Solutions to

H.W. 7

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KEY BRIDGE

1 This is an even function. Hence only nonzero terms are the cosine terms.

$$p = \pi/2$$

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \sin x\,dx$$

$$= \frac{4}{\pi}\Big[-\cos x\Big]_0^{\pi/2}$$

$$= \frac{4}{\pi}\left[-\cos\frac{\pi}{2}+1\right] = \frac{4}{\pi}$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2} f(x)\cos 2nx\,dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \sin x\cos 2nx\,dx$$

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$$\sin x \cos 2nx =$$

$$\frac{1}{2}\left[\sin(2n+1)x + \sin(1-2n)x\right]$$

$$= \frac{1}{2}\left[\sin(2n+1)x - \sin(2n-1)x\right]$$

$$\int_0^{\pi/2} \sin x \cos 2nx \, dx =$$

$$\frac{1}{2}\left[-\frac{\cos(2n+1)x}{2n+1} + \frac{\cos(2n-1)x}{2n-1}\right]\Bigg|_0^{\pi/2}$$

$$= +\frac{1}{2}\left[+\frac{1}{2n+1} - \frac{1}{2n-1}\right]$$

$$= \frac{1}{2}\,\frac{\cancel{2n}-1-\cancel{2n}-1}{4n^2-1} = \frac{1}{1-4n^2}$$

$$\therefore\ a_n = \frac{1}{\pi}\,\frac{4}{1-4n^2} = -\frac{1}{\pi}\,\frac{4}{4n^2-1}$$

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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{p}$$

$$= \frac{2}{\pi} + \sum_{n=1}^{\infty} - \frac{1}{\pi} \frac{4}{4n^2-1} \cos 2nx$$

$$= \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1}$$

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2 $p = \pi$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin x\,dx$$

$$= \frac{1}{\pi}\left(-\cos x\right)\Big|_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[1+1\right] = 2/\pi$$

$$a_n = \frac{1}{\pi}\int_{0}^{\pi} \sin x \cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{0}^{\pi} \sin x \sin nx\,dx$$

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for $n > 1$

$\sin x \cos nx =$

$$\frac{1}{2}\left[\sin(1+n)x + \sin(1-n)x\right]$$

$$= \frac{1}{2}\left[\sin(n+1)x - \sin(n-1)x\right]$$

$$\int_0^{\pi} \sin x \cos nx \, dx =$$

$$\frac{1}{2}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]\Bigg|_0^{\pi}$$

$$= \frac{1}{2}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right]$$

$$+ \frac{1}{2}\left[+\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$= 0$ when n is odd

$$= \frac{1}{n+1} - \frac{1}{n-1} = \frac{n-1-n-1}{n^2-1} = -\frac{2}{n^2-1}$$

n is even

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For $n>1$,

$$a_n = -\frac{2}{\pi}\frac{1}{n^2-1} \quad \text{for } n \text{ even}$$

For $n=1$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cos x \, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{-1}{2\pi} \left.\frac{\cos 2x}{2}\right|_0^{\pi}$$

$$= -\frac{1}{4\pi}\left[\cos 2\pi - 1\right] = 0$$

$\therefore\ a_0 = \frac{2}{\pi}$, $a_1 = 0$, $a_n = 0$ for n odd.

$$a_n = -\frac{2}{\pi}\frac{1}{n^2-1} \quad \text{for } n \text{ even}$$

$$= -\frac{2}{\pi}\frac{1}{4m^2-1} \quad \text{for } n=2m$$

$m = 1, 2, 3, \ldots$

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For $n > 1$

$\sin nx \sin x =$

$$\frac{1}{2}\left[\cos(n-1)x - \cos(n+1)x\right]$$

$$\int_0^{\pi} \sin nx \sin x \, dx =$$

$$\frac{1}{2}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]\Big|_0^{\pi}$$

$$= 0$$

$b_n = 0 \quad n > 1.$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[1 - \cos 2x\right] dx$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\pi\right] = \frac{1}{2}$$

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$$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{1}{4m^2-1}\cos 2mx$$

$$+ \frac{1}{2}\sin x$$

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3 $p = \pi/2$

$f(x)$ is an odd function.

