Answers to H.W1

① ~~拼~~

Let A be a $n \times n$ matrix.

If A has distinct eigenvalues at $\lambda_1, \lambda_2, \ldots, \lambda_n$, it follows that $\exists\, T$:

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$\Rightarrow \det(T^{-1}AT) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$$

$$= \prod_{j=1}^{n} \lambda_j$$

$$\operatorname{trace}(T^{-1}AT) = \sum_{j=1}^{n} \lambda_j$$

"The det & trace remains unchanged by similarity transformation"

i.e $\det(T^{-1}AT) = \det A$; $\operatorname{trace}(T^{-1}AT) = \operatorname{trace} A$.

Lemma:

Let $B = T^{-1}AT$

and define

$$p(\cdot) = \text{char. poly of } (\cdot)$$
$$= \det(\lambda I - (\cdot))$$

$$p(A) = \det(\lambda I - A)$$
$$p(B) = \det(\lambda I - B)$$

then

$$p(A) = p(B)$$

Proof: $p(B) = \det(\lambda I - B)$

$$= \det(\lambda I - T^{-1}AT)$$

$$= \det\left(T^{-1}[\lambda I - A]T\right)$$

$$= \det(T^{-1}) \det(\lambda I - A) \det T$$

$$= \frac{1}{\det T} \det(\lambda I - A) \det T$$
$$= \det(\lambda I - A)$$

Thus characteristic polynomials remain unchanged by similarity transformation.

Lemma:

Let A be a $n \times n$ matrix. then.

$$\det(\lambda I - A) =$$
$$\lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \cdots + \alpha_n$$

where

$\alpha_1 = (-1)$ trace A

$\alpha_2 = (-1)^2 \sum$ (2x2 principal minors of A)

$\alpha_n = (-1)^n \det A$.

Thus if $\exists T$:

$$T^{-1} A T = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

then

$$\det A = \det \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \lambda_1 \lambda_2 \cdots \lambda_n$$

$$\operatorname{trace} A = \operatorname{trace} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

$\sum$ 2x2 principal minors of A =

$$\sum_{\substack{j=1 \\ i>j}}^{n} \sum_{i=1}^{n} \lambda_i \lambda_j = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_1\lambda_n$$

$$+ \lambda_2\lambda_3 + \cdots + \lambda_2\lambda_n$$

$$- \text{———————}$$

$$+ \lambda_{n-1}\lambda_n$$

② Let $A, B, C \in \mathbb{R}^{n \times n}$

$\alpha, \beta \in \mathbb{R}$

we can see that

① $(A+B)+C = A+(B+C)$

② $0+A = A$

③ $A+(-A) = 0$

④ $A+B = B+A$

⑤ $\alpha(A+B) = \alpha A + \alpha B$

⑥ $(\alpha+\beta)A = \alpha A + \beta A$

⑦ $(\alpha\beta)A = \alpha(\beta A)$.

⑧ $1A = A$.

To show that L is a linear transformation we need to show that

- $L(A) + L(B) = L(A+B)$

$L(A) = MA + AN$.
$L(B) = MB + BN$.

$$
\begin{aligned}
L(A) + L(B) &= MA + AN + MB + BN. \\
&= M(A+B) + (A+B)N. \\
&= L(A+B).
\end{aligned}
$$

- $L(\alpha A) = \alpha L(A)$

$$
\begin{aligned}
L(\alpha A) &= M(\alpha A) + (\alpha A)N. \\
&= \alpha MA + \alpha AN \\
&= \alpha [MA + AN] = \alpha L(A).
\end{aligned}
$$

7

3 a $R(L) =$

$$\{ z \in \mathbb{R}^3 : \exists x \in \mathbb{R}^4 \ \& \ Ax = z \}.$$

choose $x = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$ we have

$$Ax = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$$

$$= \alpha_1 \underbrace{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}_{v_1} + \alpha_2 \underbrace{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}_{v_2} + \alpha_3 \underbrace{\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}}_{v_3} + \alpha_4 \underbrace{\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}}_{v_4}$$

we notice that $v_3 = v_1 + v_2$ & $v_4 = 2v_2$

thus we have

$$Ax = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 (v_1 + v_2) + \alpha_4 (2 v_2)$$

$$Ax = (\alpha_1 + \alpha_3) v_1 + (\alpha_2 + 2\alpha_4) v_2$$

Thus $R(L) =$ l.c. of vectors v_1 & v_2

$= \operatorname{span}\{v_1, v_2\}$.

