Solutions to H.W. 3

(a) $\dddot{x} - \dot{x} = \cos 3t$

$x(0) = 2,\ \dot{x}(0) = 0,\ \ddot{x}(0) = -3.$

- Hom eq is

$$\dddot{x} - \dot{x} = 0$$

- char p-ly is

$$\lambda^3 - \lambda = \lambda(\lambda^2 - 1) = \lambda(\lambda+1)(\lambda-1).$$

Roots are at $\lambda = 0, 1, -1$

- $\phi_1(t) = e^{0t} = 1$

$\phi_2(t) = e^{1t} = e^t$

$\phi_3(t) = e^{-1t} = e^{-t}.$

- $x_h(t) = \alpha + \beta e^t + \gamma e^{-t}.$

$f(t) = \cos 3t$.

- $x_p(t) = A\cos 3t + B\sin 3t$

(from Book).

- $\dot{x}_p = -3A\sin 3t + 3B\cos 3t$.

$\ddot{x}_p = -9A\cos 3t - 9B\sin 3t$.

$\dddot{x}_p = +27A\sin 3t - 27B\cos 3t$.

$\dddot{x}_p - \dot{x}_p = 30A\sin 3t - 30B\cos 3t =$
$\cos 3t$.

$\therefore A = 0 \quad B = -\frac{1}{30}$

$\therefore x_p(t) = -\frac{1}{30}\sin 3t$.

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$$x(t) = x_h(t) + x_p(t)$$

$$= \alpha + \beta e^{t} + \gamma e^{-t} - \frac{1}{30} \sin 3t.$$

$$x(0) = \boxed{\alpha + \beta + \gamma = 2}$$

$$\dot{x}(t) = \beta e^{t} - \gamma e^{-t} - \frac{1}{10} \cos 3t.$$

$$\dot{x}(0) = \boxed{\beta - \gamma - \frac{1}{10} = 0}$$

$$\ddot{x}(t) = \beta e^{t} + \gamma e^{-t} + \frac{3}{10} \sin 3t.$$

$$\ddot{x}(0) = \boxed{\beta + \gamma = -3}$$

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$$2\beta = -3 + \frac{1}{10} = -\frac{30}{10} + \frac{1}{10}$$

$$= \frac{-29}{10}$$

$$\boxed{\beta = -\frac{29}{20}}$$

$$\gamma = -3 - \beta$$

$$= -3 + \frac{29}{20} = -\frac{60}{20} + \frac{29}{20}$$

$$= \frac{-60+29}{20}$$

$$\boxed{\gamma = \frac{-31}{20}}$$

$$\alpha = 2 - \beta - \gamma$$

$$= 2 + \frac{29}{20} + \frac{31}{20} = 5.$$

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$x(t) =$

$$5 - \frac{29}{20}e^{t} - \frac{31}{20}e^{-t} - \frac{1}{30}\sin 3t.$$

Final Answer

(b)

$$\dddot{x} + 3\ddot{x} + 4\dot{x} + 2x = \sin t.$$

$$x(0) = 1, \quad \dot{x}(0) = 0, \quad \ddot{x}(0) = -1.$$

- Hom Eq is.

$$\dddot{x} + 3\ddot{x} + 4\dot{x} + 2x = 0.$$

- char poly is.

$$\lambda^3 + 3\lambda^2 + 4\lambda + 2$$

$\lambda + 1$ is one of the factor.

Dividing:

$$\begin{array}{r|l} \lambda+1 & \lambda^3 + 3\lambda^2 + 4\lambda + 2 \;\big(\; \lambda^2 + 2\lambda + 2 \\ & \underline{\lambda^3 \pm \lambda^2} \\ & \quad 2\lambda^2 + 4\lambda + 2 \\ & \quad \underline{2\lambda^2 + 2\lambda} \\ & \qquad 2\lambda + 2 \\ & \qquad \underline{2\lambda + 2} \\ & \qquad\quad \times\times \end{array}$$

$$\lambda^2 + 2\lambda + 1 + 1$$
$$= (\lambda+1)^2 + 1$$
$$= (\lambda + 1 + i)(\lambda + 1 - i)$$

- char poly has roots at.

$$\lambda = -1,\ -1+i,\ -1-i$$

$\therefore$ $\phi_1(t) = e^{-1t} = e^{-t}$

$$\phi_2(t) = e^{-t}\cos t$$
$$\phi_3(t) = e^{-t}\sin t$$
$$x_h(t) = \alpha e^{-t} + \beta e^{-t}\cos t + \gamma e^{-t}\sin t.$$

