Fourier Series

&

Solving o.d.es.

Q: Using Fourier Series expansion and other techniques we have learnt, solve the following

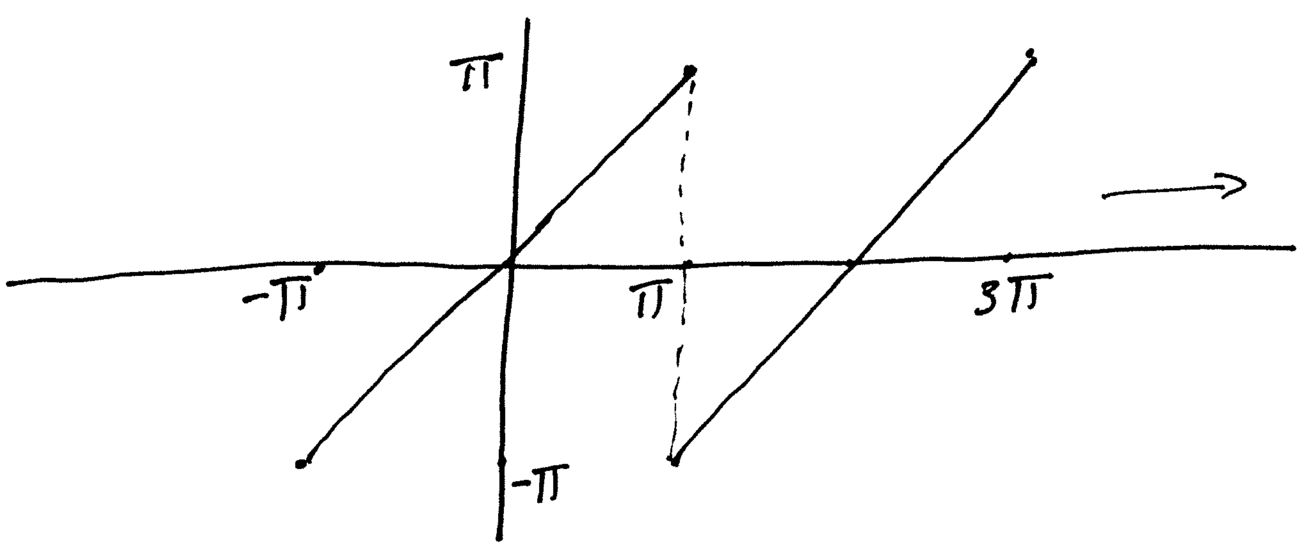
$$\ddot{y}(t) + 7\dot{y} + 2y = f(t)$$

where $y(0) = y_0$, $v(0) = \dot{y}(0)$

and where

$$f(t) = t \qquad -\pi < t < \pi$$

$$f(t + 2\pi) = f(t) \quad \forall t.$$



The function $f(t)$ is an odd function.

we can expand using Fourier Sine Series

as

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

where

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt$$

$$= \frac{2}{\pi} \int_{0}^{\pi} t \sin nt \, dt$$

check to show that

$$b_n = \frac{2}{n} (-1)^{n+1}$$

Hence

$$f(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nt$$

For each integer $n = 1, 2, \ldots$

we would solve the equation

$$\ddot{y} + 7\dot{y} + 2y = \sin nt \quad \text{(A)}$$

Let us denote $y_{p,n}(t)$ be the associated particular solution. It would then follow "using linearity" that

$$y_p(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} y_{p,n}(t).$$

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We now proceed to calculate $y_{p,n}(t)$.

Writing

$$y_{p,n}(t) = A_n \sin nt + B_n \cos nt \quad (*)$$

Plugging $(*)$ into (A) and equating co-efficients we obtain

$$\begin{pmatrix} 2-n^2 & -7n \\ 7n & 2-n^2 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

If we denote

$$\Delta_n = \sqrt{(2-n^2)^2 + 49n^2}$$

we have

$$A_n = \frac{2-n^2}{\Delta_n^2} \; ; \; B_n = \frac{-7n}{\Delta_n^2}$$

we can write

$$y_{p,n}(t) = \frac{2-n^2}{\Delta_n^2} \sin nt + \frac{-7n}{\Delta_n^2} \cos nt.$$

we define θ_n as follows.

$$\cos\theta_n = \frac{2-n^2}{\Delta_n} \; , \quad \sin\theta_n = \frac{-7n}{\Delta_n}$$

It follows that

$$y_{p,n}(t) = \frac{1}{\Delta_n}\left[\sin nt \cos\theta_n + \cos nt \sin\theta_n\right]$$

$$y_{p,n} = \frac{1}{\Delta_n}\left[\sin(nt + \theta_n)\right]$$

The particular solution $y_p(t)$ is given by

$$y_p(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \frac{1}{\Delta_n} \sin(nt + \theta_n)$$

where Δ_n and θ_n are already defined.

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The two roots of the characteristic polynomial are given by

$$\lambda_1 = -\frac{7}{2} + \sqrt{\frac{41}{4}} \; ; \; \lambda_2 = -\frac{7}{2} - \sqrt{\frac{41}{4}}$$

The homogeneous solution are given by

$$y_h(t) = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t}$$

where α and β are unknown constants.