Finally note that v_1 & v_2 are l.i.

To see that we set

$$\beta_1 v_1 + \beta_2 v_2 = 0$$

$$\Rightarrow \left.\begin{aligned} \beta_1 \cdot 0 + \beta_2 1 &= 0 \\ \beta_1 \cdot 1 + \beta_2 \cdot 2 &= 0 \\ \beta_1 2 + \beta_2 \cdot 0 &= 0. \end{aligned}\right\} \Rightarrow \begin{aligned} \beta_2 &= 0 \\ \beta_1 + 2\beta_2 &= 0. \\ 2\beta_1 &= 0. \end{aligned}$$

$$\Rightarrow \boxed{\beta_1 = \beta_2 = 0}$$

3 b

$$\mathcal{N}(A^T) =$$

$$\left\{ x \in \mathbb{R}^3 : \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 3 & 2 \\ 2 & 4 & 0 \end{pmatrix} x = 0 \right\}$$

$$A^T x = 0$$

$$\Rightarrow \left. \begin{array}{l} v_1^T x = 0 \\ v_2^T x = 0 \\ v_3^T x = 0 \\ v_4^T x = 0 \end{array} \right\} \quad \because \begin{array}{l} v_3 = v_1 + v_2 \\ v_4 = 2 v_2 \end{array}$$

The last two eqns are redundant

Hence

$$\mathcal{N}(A^T) = \left\{ x \in \mathbb{R}^3 : v_1^T x = 0,\ v_2^T x = 0 \right\}.$$

$$\begin{array}{l} x_2 + 2x_3 = 0 \\ x_1 + 2x_2 = 0 \end{array} \Rightarrow \begin{array}{l} x_2 = -2x_3 \\ x_1 = -2x_2 = +4x_3 \end{array}$$

10

$\mathcal{N}(A^T) =$

$$\left\{ x \in \mathbb{R}^3 : \; x = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} x_3 \right\}$$

$$= \operatorname{span}\left\{ \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \right\}$$

3 c

$$R(L) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}$$

$$\mathcal{N}(A^T) = \text{span}\left\{ \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \right\}$$

To see that $R(L)$ is perp. to $\mathcal{N}(A^T)$ we verify that

$$\underset{u}{\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}} \text{ is perp to } \underset{v_1}{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}} \text{ \& } \underset{v_2}{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}$$

check that $u \cdot v_1 = 0$, $u \cdot v_2 = 0$ $\Rightarrow u \perp v_1$, $u \perp v_2$

$\blacksquare$

④ Ans:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \int_0^t \begin{pmatrix} 1 \\ \tau \end{pmatrix} u(\tau)\, d\tau .$$

set $t = 5$ we have

$$\begin{pmatrix} 10 \\ 20 \end{pmatrix} = \int_0^5 \begin{pmatrix} 1 \\ \tau \end{pmatrix} u(\tau)\, d\tau .$$

choose $u(\tau) = (1 \quad \tau)\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ we obtain

$$\begin{pmatrix} 10 \\ 20 \end{pmatrix} = \int_0^5 \begin{pmatrix} 1 & \tau \\ \tau & \tau^2 \end{pmatrix} d\tau \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \begin{pmatrix} \tau & \tau^2/2 \\ \tau^2/2 & \tau^3/3 \end{pmatrix}\Bigg|_0^5 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 25/2 \\ 25/2 & 125/3 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Solve for α, β as follows. 13

$$\begin{pmatrix} \frac{1}{5}25 & \frac{1}{2}25 \\ \frac{1}{2}25 & \frac{5}{3}25 \end{pmatrix} = 25 \begin{pmatrix} \frac{1}{5} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{3} \end{pmatrix}$$

$$\begin{pmatrix} 1/5 & 1/2 \\ 1/2 & 5/3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 2/5 \\ 4/5 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 15 \\ 15 & 50 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 12 \\ 24 \end{pmatrix}$$

$$\begin{array}{r} 15 \\ 24 \\ \hline 360 \end{array}$$

$$\alpha = \frac{300-225}{600-360} = \frac{75}{240} = \frac{15}{48} = \frac{5}{16}$$

$$\beta = \frac{6 \cdot 24 - 15 \cdot 12}{240} = \frac{144-180}{240} = \frac{-36}{240}$$
$$= -3/20$$

$$u(r) = \alpha + \beta r.$$

$$= \frac{5}{16} - \frac{3}{20} r.$$

5

$p(A) =$

$$\det \begin{pmatrix} \lambda & -5 & -1 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda+3 \end{pmatrix}$$

$$= \lambda^2 (\lambda + 3)$$

Eigenvalues at $\lambda = 0, 0, -3$.