$\therefore a_n = 0$ for $n \geq 0$.

$$b_n = \frac{4}{\pi}\int_0^{\pi/2} \cos x \sin 2nx \, dx$$

$$\sin 2nx \cos x = \frac{1}{2}\left[\sin(2n+1)x + \sin(2n-1)x\right]$$

$$\int_0^{\pi/2} \sin 2nx \cos x \, dx =$$

$$-\frac{1}{2}\left[\frac{\cos(2n+1)x}{2n+1} + \frac{\cos(2n-1)x}{2n-1}\right]\Bigg|_0^{\pi/2}$$

$$= -\frac{1}{2}\left[\frac{\cos\left[(2n+1)\frac{\pi}{2}\right]}{2n+1} + \frac{\cos\left[(2n+1)\frac{\pi}{2}\right]}{2n-1}\right] = 0$$

$$+\frac{1}{2}\left[\frac{1}{2n+1} + \frac{1}{2n-1}\right]$$

$$= \frac{1}{2}\,\frac{2n-\cancel{1}+2n+\cancel{1}}{4n^2-1} = \frac{2n}{4n^2-1}$$

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$$b_n = \frac{4}{\pi} \frac{2n}{4n^2-1}$$

$$= \frac{8n}{4n^2-1} \frac{1}{\pi}$$

$$\therefore f(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{8n}{4n^2-1} \sin 2nx$$

$$\boxed{f(x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2-1} \sin 2nx}$$

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4 Ans:

since

$$y(t) = 3e^{-3t} + 4e^{-6t} + 9\sin 10t$$

$$y(0) = 3 + 4 + 0 = 7$$

$$\dot{y}(t) = -9e^{-3t} - 24e^{-6t} + 90\cos 10t$$

$$\dot{y}(0) = -9 - 24 + 90 = 57 = V_0.$$

(a) If $y_h(t) = 9\sin 10t$

$$\dot{y}_h(t) = 90\cos 10t$$

$$\ddot{y}_h(t) = -900\sin 10t$$

$$\ddot{y}_h + a\dot{y}_h + by_h = 0$$

$$\Rightarrow -900\sin 10t + 90a\cos 10t + 9b\sin 10t = 0$$

$$\therefore 9b - 900 = 0;\ 90a = 0$$

$$\Rightarrow b = 100,\ a = 0$$

$\therefore$ o.d.e is $\ddot{y} + 100y = f(t)$

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If
$$y_p(t) = 3e^{-3t} + 4e^{-6t}.$$
$$\dot{y}_p(t) = -9e^{-3t} - 24e^{-6t}$$
$$\ddot{y}_p(t) = 27e^{-3t} + 144e^{-6t}$$

$$f(t) = \ddot{y}_p + 100y_p$$
$$= 27e^{-3t} + 144e^{-6t}$$
$$+ 300e^{-3t} + 400e^{-6t}.$$
$$= 327e^{-3t} + 544e^{-6t}.$$

$\therefore$ $a=0$, $b=100$, $y_0 = 7$, $v_0 = 57$
$$f(t) = 327e^{-3t} + 544e^{-6t}.$$

(b) If

$$y_h(t) = 3e^{-3t} + 4e^{-6t}$$

$$\dot{y}_h(t) = -9e^{-3t} - 24e^{-6t}$$

$$\ddot{y}_h(t) = 27e^{-3t} + 144e^{-6t}$$

$$\ddot{y}_h + a\dot{y}_h + by_h = 0$$

$$\Rightarrow 27e^{-3t} + 144e^{-6t}$$
$$-9ae^{-3t} - 24ae^{-6t}$$
$$3be^{-3t} + 4be^{-6t} = 0$$

$$\therefore 9a - 3b = 27$$
$$24a - 4b = 144$$

$$\Rightarrow \begin{array}{l} 36a - 12b = 108 \\ 72a - 12b = 432 \end{array}$$

$$\Rightarrow 36a = 324 \Rightarrow a = 9$$

$$3b = 81 - 27 \Rightarrow b = 18$$

$\therefore$ ode is

$$\ddot{y} + 9\dot{y} + 18y = f(t).$$

$$y_p(t) = 9\sin 10t$$
$$\dot{y}_p(t) = 90\cos 10t$$
$$\ddot{y}_p(t) = -900\sin 10t.$$

$$\begin{aligned} f(t) = &-900\sin 10t \\ &+ 810\cos 10t \\ &+ 162\sin 10t \\ = &\ 810\cos 10t - 738\sin 10t. \end{aligned}$$

(15)

