H.W. 2

Solutions

1 Ans

Define

$$B(t) = f(t)\, A$$

we remark that

$$\int_0^t B(\sigma)\, d\sigma = \int_0^t f(\sigma)\, A\, d\sigma$$

$$= \underbrace{\left(\int_0^t f(\sigma)\, d\sigma\right)}_{=\alpha(t)} A$$

$$= \alpha(t)\, A$$

$$B(t) \int_0^t B(\sigma)\, d\sigma = f(t)\, \alpha(t)\, A^2$$

$$\left[\int_0^t B(\sigma)\, d\sigma\right] B(t) = \alpha(t)\, f(t)\, A^2$$

Hence

$$B(t) \int_0^t B(\sigma)\, d\sigma = \int_0^t B(\sigma)\, d\sigma \cdot B.$$

Hence

$$\phi(t,0) = \exp\left(\int_0^t B(\sigma)\,d\sigma\right)$$

$$= e^{\alpha(t) A}$$

2) Ans:

char poly of $A$ =

$$\begin{vmatrix} \lambda & -1 \\ 1 & \lambda+2\delta \end{vmatrix} = \lambda^2 + 2\delta\lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{-2\delta \pm \sqrt{4\delta^2 - 4}}{2}$$

$$= -\delta \pm \sqrt{\delta^2 - 1}$$

There are 3 cases

Case 1 ($\delta = \pm 1$)

$$\lambda = -\delta, -\delta$$

$$e^{At} = \alpha_0 I + \alpha_1 A$$

$$e^{-\delta t} = \alpha_0 - \alpha_1 \delta$$

$$t e^{-\delta t} = \alpha_1$$

Hence $\alpha_1 = t e^{-\delta t}$

$$\alpha_0 = e^{-\delta t} + t\delta e^{-\delta t}$$
$$= (1+\delta t) e^{-\delta t}$$

$$\therefore e^{At} = \begin{pmatrix} 1+\delta t & t \\ -t & 1-\delta t \end{pmatrix} e^{-\delta t}$$

## Case 2 $(|\delta| > 1)$

Eigenvalues are real

$$\lambda_1 = -\delta + \sqrt{\delta^2 - 1}$$

$$\lambda_2 = -\delta - \sqrt{\delta^2 - 1}$$

$$e^{At} = \alpha_0 I + \alpha_1 A.$$

$$e^{\lambda_1 t} = \alpha_0 + \lambda_1 \alpha_1$$

$$e^{\lambda_2 t} = \alpha_0 + \lambda_2 \alpha_1$$

$$\begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{pmatrix}$$

$$\alpha_0 = \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1}.$$

$$= \cancel{\frac{\lambda_1 e^{\lambda_2 t}\left[1 - \frac{\lambda_2}{\lambda_1} e^{(\lambda_1 - \lambda_2)t}\right]}{\lambda_1 - \lambda_2}}.$$

$$\cancel{= \frac{\lambda_1 e^{\lambda_2 t}\left[1 - \frac{\lambda_2}{\lambda_1} e^{2\sqrt{\delta^2-1}\, t}\right]}{2\sqrt{\delta^2-1}}}$$

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$$\alpha_1 = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}$$

$$e^{At} = \begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_1 \\ -\alpha_1 & -2\delta\alpha_1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0 & \alpha_1 \\ -\alpha_1 & \alpha_0 - 2\delta\alpha_1 \end{pmatrix}$$

$$= \frac{\begin{pmatrix} \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} & e^{\lambda_2 t} - e^{\lambda_1 t} \\ e^{\lambda_1 t} - e^{\lambda_2 t} & \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} - 2\delta e^{\lambda_2 t} + 2\delta e^{\lambda_1 t} \end{pmatrix}}{}$$

(6)

$$e^{At} = \begin{pmatrix} \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} & e^{\lambda_2 t} - e^{\lambda_1 t} \\ e^{\lambda_1 t} - e^{\lambda_2 t} & (\lambda_2 + 2\delta) e^{\lambda_1 t} - (\lambda_1 + 2\delta) e^{\lambda_2 t} \end{pmatrix}$$