For $\lambda_3 = -3$ solve

$$A v_3 = \lambda_3 v_3$$

$$\begin{pmatrix} 0 & 5 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -3a \\ -3b \\ -3c \end{pmatrix}$$

$$\Rightarrow 5b + c = -3a$$

$$c = -3b \quad \Rightarrow 2b = -3a$$

$$-3c = -3c \quad \Rightarrow \boxed{b = -\frac{3}{2} a} \; \boxed{c = \frac{9}{2} a}$$

16

$$v_3 = \begin{pmatrix} a \\ -\frac{3}{2}a \\ \frac{9}{2}a \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{3}{2} \\ \frac{9}{2} \end{pmatrix} a$$

set $a=2$ we get

$$v_3 = \begin{pmatrix} 2 \\ -3 \\ 9 \end{pmatrix} \leftarrow$$ Eigenvector for the eigenvalue $\lambda = -3$

— x —

For $\lambda_1 = 0$ solve

$$A v_1 = 0 v_1 = 0$$

$$\begin{pmatrix} 0 & 5 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} 5b + c &= 0 \\ c &= 0 \\ -3c &= 0 \end{aligned}$$

$\Rightarrow b = c = 0$

a is arbitrary.

$$V_1 = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$$

Choose $a = 5 \Rightarrow V_1 = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$

— x —

The other eigenvector for $\lambda_2 = 0$ must be a generalized eigenvector and we need to solve

$$A V_2 = \lambda_2 V_2 + V_1 , \quad \lambda_2 = 0$$

$$= V_1$$

$$\begin{pmatrix} 0 & 5 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$$

(18)

$$5b + c = 5 \qquad \qquad b = 1$$
$$c = 0 \qquad \Rightarrow \qquad c = 0$$
$$-3c = 0$$

$$V_2 = \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix} \qquad a \text{ is arbitrary}$$

Choose $a=0$ we obtain.

$$V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

——— x ———

For $\lambda = 0$ we have a chain of gen. eigenvectors $\left\{ \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

For $\lambda = -3$, we have one eigenvector

at $\begin{pmatrix} 2 \\ -3 \\ 9 \end{pmatrix}$

λ

Define

$$T = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 9 \end{pmatrix}$$

We have

$$T^{-1} A T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

$$\dot{z} = T^{-1}\dot{x}$$

$$= T^{-1} A T z$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix} z$$

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}(t) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}(t) = T A T^{-1} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}$$

21

$$T^{-1} = \frac{\begin{pmatrix} 9 & 0 & 0 \\ 0 & 45 & 0 \\ -2 & 15 & 5 \end{pmatrix}^T}{45} = \frac{\begin{pmatrix} 9 & 0 & -2 \\ 0 & 45 & 15 \\ 0 & 0 & 5 \end{pmatrix}}{45}$$

$$TAT^{-1} =$$

$$\frac{\begin{pmatrix} 5 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 9 \end{pmatrix}\begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix}\begin{pmatrix} 9 & 0 & -2 \\ 0 & 45 & 15 \\ 0 & 0 & 5 \end{pmatrix}}{45}$$

$$= \begin{pmatrix} 5 & 5t & 2e^{-3t} \\ 0 & 1 & -3e^{-3t} \\ 0 & 0 & 9e^{-3t} \end{pmatrix} \cdot \begin{pmatrix} 9 & 0 & -2 \\ 0 & 45 & 15 \\ 0 & 0 & 5 \end{pmatrix} / 45$$

$$\begin{pmatrix} 45 & 225t & -10+75t+10e^{-3t} \\ 0 & 45 & 15-15e^{-3t} \\ 0 & 0 & 45e^{-3t} \end{pmatrix} \Big/\!\Big/ 45$$

