Solutions to
H. W. 5

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a) char. poly $\lambda^2+4\lambda+3$.

Roots are at

$$\lambda = \frac{-4 \pm \sqrt{16-12}}{2}$$

$$= \frac{-4 \pm 2}{2} = \frac{-2}{2}, \frac{-6}{2}$$

$$= -1, -3$$

$\therefore \lambda^2+4\lambda+3 = (\lambda+1)(\lambda+3)$.

$$y_h(t) = Ae^{-t} + Be^{-3t}$$

$$y_p(t) = A$$

$\ddot{A} + 4\dot{A} + 3A = 6 \Rightarrow \boxed{A=2}$.

$y_p(t) = 2$.

$$y(t) = 2 + Ae^{-t} + Be^{-3t}.$$

$$y(0) = 2 + A + B = 0 \Rightarrow \boxed{A + B = -2}$$

$$\dot{y}(t) = -Ae^{-t} - 3Be^{-3t}.$$

$$\dot{y}(0) = -A - 3B = 10 \Rightarrow \boxed{A + 3B = -10}$$

$$2B = -10 + 2 = -8$$

$$\boxed{B = -4} \quad \boxed{A = 2}$$

$$\therefore \boxed{y(t) = 2 + 2e^{-t} - 4e^{-3t}}.$$

$$\mathcal{L}(y) = Y(s)$$

$$\mathcal{L}(\dot{y}) = sY(s) - y(0) = sY(s).$$

$$\mathcal{L}(\ddot{y}) = s\mathcal{L}(\dot{y}) - \dot{y}(0).$$

$$= s^2 Y(s) - 10.$$

$$\therefore \mathcal{L}(\ddot{y}) + 4\mathcal{L}(\dot{y}) + 3\mathcal{L}(y) = \mathcal{L}(6)$$

$$= \frac{6}{s}.$$

$$s^2 Y(s) - 10 + 4sY(s) + 3Y(s) = \frac{6}{s}.$$

$$(s^2 + 4s + 3)Y(s) = 10 + \frac{6}{s} = \frac{10s + 6}{s}.$$

$$\therefore Y(s) = \frac{10s + 6}{s(s+3)(s+1)}.$$

④

$$\frac{10s+6}{s(s+3)(s+1)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+1}.$$

$$A = \left.\frac{10s+6}{(s+3)(s+1)}\right|_{s=0} = \frac{6}{3} = 2.$$

$$B = \left.\frac{10s+6}{s(s+1)}\right|_{s=-3} = \frac{-30+6}{(-3)(-2)}.$$

$$= \frac{-24}{6} = -4.$$

$$C = \left.\frac{10s+6}{s(s+3)}\right|_{s=-1} = \frac{-10+6}{(-1)(2)} = \frac{-4}{-2}$$

$$= 2.$$

$$\therefore\ y(t) = \mathcal{L}^{-1}\left[\frac{2}{s} - \frac{4}{s+3} + \frac{2}{s+1}\right]$$

$$= 2 - 4e^{-3t} + 2e^{-t}.$$

$\blacksquare$

(2) Ans:

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~~$\emptyset$~~ Char poly

$\lambda^2 + 4\lambda + 6$.

Roots at

$$\frac{-4 \pm \sqrt{16-24}}{2}$$

$$= -2 \pm \sqrt{-\frac{\cancel{8}\,2}{\cancel{4}}}$$

$$= -2 \pm \sqrt{2}\, i$$

$$\therefore\ y_h(t) = e^{-2t}\left(A \sin\sqrt{2}t + B\cos\sqrt{2}t\right).$$

$$y_p(t) = 1 .$$

⑥

$$y(t) = 1 + e^{-2t}\left[A \sin\sqrt{2}\,t + B\cos\sqrt{2}\,t\right]$$

$$y(0) = 1 + B = 0 \Rightarrow B = -1.$$

$$\therefore\ y(t) = 1 + e^{-2t}\left[A\sin\sqrt{2}\,t - \cos\sqrt{2}\,t\right].$$