(5) Ans:

$$\mathcal{L}\left(f(t)\right) = F(s)$$

$$\mathcal{L}\left(t^2 f(t)\right) = \frac{d^2}{ds^2} F(s).$$

$$\frac{d}{ds}\frac{s}{s^2+25} = \frac{(s^2+25)1 - s\,2s}{(s^2+25)^2}$$

$$= \frac{-s^2+25}{(s^2+25)^2}$$

$$\frac{d^2}{ds^2}\frac{s}{s^2+25} = \frac{(s^2+25)^2(-2s) - [-s^2+25]\,4[s^2+25]s}{(s^2+25)^4}$$

$$= \frac{(s^2+25)(-2s) + (s^2-25)4s}{(s^2+25)^3}$$

$$= \frac{-2s^3 - 50s + 4s^3 - 100s}{(s^2+25)^3}$$

$$= (2s^3 - 150s)\big/(s^2+25)^3$$

$$= 2s\left[s^2 - 75\right] \Big/ (s^2+25)^3$$

$$\mathcal{L}\left(t^2 \cos 5t\right) = \frac{2s\left[s^2-75\right]}{(s^2+25)^3}$$

$$\mathcal{L}\left(t^2 e^{-3t} \cos 5t\right) = .$$

$$\frac{2(s+3)\left[(s+3)^2 - 75\right]}{\left[(s+3)^2 + 25\right]^3}$$

6 A

$$\boxed{\mathcal{L}^{-1}\frac{5}{s^2+25} = \sin 5t}$$

$$\boxed{\mathcal{L}^{-1}\frac{s}{s^2+25} = \cos 5t}.$$

$$\mathcal{L}(t\sin 5t) = -\frac{d}{ds}\frac{5}{s^2+25}.$$

$$= +5\frac{1}{(s^2+25)^2}2s.$$

$$= \frac{10s}{(s^2+25)^2}.$$

$$\therefore \boxed{\mathcal{L}^{-1}\frac{10s}{(s^2+25)^2} = t\sin 5t}.$$

$$\mathcal{L}(t\cos 5t) = -\frac{d}{ds}\frac{s}{s^2+25}$$

$$= \frac{s^2-25}{(s^2+25)^2}$$

$$\therefore \boxed{\mathcal{L}^{-1}\frac{s^2-25}{(s^2+25)^2} = t\cos 5t}.$$

(B) $AF_1 + BF_2 + CF_3 + DF_4 =$.

$$\frac{[5A+Bs](s^2+25) + 10Cs + (s^2-25)D}{(s^2+25)^2}$$

$$\therefore (5A+Bs)(s^2+25) + [10Cs + (s^2-25)D]$$
$$= 3s^3 + 4s^2 + 5s + 6.$$

co-eff of s^3 is $B = 3$

" " " s^2 is $5A + D = 4$

" " " s is $25B + 10C = 5$

$$\Rightarrow 10C = 5 - 75 = -70$$

$$\Rightarrow C = -7$$

" " " 1 is $125A - 25D = 6$.

$$125A + 25D = 100 \Rightarrow 250A = 106$$
$$125A - 25D = 6$$

$$\Rightarrow \boxed{A = \frac{106}{250}}$$

$$50D = 94 \Rightarrow \boxed{D = \frac{94}{50}}$$

$$\therefore \boxed{A = \frac{53}{125},\ B = 3,\ C = -7,\ D = \frac{47}{25}}$$

$$\therefore \boxed{\mathcal{L}^{-1} F_5(s) = \frac{53}{125}\sin 5t + 3\cos 5t - 7t\sin 5t + \frac{47}{25}t\cos 5t}.$$

(c) substituting $s_1 = s+1$ we have

$$3s^3 + 4s^2 + 5s + 6$$

$$= 3(s_1-1)^3 + 4(s_1-1)^2 + 5(s_1-1) + 6.$$

$$= 3(s_1^3 - 3s_1^2 + 3s_1 - 1)$$

$$+ 4(s_1^2 - 2s_1 + 1)$$

$$+ 5(s_1 - 1) + 6$$

$$= 3s_1^3 - 9s_1^2 + 9s_1 - 3$$

$$+ 4s_1^2 - 8s_1 + 4$$

$$+ 5s_1 - 5$$

$$+ 6$$

$$= 3s_1^3 - 5s_1^2 + 6s_1 + 2$$

$$F_6(s) = \frac{3s_1^3 - 5s_1^2 + 6s_1 + 2}{(s_1^2 + 25)^2}.$$

$$\mathcal{L}^{-1} F_6(s) =$$

$$e^{-t} \mathcal{L}^{-1} \left(\frac{3s^3 - 5s^2 + 6s + 2}{(s^2+25)^2} \right)$$

proceeding as in B we get

$B = 3$

$5A + D = -5$

$25B + 10C = 6$

$125A - 25D = 2$

which can now be solved.