## Case 3 $(|\delta| < 1)$

$$\lambda = -\delta \pm i\underbrace{\sqrt{1-\delta^2}}_{\omega}$$

$$= -\delta \pm i\omega \qquad \omega = \sqrt{1-\delta^2}$$

$$e^{(-\delta+i\omega)t} = \alpha_0 + \alpha_1(-\delta+i\omega).$$

$$e^{(-\delta-i\omega)t} = \alpha_0 + \alpha_1(-\delta-i\omega).$$

$$2e^{-\delta t}\cos\omega t = 2\alpha_0 - 2\delta\alpha_1.$$

$$\cancel{2}\cancel{i}e^{-\delta t}\sin\omega t = \cancel{i}\cancel{2}\omega\alpha_1.$$

$$\alpha_1 = e^{-\delta t}\frac{\sin\omega t}{\omega}$$

$$\alpha_0 - \alpha_1\delta = e^{-\delta t}\cos\omega t.$$

$$\alpha_0 = \delta e^{-\delta t}\frac{\sin\omega t}{\omega} + e^{-\delta t}\cos\omega t.$$

$$\alpha_0 = \left[\frac{\delta \sin \omega t}{\omega} + \cos \omega t\right] e^{-\delta t}.$$

$$\alpha_1 = \frac{\sin \omega t}{\omega} e^{-\delta t}.$$

$$e^{At} =$$

$$\begin{pmatrix} \frac{\delta \sin \omega t}{\omega} + \cos \omega t & \frac{\sin \omega t}{\omega} \\ -\frac{\sin \omega t}{\omega} & -\frac{\delta \sin \omega t}{\omega} + \cos \omega t \end{pmatrix} e^{-\delta t}$$

3 Ans

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$$[B, C] = BC - CB.$$

$$[A, [B, C]] = A(BC - CB) - (BC - CB)A$$

$$= ABC - ACB - BCA + CBA$$

Likewise

$$[B, [C, A]] = BCA - BAC - CAB + ACB.$$

&

$$[C, [A, B]] = CAB - CBA - ABC + BAC.$$

Hence

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]]$$

$$= \cancel{ABC} - \cancel{ACB} - \cancel{BCA} + \cancel{CBA}$$

$$+ \cancel{BCA} - \cancel{BAC} - \cancel{CAB} + \cancel{ACB}. = 0.$$

$$+ \cancel{CAB} - \cancel{CBA} - \cancel{ABC} + \cancel{BAC}.$$

4 Ans:

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots$$

$$\begin{aligned} e^A e^B = {} & I \\ & + (A+B) \\ & + \frac{A^2}{2} + AB + \frac{B^2}{2} \\ & + \frac{A^3}{6} + \frac{A^2B}{2} + \frac{AB^2}{2} + \frac{B^3}{6} \\ & + \cdots \end{aligned}$$

~~Let~~

$$\cancel{C = C_1 + C_2 + C_3 + \cdots}$$

$$\cancel{e^C = I +}$$

~~Let~~

$$\cancel{C = (A+B) + C_1}$$

Denote

$$e^A e^B = e^C$$

and write

$$C = (A+B) + C_1 + \text{h.o.t.}$$

quadratic terms.

It follows that

$$e^C = I + (A+B) + C_1 + \frac{(A+B)^2}{2} + \text{h.o.t}$$

Equating the quadratic terms we have

$$C_1 = \frac{A^2}{2} + AB + \frac{B^2}{2} - \frac{A^2 + B^2 + AB + BA}{2}$$

$$= \frac{[A,B]}{2}$$

Let us now write

$$C = (A+B) + \frac{[A,B]}{2} + \underbrace{C_2}_{\text{cubic terms.}} + \text{h.o.t.}$$

$$e^C = I + (A+B) + \frac{[A,B]}{2} + C_2 + \frac{(A+B)[A,B] + [A,B](A+B)}{4} + \frac{(A+B)^3}{6} + \text{h.o.t.}$$

Equating the cubic terms we obtain.