- $f(t) = \sin t$

$$x_p(t) = A\sin t + B\cos t.$$

$$\dot{x}_p = A\cos t - B\sin t.$$
$$\ddot{x}_p = -A\sin t - B\cos t$$
$$\dddot{x}_p = -A\cos t + B\sin t.$$

Plugging back into the eqn we have

$$\begin{aligned}
&-A\cos t + B\sin t. \\
&-3B\cos t - 3A\sin t. \qquad = \sin t. \\
&4A\cos t - 4B\sin t. \\
&2B\cos t + 2A\sin t
\end{aligned}$$

$$\Rightarrow \quad 3A - B = 0 \qquad \boxed{B = 3A}$$

$$-3B - A = 1$$

$$\Rightarrow -9A - A = 1 \Rightarrow A = -\frac{1}{10}.$$

$$B = -\frac{3}{10}.$$

$$x_p(t) = -\frac{1}{10}\sin t - \frac{3}{10}\cos t$$

$x(t) =$

$$\alpha e^{-t} + \beta e^{-t}\cos t + \gamma e^{-t}\sin t$$
$$-\frac{1}{10}\sin t - \frac{3}{10}\cos t.$$

$$x(0) = \alpha + \beta - \frac{3}{10} = 1$$
$$\alpha + \beta = 1 + \frac{3}{10} = \frac{13}{10}$$

$$\boxed{\alpha + \beta = \frac{13}{10}}$$

$$\dot{x}(t) = -\alpha e^{-t} + \beta e^{-t}(-\sin t) + \beta \cos t(-e^{-t})$$
$$+ \gamma e^{-t}\cos t + \gamma \sin t(-e^{-t})$$
$$-\frac{1}{10}\cos t + \frac{3}{10}\sin t.$$

$$\dot{x}(t) =$$

$$-\alpha e^{-t} + e^{-t}\sin t(-\beta-\gamma)$$

$$+ e^{-t}\cos t(-\beta+\gamma)$$

$$-\frac{1}{10}\cos t + \frac{3}{10}\sin t.$$

$$= e^{-t}\left[-\alpha - (\beta+\gamma)\sin t - (\beta-\gamma)\cos t\right]$$

$$-\frac{1}{10}\cos t + \frac{3}{10}\sin t.$$

$$\dot{x}(0) = \left[-\alpha - (\beta-\gamma)\right] - \frac{1}{10} = 0$$

$$\boxed{-\alpha-\beta+\gamma = \frac{1}{10}}$$

$$\because \alpha+\beta = \frac{13}{10} \qquad -\frac{13}{10}+\gamma = \frac{1}{10}$$

$$\gamma = \frac{14}{10} = \frac{7}{5}$$

$$\therefore \boxed{\gamma = \frac{7}{5}} \quad \boxed{\alpha+\beta = \frac{13}{10}}$$

$$\ddot{x}(t) =$$

$$-e^{-t}\left[-\alpha - (\beta+\gamma)\sin t - (\beta-\gamma)\cos t\right]$$

$$+ e^{-t}\left[-(\beta+\gamma)\cos t + (\beta-\gamma)\sin t\right]$$

$$+ \frac{1}{10}\sin t + \frac{3}{10}\cos t.$$

$$\ddot{x}(0) = (-1)\underbrace{\left[-\alpha-\beta+\gamma\right]}_{=\frac{1}{10}}$$

$$+ 1\left[-(\beta+\gamma)\right] + \frac{3}{10} = -1$$

$$\Rightarrow -\frac{1}{10} - (\beta+\gamma) = -1 - \frac{3}{10}$$

$$= -\frac{13}{10}.$$

$$\Rightarrow \beta+\gamma = \frac{12}{10} = \frac{6}{5}.$$

Hence $\beta = \frac{6}{5} - \gamma = \frac{6}{5} - \frac{7}{5} = -\frac{1}{5}.$

$$\beta = -\frac{1}{5}, \quad \gamma = \frac{7}{5}.$$

$$\alpha = \frac{13}{10} - \beta = \frac{13}{10} + \frac{1}{5}.$$

$$= \frac{13+2}{10} = \frac{15}{10}$$

$$= \frac{7.5}{5}.$$

Hence

$$x(t) = \frac{15}{10} e^{-t} - \frac{1}{5} e^{-t} \cos t + \frac{7}{5} e^{-t} \sin t$$

$$- \frac{1}{10} \sin t - \frac{3}{10} \cos t.$$

Final Answer