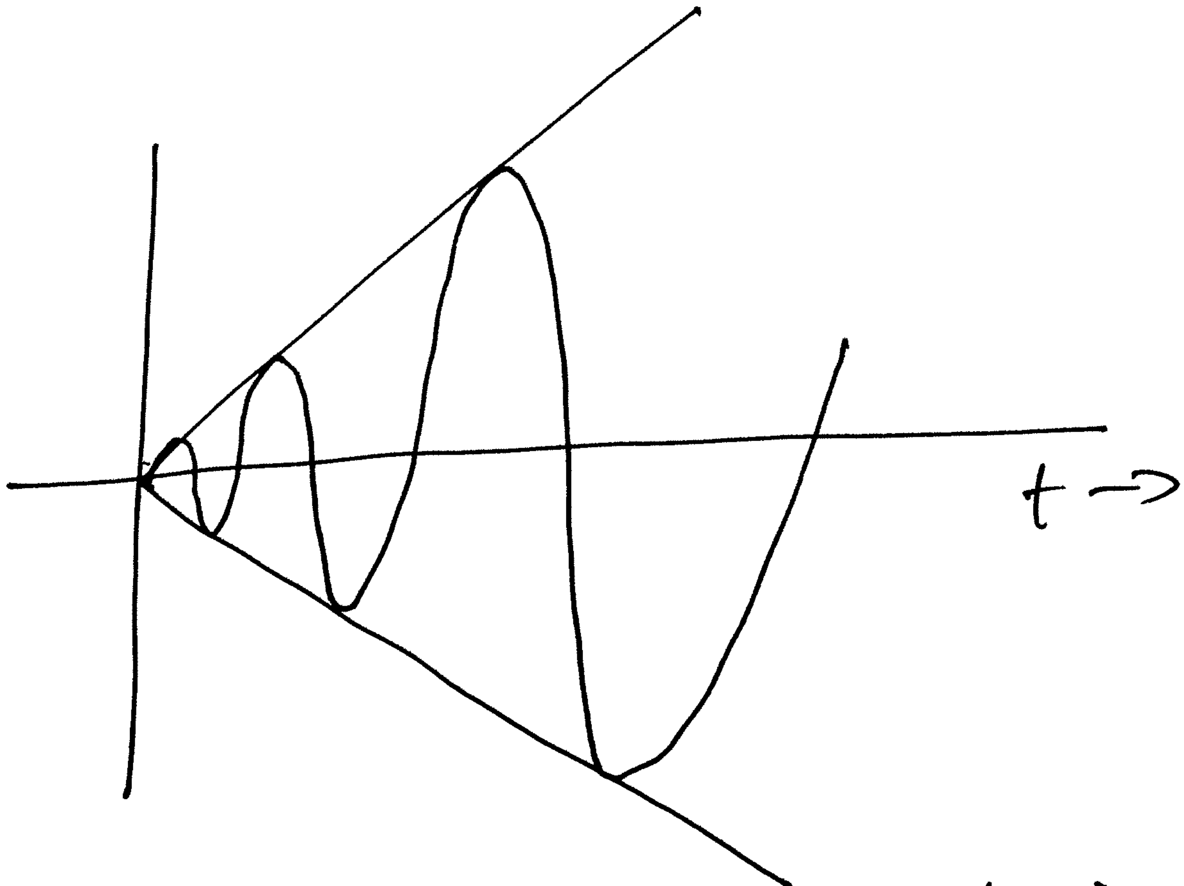


$\omega \neq 1$



$\omega \neq 1$

As ω approaches 1, the amplitude increases.



when $\omega = 1$, the amplitude is unbounded, in fact increases linearly as a $f^{\underline{n}}$ of t.