$$C_2 = \frac{A^3}{6} + \frac{A^2B}{2} + \frac{AB^2}{2} + \frac{B^3}{6} - \frac{(A+B)[A,B]}{4} - \frac{[A,B](A+B)}{4} - \frac{(A+B)^3}{6}$$

$$C_2 = \frac{1}{12}\left(A^2B + BA^2 + AB^2 + B^2A - 2ABA - 2BAB\right).$$

$$= \frac{1}{12}\left\{\left[[B,A],A\right] - \left[[B,A],B\right]\right\}.$$

4 B

writing

$$W_1 = e^{At} e^{Bt} = e^{Z_1(t)}$$

$$W_2 = e^{Bt} e^{At} = e^{Z_2(t)}$$

we have

$$Z_1(t) = At + Bt + \frac{1}{2}[A, B]t^2 + \cdots$$

$$Z_2(t) = Bt + At + \frac{1}{2}[B, A]t^2 + \cdots$$

$$W_1 W_2^{-1} = e^{Z_1} e^{-Z_2}$$

$$= Z_1 - Z_2 + \frac{1}{2}[Z_1, -Z_2] + \cdots$$

$$= \frac{1}{2}[A, B]t^2 - \frac{1}{2}[B, A]t^2.$$

$$+ \frac{1}{2}(-Z_1 Z_2 + Z_2 Z_1)$$

$$Z_2 Z_1 - Z_1 Z_2 = (Bt + At)(At + Bt) - (At + Bt)(Bt + At)$$

$$= 0$$

$$\therefore D(t) = [A, B]t^2$$

$$\therefore W_1 W_2^{-1} = [A, B]t^2$$

5 Ans:

$$\underline{x}(t) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}(t)$$

$$\dot{\underline{x}} = \Omega(t)\, x(t)$$

$$\Omega(t) = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \cos \omega t \\ -\omega_3 & 0 & +\omega_2 \sin \omega t \\ +\omega_2 \cos \omega t & -\omega_2 \sin \omega t & 0 \end{pmatrix}$$

$$\phi(t, 0) = e^{\begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} t} e^{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \theta \\ 0 & -\theta & 0 \end{pmatrix}} e^{\begin{pmatrix} 0 & \eta & 0 \\ -\eta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} t}$$

$$\eta \cos \theta = \omega_3 - \omega$$

$$\eta \sin \theta = \omega_2$$

verify that

$$\phi(t,0) =$$

$$\begin{pmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \cdot$$

$$\begin{pmatrix} \cos\eta t & \sin\eta t & 0 \\ -\sin\eta t & \cos\eta t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\omega t\cos\eta t - \sin\omega t\sin\eta t\cos\theta & \cos\omega t\sin\eta t + \sin\omega t\cos\eta t\cos\theta & \sin\omega t\sin\theta \\ -\sin\omega t\cos\eta t - \cos\omega t\sin\eta t\cos\theta & -\sin\omega t\sin\eta t + \cos\omega t\cos\eta t\cos\theta & \cos\omega t\sin\theta \\ \sin\eta t\sin\theta & -\cos\eta t\sin\theta & \cos\theta \end{pmatrix}$$

Also verify that

$$\dot{\phi}(t,0) =$$

$$\begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \phi(t,0) +$$

$$\phi(t,0) \begin{pmatrix} 0 & \eta & 0 \\ -\eta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let us write

$\Omega(t)\,\phi(t,0)$ as

$$\begin{pmatrix} 0 & \omega_3 & -\omega_2 \cos\omega t \\ -\omega_3 & 0 & +\omega_2 \sin\omega t \\ \omega_2 \cos\omega t & -\omega_2 \sin\omega t & 0 \end{pmatrix} \phi(t,0).$$