$$= \begin{pmatrix} 1 & 5t & \frac{1}{45}\left(-10+75t+10e^{-3t}\right) \\ 0 & 1 & \frac{\cancel{3}\,\cancel{15}}{\cancel{45}\,\cancel{9}\,3}\left(1-e^{-3t}\right) \\ 0 & 0 & e^{-3t} \end{pmatrix}$$

$$\therefore\ x_3(t) = x_3(0)e^{-3t}$$

$$x_2(t) = x_2(0) + \frac{1}{3}\left(1-e^{-3t}\right)x_3(0)$$

$$x_1(t) = x_1(0) + 5t\,x_2(0) + \frac{1}{45}\left(-10+75t+10e^{-3t}\right)x_3(0).$$

6

$$A = \left(\begin{array}{cc|cc} 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 7 \\ 0 & 0 & -7 & 0 \end{array}\right) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$A_{11} = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$A_{22} = \begin{pmatrix} 0 & 7 \\ -7 & 0 \end{pmatrix}$$

$$p(A) = \det(\lambda I - A) =$$

$$\det \begin{pmatrix} \lambda I - A_{11} & -A_{12} \\ & \\ 0 & \lambda I - A_{22} \end{pmatrix}$$ ← Notice the block diagonal structure

$$= \det(\lambda I - A_{11}) \cdot \det(\lambda I - A_{22})$$

$$\det(\lambda I - A_{11}) = \lambda^2$$

$$\det(\lambda I - A_{22}) = \lambda^2 + 49$$

$$\therefore \; p(A) = \lambda^2(\lambda^2 + 49)$$

Eigenvalues at $\lambda = 0, 0, \pm i7$

For $\lambda = 0$, solve

$$A v_1 = 0$$

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0 \Rightarrow A_{11} \begin{pmatrix} a \\ b \end{pmatrix} + A_{12} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A_{22} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A_{22} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 7d = 0, \; -7c = 0 \Rightarrow d = c = 0$$

$$\therefore A_{11} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 3b = 0 \Rightarrow b = 0$$

$$\therefore v_1 = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{choose } a = 3$$

$$v_1 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

There are no other eigenvector v_2 of A for $\lambda = 0$ such that v_1 & v_2 are independent.

To find a gen. eigenvector we solve

$$A v_2 = 0 v_2 + v_1$$

$$\Rightarrow \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow A_{11} \begin{pmatrix} a \\ b \end{pmatrix} + A_{12} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$A_{22} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow c = d = 0$$

$$\Rightarrow A_{11} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \Rightarrow 3b = 3 \Rightarrow b = 1$$

$$v_2 = \begin{pmatrix} a \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{choose } a = 0 \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda = i7$ we solve

$$A\left[\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} + i\begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}\right] = i7\left[\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} + i\begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}\right]$$

$$\Rightarrow \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} = -7\begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} = 7\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix}$$

we have

$$A_{22}\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = -7\begin{pmatrix} c_2 \\ d_2 \end{pmatrix} ; \quad A_{22}\begin{pmatrix} c_2 \\ d_2 \end{pmatrix} = 7\begin{pmatrix} c_1 \\ d_1 \end{pmatrix}$$

$$A_{22}^2\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = -49\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} \quad \text{so}$$

$$\begin{pmatrix} 0 & 7 \\ -7 & 0 \end{pmatrix}\begin{pmatrix} 0 & 7 \\ -7 & 0 \end{pmatrix} = \begin{pmatrix} -49 & 0 \\ 0 & -49 \end{pmatrix}$$

Thus $A_{22}^2 \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = -49 \begin{pmatrix} c_1 \\ d_1 \end{pmatrix}$ Hence c_1 & d_1 are arbitrary.

$$\Rightarrow -49c_1 = -49c_1$$
$$-49d_1 = -49d_1$$

But $-7c_2 = 7d_1$ $\Rightarrow$ $\boxed{\begin{matrix} d_1 = -c_2 \\ d_2 = c_1 \end{matrix}}$

$-7d_2 = -7c_1$.

$$\begin{pmatrix} c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} -d_1 \\ c_1 \end{pmatrix}$$

