H.W. 3

MATH 3350

Look at the practice problem on pages 1 to 8 and solve the two problems on page 9.

(I) We want to solve the following equation

$$\frac{d^3x}{dt^3} + 6\frac{d^2x}{dt^2} + 11\frac{dx}{dt} + 6x = f(t)$$

(a) Write down the associated homogeneous equation by setting $f(t) = 0$.

(b) Write down the characteristic polynomial.

(c) Factor the polynomial and find out the roots. (Hint: The roots are integers).

(d) Write down the three independent solutions of the homogeneous equation. Call them $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$.

(e) Hence write down

$$x_h(t) = \alpha\,\phi_1(t) + \beta\,\phi_2(t) + \gamma\,\phi_3(t)$$

where α, β, γ are arbitrary constants.

Let us now assume

$$f(t) = \sin t.$$

(f) Using a suitable table in the book write down the expression of $x_p(t)$ where

$$x_p(t) = A\,\psi_1(t) + B\,\psi_2(t).$$

A, B are constants. $\psi_1(t)$, $\psi_2(t)$ are in the book.

Find it.

(g) Calculate A, B by plugging $x_p(t)$ into the differential equation

$$\frac{d^3 x_p}{dt^3} + 6\frac{d^2 x_p}{dt^2} + 11\frac{dx_p}{dt} + 6x_p = \sin t.$$

(h) Write down the overall solution

$$x(t) = x_h(t) + x_p(t)$$

$$= \alpha\,\phi_1(t) + \beta\,\phi_2(t) + \gamma\,\phi_3(t) + A\,\psi_1(t) + B\,\psi_2(t).$$

We know $\phi_1, \phi_2, \phi_3, \psi_1, \psi_2$ and A, B

Need to find α, β, γ

i choose initial conditions

$$x(0) = 1$$
$$\dot{x}(0) = 2$$
$$\ddot{x}(0) = -1$$

and find α, β, γ from this data.

— x —

Ans

(a) $$\frac{d^3x}{dt^3} + 6\frac{d^2x}{dt^2} + 11\frac{dx}{dt} + 6x = 0$$

(b) $\lambda^3 + 6\lambda^2 + 11\lambda + 6$

(c) $(\lambda+1)(\lambda+2)(\lambda+3) = 0$

$\Rightarrow \lambda = -1, -2, -3.$

(d) $\phi_1(t) = e^{-t}, \quad \phi_2(t) = e^{-2t}, \quad \phi_3(t) = e^{-3t}$

(e) $x_h(t) = \alpha e^{-t} + \beta e^{-2t} + \gamma e^{-3t}$

(f) from the book:

$\psi_1(t) = \sin t$

$\psi_2(t) = \cos t$

$x_p(t) = A \sin t + B \cos t.$

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$$x_p(t) = A\sin t + B\cos t$$

$$\dot{x}_p(t) = -B\sin t + A\cos t$$

$$\ddot{x}_p(t) = -A\sin t - B\cos t$$

$$\dddot{x}_p(t) = B\sin t - A\cos t$$

Since

$$\dddot{x}_p + 6\ddot{x}_p + 11\dot{x}_p + 6x_p = \sin t$$

we have

$$[B - 6A - 11B + 6A]\sin t + [-A - 6B + 11A + 6B]\cos t = \sin t$$

$$\Rightarrow B - \cancel{6A} - 11B + \cancel{6A} = 1 \qquad B = -\frac{1}{10}$$

$$-A - \cancel{6B} + 11A + \cancel{6B} = 0 \qquad A = 0$$

$$\therefore \boxed{x_p(t) = -\frac{1}{10}\cos t}$$

(h)

$$x(t) = x_h(t) + x_p(t)$$

$$= \alpha e^{-t} + \beta e^{-2t} + \gamma e^{-3t} - \frac{1}{10}\cos t.$$

$$\dot{x}(t) = -\alpha e^{-t} - 2\beta e^{-2t} - 3\gamma e^{-3t} + \frac{1}{10}\sin t$$

$$\ddot{x}(t) = \alpha e^{-t} + 4\beta e^{-2t} + 9\gamma e^{-3t} + \frac{1}{10}\cos t.$$

$$\begin{aligned} x(0) &= \alpha + \beta + \gamma - \tfrac{1}{10} = 1 \\ \dot{x}(0) &= -\alpha - 2\beta - 3\gamma = 2 \\ \ddot{x}(0) &= \alpha + 4\beta + 9\gamma + \tfrac{1}{10} = -1 \end{aligned}$$

← Need to solve these equations for α, β, γ

Adding 1st and 2nd equation we get

$$-\beta - 2\gamma - \frac{1}{10} = 3$$

Subtracting 1st from 3rd we get

$$3\beta + 8\gamma + \frac{1}{5} = -2$$

$$\boxed{\begin{aligned} -\beta - 2\gamma &= \tfrac{31}{10} \\ 3\beta + 8\gamma &= -\tfrac{9}{5} \end{aligned}}$$

Need to solve for β and γ

$$\boxed{\begin{aligned} -3\beta - 6\gamma &= \frac{93}{10} \\ 3\beta + 8\gamma &= -\frac{18}{10} \end{aligned}}$$

Adding : $2\gamma = \frac{93-18}{10} = \frac{75}{10} = \frac{15}{2}$

$$\boxed{\gamma = \frac{15}{4}}$$

Plug the value of γ into

$$3\beta + 8\gamma = -\frac{9}{5}$$

$$\Rightarrow 3\beta + 30 = -\frac{9}{5}$$

$$\Rightarrow \beta + 10 = -\frac{3}{5}$$

$$\Rightarrow \beta = -10 - \frac{3}{5} = -\frac{50}{5} - \frac{3}{5} = -\frac{53}{5}.$$

$$\boxed{\beta = -\frac{53}{5}}$$

$$\gamma = \frac{75}{20}, \quad \beta = -\frac{212}{20}$$

Plugging γ, β into

$$\alpha + \beta + \gamma - \frac{1}{10} = 1$$

we get

$$\alpha - \frac{212}{20} + \frac{75}{20} - \frac{2}{20} = \frac{20}{20}$$

$$\alpha + \frac{-212+75-2-20}{20} = 0$$

$$-234 + 75$$

$$\alpha = \frac{234-75}{20} = \frac{159}{20}$$

$$\boxed{\alpha = \frac{159}{20}}$$

$$\boxed{X(t) = \frac{159}{20}e^{-t} - \frac{53}{5}e^{-2t} + \frac{15}{4}e^{-3t} - \frac{1}{10}\cos t}$$

final answer

Now solve the following two problems:

(a) $$\frac{d^3x}{dt^3} - \frac{dx}{dt} = \cos 3t$$

Initial conditions

$$x(0) = 2, \quad \dot{x}(0) = 0, \quad \ddot{x}(0) = -3$$

(b) $$\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 2x = \sin t$$

Initial conditions

$$x(0) = 1, \quad \dot{x}(0) = 0, \quad \ddot{x}(0) = -1$$

Hint: One of the root of the characteristic polynomial is $\lambda = -1$.