In order that

$$\dot{\phi}(t,0) = \Omega(t)\phi(t,0).$$

we must have

$$\phi(t,0)\begin{pmatrix} 0 & \eta & 0 \\ -\eta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & \omega_3-\omega & -\omega_2\cos\omega t \\ \omega-\omega_3 & 0 & \omega_2\sin\omega t \\ \omega_2\cos\omega t & -\omega_2\sin\omega t & 0 \end{pmatrix}\phi(t,0).$$

which is equivalent to

$$\phi(t,0)\begin{pmatrix} 0 & \eta & 0 \\ -\eta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & \eta\cos\alpha & -\eta\sin\alpha\cos\omega t \\ -\eta\cos\alpha & 0 & \eta\sin\alpha\sin\omega t \\ \eta\sin\alpha\cos\omega t & -\eta\sin\alpha\sin\omega t & 0 \end{pmatrix}\phi(t,0).$$

1st column of l.h.s

$$\begin{pmatrix} -\eta\cos\omega t\sin\eta t-\eta\sin\omega t\cos\eta t\cos\alpha \\ \eta\sin\omega t\sin\eta t-\eta\cos\omega t\cos\eta t\cos\alpha \\ \eta\cos\eta t\sin\alpha \end{pmatrix}$$

1st column of r.h.s.

$$\begin{pmatrix} -\eta\cos\alpha\left[\sin\omega t\cos\eta t+\cos\omega t\sin\eta t\cos\alpha\right] \\ -\eta\sin^2\alpha\sin\eta t\cos\omega t \end{pmatrix}$$

$$\begin{pmatrix} -\eta\cos\alpha\left[\cos\omega t\cos\eta t-\sin\omega t\sin\eta t\cos\alpha\right] \\ +\eta\sin^2\alpha\sin\omega t\sin\eta t \end{pmatrix}$$

$$\begin{pmatrix} \eta\sin\alpha\cos\omega t\left[\cos\omega t\cos\eta t-\sin\omega t\sin\eta t\cos\alpha\right] \\ +\eta\sin\alpha\sin\omega t\left[\sin\omega t\cos\eta t+\cos\omega t\sin\eta t\cos\alpha\right] \end{pmatrix}$$

Likewise verify all the other columns.

6 Ans:

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad A^2B = \begin{pmatrix} \alpha^2-\beta^2 & \alpha\beta+\beta\gamma \\ -\alpha\beta-\beta\gamma & \gamma^2-\beta^2 \\ -\alpha^2+\beta\gamma & -\alpha\beta-\gamma^2 \end{pmatrix}$$

$$AB = \begin{pmatrix} -\alpha & -\beta \\ \beta & -\gamma \\ \alpha & \gamma \end{pmatrix}$$

$$\text{rank}\left[B \mid AB \mid A^2B\right] = 3$$

unless $(\alpha, \gamma) = (0, 0)$.

Hence the state eqn is controllable if we assume that $(\alpha, \gamma) \neq 0$.

— x —

writing the state eqn as.

$$\dot{\underline{x}} = A\underline{x} + Bu$$

we obtain

$$\underline{x}(t) = e^{At}\underline{x}(0) + \int_0^t e^{A(t-\sigma)} B u(\sigma) d\sigma$$

Assume that for some initial condition $\underline{x}(0)$, we want to drive the state to $(0,0,0)$ at $t = T$, we have

$$0 = e^{AT} x(0) + \int_0^T e^{A(T-\sigma)} B u(\sigma) d\sigma$$

Multiplying by $e^{-AT}$ we obtain

$$0 = x(0) + \int_0^T e^{-A\sigma} B u(\sigma) d\sigma$$

or $$\int_0^T e^{-A\sigma} B u(\sigma) d\sigma = -\underline{x}(0).$$

Assume

$$u(\sigma) = B^T e^{-A^T\sigma} \xi.$$

we have

$$\underbrace{\left[\int_0^T e^{-A\sigma} B B^T e^{-A^T\sigma}\, d\sigma\right]}_{M} \xi = -\underline{X}(0).$$

For $\alpha = 1$, $\beta = 2$, $T = 10$ we obtain $M$

and write

$$\xi = -M^{-1}\underline{X}(0)$$

and obtain.

$$u(\sigma) = -B^T e^{-A^T\sigma} M^{-1}\underline{X}(0)$$

Use matlab to find $\xi$ and solve

for $u(\sigma)$.