— x —

$$A_{11}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + A_{12}\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = -7\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

$$A_{11}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} + A_{12}\begin{pmatrix} -d_1 \\ c_1 \end{pmatrix} = 7\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

$$A_{11}^2 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + A_{11} A_{12} \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = -7 A_{11} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

$$= -7 \left[-A_{12} \begin{pmatrix} -d_1 \\ c_1 \end{pmatrix} + 7 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \right]$$

$$= 7 A_{12} \begin{pmatrix} -d_1 \\ c_1 \end{pmatrix} - 49 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

$$\left[A_{11}^2 + 49 I \right] \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = 7 A_{12} \begin{pmatrix} -d_1 \\ c_1 \end{pmatrix} - A_{11} A_{12} \begin{pmatrix} c_1 \\ d_1 \end{pmatrix}$$

$$\cancel{A_{11}^2 + 49 I = 0} \qquad A_{11}^2 = 0$$

~~Hence~~

$$\cancel{\begin{pmatrix} 0 & 0 \\ 7 & 7 \end{pmatrix} \begin{pmatrix} -d_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 6 \end{pmatrix}}$$

$$A_{11} A_{12} = \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 49 a_1 \\ 49 b_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 7 & 7 \end{pmatrix} \begin{pmatrix} -d_1 \\ c_1 \end{pmatrix} - \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ d_1 \end{pmatrix}$$

$$49\, a_1 = -3c_1 - 3d_1$$

$$49\, b_1 = 7c_1 - 7d_1$$

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{matrix} -\frac{3}{49}(c_1 + d_1) \\ \frac{7}{49}(c_1 - d_1) \end{matrix}$$

$$-7\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ d_1 \end{pmatrix}$$

$$= \begin{pmatrix} 3b_1 \\ c_1 + d_1 \end{pmatrix} = \begin{bmatrix} \frac{21}{49}(c_1 - d_1) \\ c_1 + d_1 \end{bmatrix}$$

$$a_2 = -\frac{3}{49}(c_1 - d_1)$$

$$b_2 = -\frac{1}{7}(c_1 + d_1)$$

Finally we have our eigenvector given by

$$\begin{bmatrix} -\frac{3}{49}(c_1+d_1) \\ \frac{7}{49}(c_1-d_1) \\ c_1 \\ d_1 \end{bmatrix} + i \begin{bmatrix} -\frac{3}{49}(c_1-d_1) \\ -\frac{1}{7}(c_1+d_1) \\ -d_1 \\ c_1 \end{bmatrix}$$

Here c_1 & d_1 are arbitrary

choose $c_1 = d_1 = 49$. We obtain

$$\begin{pmatrix} -6 \\ 0 \\ 49 \\ 49 \end{pmatrix} + i \begin{pmatrix} 0 \\ -14 \\ -49 \\ 49 \end{pmatrix} \leftarrow \text{Eigenvector for } \lambda = 7i$$

Define

$$T = \begin{pmatrix} 3 & 0 & -6 & 0 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 49 & -49 \\ 0 & 0 & 49 & 49 \end{pmatrix}$$

and

$$X = TZ$$

we have

$$\dot{Z} = T^{-1} \dot{X} = T^{-1} A T Z$$

where

$$T^{-1}AT = \left(\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 7 \\ & & -7 & 0 \end{array}\right)$$

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} ; \quad \begin{pmatrix} \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} 0 & 7 \\ -7 & 0 \end{pmatrix} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}$$

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix}$$

$$\boxed{\begin{aligned} z_1(t) &= z_1(0) + t\, z_2(0) \\ z_2(t) &= z_2(0) \end{aligned}}$$

$$\begin{pmatrix} z_3(t) \\ z_4(t) \end{pmatrix} = \begin{pmatrix} \cos 7t & \sin 7t \\ -\sin 7t & \cos 7t \end{pmatrix} \begin{pmatrix} z_3(0) \\ z_4(0) \end{pmatrix}$$

$$\boxed{\begin{aligned} z_3(t) &= z_3(0) \cos 7t + z_4(0) \sin 7t \\ z_4(t) &= -z_3(0) \sin 7t + z_4(0) \cos 7t \end{aligned}}$$

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}(t) = \underbrace{\begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos 7t & \sin 7t \\ 0 & 0 & -\sin 7t & \cos 7t \end{pmatrix}}_{R} \begin{pmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \end{pmatrix}$$