$$\dot{y}(t) = -2e^{-2t}\left[A\sin\sqrt{2}\,t - \cos\sqrt{2}\,t\right]$$

$$+ e^{-2t}\left[\sqrt{2}A\cos\sqrt{2}\,t + \sqrt{2}\sin\sqrt{2}\,t\right].$$

$$\dot{y}(0) = -2\left[-1\right] + \left[\sqrt{2}A\right]$$

$$= 2 + \sqrt{2}A = 10$$

$$\Rightarrow \sqrt{2}A = 8$$

$$\Rightarrow A = \frac{4\times 2}{\sqrt{2}} = 4\sqrt{2}.$$

$$\boxed{y(t) = 1 + e^{-2t}\left[4\sqrt{2}\sin\sqrt{2}\,t - \cos\sqrt{2}\,t\right]}$$

— x —

Using Laplace Transform we have

$$\left(s^2 Y(s) - 10\right) + 4sY(s) + 6Y(s) = \frac{6}{s}.$$

$$\left(s^2 + 4s + 6\right) Y(s) = 10 + \frac{6}{s} = \frac{10s + 6}{s}.$$

$$Y(s) = \frac{10s + 6}{s(s^2 + 4s + 6)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 6}$$

$$A = \left.\frac{10s + 6}{s^2 + 4s + 6}\right|_{s=0} = 1.$$

Hence

$$\frac{10s + 6}{s(s^2 + 4s + 6)} = \cancel{B}\,\frac{s^2 + 4s + 6 + Bs^2 + Cs}{s(s^2 + 4s + 6)}.$$

$$\therefore \quad B+1=0 \quad \Rightarrow \quad B=-1$$
$$C+4=10 \qquad\qquad C=6$$

$$Y(s) = \frac{1}{s} - \frac{s-6}{s^2+4s+6}$$

$$\frac{s-6}{s^2+4s+6} = \frac{s-6}{(s+2)^2+2}$$

$$= \frac{(s+2)-8}{(s+2)^2+(\sqrt{2})^2}$$

$$= \frac{s+2}{(s+2)^2+(\sqrt{2})^2}$$

$$-\frac{8}{\sqrt{2}} \cdot \frac{\sqrt{2}}{(s+2)^2+(\sqrt{2})^2}$$

$$\mathcal{L}^{-1}\left(\frac{s-6}{s^2+4s+6}\right) = e^{-2t}\cos\sqrt{2}\,t - \frac{8}{\sqrt{2}}\sin\sqrt{2}\,t$$
$$= e^{-2t}\left[\cos\sqrt{2}\,t - 4\sqrt{2}\sin\sqrt{2}\,t\right]$$

(9)

$$y(t) = 1 - \mathcal{L}^{-1} \frac{s-6}{s^2+4s+6}.$$

$$\boxed{y(t) = 1 + e^{-2t}\left(4\sqrt{2}\sin\sqrt{2}\,t - \cos\sqrt{2}\,t\right)}.$$

3 Ans.

char polynomial.

$\lambda^2 + 4\lambda + 3$.

$= (\lambda+1)(\lambda+3)$.

$y_h(t) = Ae^{-t} + Be^{-3t}$.

$y_p(t) = (C + Dt)e^{-5t}$. (from table in the book).

$\dot{y}_p(t) = (C + Dt)(-5)e^{-5t}$
$+ De^{-5t}$.

$= (-5C - 5Dt + D)e^{-5t}$.

$= [(D - 5C) - 5Dt]e^{-5t}$.

