H. W. 1

1. Show that the product of the eigenvalues of a matrix equal the determinant. Show that the sum of the eigenvalues of a matrix equals the sum of the elements on the main diagonal (trace).

2. Show that the space of $n \times n$ matrices is a vector space. Show that if M and N are $n \times n$ matrices then

$$L : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$$

$$\underline{X} \mapsto M\underline{X} + \underline{X}N$$

defines a linear transformation.

3. Let A be the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 0 & 2 & 0 \end{pmatrix}$$

Consider a linear transformation

$$L : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$
$$\underline{x} \longmapsto A\underline{x}$$

(a) Characterize the range space of L by writing it as a linear combination of independent vectors in $\mathbb{R}^3$.

(b) Characterize the null space of A^T by writing it as a linear combination of independent vectors in $\mathbb{R}^3$

(c) Show that the range space in (a) is perpendicular to the null space in (b).

4. Consider a state space system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 \\ t \end{pmatrix} u(t).$$

where $x_1(0) = x_2(0) = 0$. We would like to find $u(t)$, $0 \le t \le 5$ such that $x_1(5) = 10$, $x_2(5) = 20$. What would be a choice of $u(t)$.

5. We are given

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 5 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(a) Calculate the eigenvalues and generalized eigenvectors of A.

(b) Find a matrix T such that for

$$X = TZ$$

we have

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

(c) Solve (b) for $z_1(t)$, $z_2(t)$, $z_3(t)$.

(d) Now write down $x_1(t)$, $x_2(t)$, $x_3(t)$.

(6) Repeat problem (5) for

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 7 \\ 0 & 0 & -7 & 0 \end{array}\right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$