X

$\cancel{TA}$

$TR =$

$$\begin{pmatrix} 3 & 3t & -6\cos 7t & -6\sin 7t \\ 0 & 1 & 14\sin 7t & -14\cos 7t \\ 0 & 0 & 49(\cos 7t + \sin 7t) & 49(\sin 7t - \cos 7t) \\ 0 & 0 & 49(\cos 7t - \sin 7t) & 49(\cos 7t + \sin 7t) \end{pmatrix}$$

Calculate T^{-1}

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \qquad W = \begin{pmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{pmatrix}$$

$$W_{11} = T_{11}^{-1} \qquad W_{22} = T_{22}^{-1}.$$

$$T_{11} W_{12} + T_{12} W_{22} = 0$$

$$W_{12} = - T_{11}^{-1} T_{12} W_{22}$$

$$W_{11} = \begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix}$$

$$W_{22} = \frac{1}{2.49} \begin{pmatrix} 1 & +1 \\ -1 & 1 \end{pmatrix} = \frac{1}{98} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \cancel{0} \begin{pmatrix} 1/98 & 1/98 \\ -1/98 & 1/98 \end{pmatrix}$$

35

$W_{12} =$

$$\frac{1}{98}\begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 14 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{98}\begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 6 & 6 \\ -14 & 14 \end{pmatrix}$$

$$= \frac{1}{98}\begin{pmatrix} 2 & 2 \\ -14 & 14 \end{pmatrix} = \begin{pmatrix} 2/98 & 2/98 \\ -14/98 & 14/98 \end{pmatrix}$$

$$= \begin{pmatrix} 1/49 & 1/49 \\ -1/7 & 1/7 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} 1/3 & 0 & 1/49 & 1/49 \\ 0 & 1 & -1/7 & 1/7 \\ 0 & 0 & 1/98 & 1/98 \\ 0 & 0 & -1/98 & 1/98 \end{pmatrix}$$

$$TRT^{-1} =$$

$$\left(\begin{array}{ccc|c}
1 & 3t & \frac{3}{49} - \frac{3}{7}t - \frac{6}{98}\cos 7t + \frac{6}{98}\sin 7t & \frac{3}{49} + \frac{3}{7}t - \frac{6}{98}\cos 7t - \frac{6}{98}\sin 7t \\
0 & 1 & -\frac{1}{7} + \frac{14}{98}(\sin 7t + \cos 7t) & 1/7 + \frac{14}{98}\sin 7t - \frac{14}{98}\cos \\
0 & 0 & 1/2\,(\cos 7t + \sin 7t) - 1/2\,(\sin 7t - \cos 7t) & 1/2\left[2\sin 7t\right] \\
0 & 0 & 1/2\,(\cos 7t - \sin 7t) - 1/2\,(\cos 7t + \sin 7t) & \frac{1}{2}\left[2\cos 7t\right]
\end{array}\right)$$

$$= \left(\begin{array}{ccc|c}
1 & 3t & \frac{3}{49}\left[1 - 7t\right] - \frac{6}{98}(\cos 7t - \sin 7t) & \frac{3}{49}(1 + 7t) - \frac{6}{98}(\cos 7t + \sin 7t) \\
0 & 1 & -\frac{1}{7}\left[1 - (\sin 7t + \cos 7t)\right] & 1/7\,(1 + \sin 7t - \cos 7t) \\
0 & 0 & \cos 7t \quad \cancel{\sin 7t} & \sin 7t \\
0 & 0 & -\sin 7t \quad \cancel{\cos 7t} & \cos 7t
\end{array}\right)$$

$$x(t) = T R T^{-1} x(0)$$

$\therefore$ ~~$x_3(0)$~~

$$\begin{bmatrix} x_3(t) \\ x_4(t) \end{bmatrix} = \begin{pmatrix} \cos 7t & \sin 7t \\ -\sin 7t & \cos 7t \end{pmatrix} \begin{pmatrix} x_3(0) \\ x_4(0) \end{pmatrix}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 3t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} +$$

$$\frac{3}{49} \left(\begin{array}{c|c} 1 - 7t - \cos 7t + \sin 7t & 1 + 7t - \cos 7t - \sin 7t \\ \hline -\frac{7}{3}(1 - \sin 7t - \cos 7t) & +\frac{7}{3}(1 + \sin 7t - \cos 7t) \end{array} \right) \begin{pmatrix} x_3(0) \\ x_4(0) \end{pmatrix}$$