$\ddot{y}_p = [(D-5C) - 5Dt](-5)e^{-5t}$.
$-5De^{-5t}$.

$$\left[-10D + 25C + 25Dt\right] e^{-5t}.$$

$$= \left[25C - 10D + 25Dt\right] e^{-5t}.$$

$$\ddot{y}_p + 4\dot{y}_p + 3y_p = .$$

$$\left[\begin{array}{lll} 25C & -10D & +25Dt \\ -20C & +4D & -20Dt \\ 3C & & +3Dt \end{array}\right] e^{-5t} = te^{-5t}$$

$$\therefore 8C - 6D = 0 \qquad D = \frac{1}{8} = \frac{4}{32}.$$

$$8D = 1. \qquad \Rightarrow \qquad 8C = \frac{6}{8}$$

$$C = \frac{6}{64} = \frac{3}{32}.$$

$$y_p = \frac{(3+4t)e^{-5t}}{32}.$$

$$y(t) = Ae^{-t} + Be^{-3t} + \frac{(3+4t)}{32} e^{-5t}.$$

$$y(0) = A + B + \frac{3}{32} = 0.$$

$$\boxed{A + B = -\frac{3}{32}}$$

$$\dot{y}(t) = -Ae^{-t} - 3Be^{-3t} + \frac{3+4t}{32}(-5)e^{-5t} + \frac{4}{32}e^{-5t}.$$

$$\dot{y}(0) = -A - 3B - \frac{15}{32} + \frac{4}{32}.$$

$$= -A - 3B - \frac{11}{32} = 10.$$

$$-A - 3B = 10 + 11/32.$$

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$$A + 3B = -10 - \frac{11}{32}.$$

$$A + B = -\frac{3}{32}.$$

$$2B = -10 - \frac{\cancel{8}}{\cancel{32}_4} = -10 - \frac{1}{4}.$$

$$\boxed{B = -5 - \frac{1}{8}}.$$

$$A = -\frac{3}{32} + 5 + \frac{1}{8}.$$

$$\boxed{A = 5 + \frac{1}{32}}.$$

$$\therefore\ y(t) = \left(5 + \frac{1}{32}\right)e^{-t} - \left(5 + \frac{1}{8}\right)e^{-3t} + \frac{3+4t}{32}e^{-5t}.$$

(14)

(3) (Using Laplace's Transform).

$$\left[s^2 Y(s) - 10\right] + 4sY(s) + 3Y(s) = .$$

$$\mathcal{L}\left(te^{-5t}\right) = \frac{1}{(s+5)^2}$$

$$\therefore \left[s^2 + 4s + 3\right] Y(s) = 10 + \frac{1}{(s+5)^2}.$$

$$= \frac{10(s^2+10s+25)+1}{(s+5)^2}.$$

$$= \frac{10s^2+100s+251}{(s+5)^2}.$$

$$Y(s) = \frac{10s^2+100s+251.}{(s^2+4s+3)(s+5)^2}.$$

$s^2+4s+3=(s+1)(s+3).$

$$\therefore \frac{10s^2+100s+251}{(s+1)(s+3)(s+5)^2}$$

$$= \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{s+5} + \frac{D}{(s+5)^2} \quad (*)$$

$$A = \left.\frac{10s^2+100s+251}{(s+3)(s+5)^2}\right|_{s=-1}.$$

$$B = \left.\frac{10s^2+100s+251}{(s+1)(s+5)^2}\right|_{s=-3}.$$

$$D = \left.\frac{10s^2+100s+251}{(s+1)(s+3)}\right|_{s=-5}.$$

C cannot be computed using short cut.

From ⊛ we write down.

$$
\begin{aligned}
& A(s+3)(s+5)^2 \\
& + B(s+1)(s+5)^2 \quad = 10s^2 + 100s + 251. \\
& + C(s+1)(s+3)(s+5). \\
& + D(s+1)(s+3).
\end{aligned}
$$

comparing the coefficient of s^3

we get.

$$A + B + C = 0$$

$$\Rightarrow C = -(A+B).$$

We have.

$y(t) =$

$$Ae^{-t} + Be^{-3t} - (A+B)e^{-5t}$$

$$+ Dte^{-5t}.$$

where A, B, D. are described on page 15.