We write

$$y(t) = y_h(t) + y_p(t)$$

and compute α, β from the initial conditions $y(0), \dot{y}(0)$.

$$y(0) = y_h(0) + y_p(0)$$

$$= \alpha + \beta + a$$

where

$$a = \sum_{n=1}^{\infty} (-1)^n \frac{14}{(2-n^2)^2 + 49n^2}$$

$$\dot{y}(0) = \dot{y}_h(0) + \dot{y}_p(0)$$

$$= \lambda_1 \alpha + \lambda_2 \beta + b$$

where

$$b = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2(2-n^2)}{(2-n^2)^2 + 49n^2}$$

α & β are computed from.

$$\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} y_0 - a \\ v_0 - b \end{pmatrix}$$

where $y(0) = y_0$, $\dot{y}(0) = v_0$.

We obtain.

$$\alpha = \frac{\lambda_2 (y_0 - a) - (v_0 - b)}{\lambda_2 - \lambda_1}.$$

$$\beta = \frac{(v_0 - b) - \lambda_1 (y_0 - a)}{\lambda_2 - \lambda_1}.$$

$$y(t) = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n \Delta_n} \sin(nt + \theta_n)$$

sol^n to the o.d.e.

It can be checked that both λ_1 & λ_2 are negative so that

$$e^{\lambda_1 t},\ e^{\lambda_2 t}$$

are exponentially decaying functions.

If we wait long enough. then

$$y(t) \approx \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n\Delta_n} \sin(nt + \theta_n)$$

The right hand side is called the steady state response. $y_{ss}(t)$

$$y_{ss}(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n\Delta_n} \sin(nt + \theta_n)$$

Remark:

Note that

$$\frac{1}{\Delta_n}(\sin nt + \theta_n)$$

is the particular solution when the input $f(t) = \sin nt$.

Δ_n and θ_n has the following interpretation.

write

$$Y(s) = \frac{1}{s^2 + 7s + 2} F(s)$$

where we assume that the initial conditions are zero.

If we define

$$h(s) = \frac{1}{s^2 + 7s + 2}.$$

and calculate.

$$h(in) = \frac{1}{(in)^2 + 7(in) + 2}$$

$$= \frac{1}{(2-n^2) + i7n}$$

$$= \frac{(2-n^2) - i7n}{(2-n^2)^2 + (7n)^2}$$

$$= \frac{(2-n^2) - i7n}{\Delta_n^2}$$

$$|h(in)| = \frac{|(2-n^2) - i7n|}{\Delta_n^2}$$

$$= \frac{\sqrt{(2-n^2)^2 + 49n^2}}{\Delta_n^2} = \frac{\Delta_n}{\Delta_n^2} = \frac{1}{\Delta_n}.$$

$$\therefore \boxed{\frac{1}{\Delta_n} = |h(in)|}$$

If $\arg(h(in)) = \theta_n$ we have.

$$\sin\theta_n = -\frac{7n}{\Delta_n}, \cos\theta_n = \frac{2-n^2}{\Delta_n}$$

It would follow that the particular sol$^{\underline{n}}$ due to $f(t) = \sin nt$ is given by

$$y_{p,n}(t) = |h(in)| \sin(nt + \arg(h(in)))$$

It follows that

$$\boxed{y_{ss}(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} |h(in)| \sin(nt + \arg[h(in)]).}$$

The problem posed on page 1 is over.

Remark:

If we have

$$\ddot{y}(t) + a\dot{y}(t) + b\,y(t) = \cos nt$$

then $y_{p,n}(t)$ can be computed likewise and I state the result as follows:

$$y_{p,n}(t) = \frac{1}{\Delta_n}\left[\cos(nt + \theta_n)\right]$$

(see page 5)

where Δ_n and θ_n are given precisely by the formula on page 4 if $a = 7$ and $b = 2$.

For a, b in general

$$\frac{1}{\Delta_n} = |h(in)|, \quad \theta_n = \arg|h(in)|$$

where

$$h(s) = \frac{1}{s^2 + as + b}.$$

In fact.

$$h(in) = \frac{1}{(b-n^2)+ian}$$

$$= \frac{(b-n^2)-ian}{(b-n^2)^2+a^2n^2}$$

Define $\Delta_n = \sqrt{(b-n^2)^2+a^2n^2}$.

$$\sin \theta_n = -\frac{an}{\Delta_n}$$

$$\cos \theta_n = \frac{b-n^2}{\Delta_n}.$$

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Remark

$$\ddot{y} + a\dot{y} + by = f(t)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nt + b_n \sin nt\right]$$

One can write

$$y_p(t) = |h(0)| \frac{a_0}{2} + \sum_{n=1}^{\infty} |h(in)| \left[a_n \cos(nt + \theta_n) + b_n \sin(nt + \theta_n)\right]$$

where

$$\theta_n = \arg\left[h(in)\right]$$

and where

$$h(s) = \frac{1}{s^2 + as + b}.$$

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