Math 3350

Final Exam

Due by 11:00 AM in my office

The exam is open book, open notes
but no consultation is allowed.

1. Solve the 1st order equation

$$\frac{dy}{dt} = y^2 - 9$$

$$y(0) = 6$$

$$\boxed{\text{Ans: } y(t) = 3\,\frac{3+e^{6t}}{3-e^{6t}}}$$

2. Solve the 1st order linear equation

$$\frac{dy(t)}{dt} + \sin t \; y(t) = \sin t$$

$$y(0) = 2e + 1$$

$$\boxed{\text{Ans: } y(t) = 1 + 2e^{\cos t}}$$

3. Solve

$$(1+3x^2y^2)\,dx + (1+2yx^3)\,dy = 0$$

$$y(0) = 1, \text{ i.e. } y(x=0) = 1$$

$$\boxed{\text{Ans: } x + x^3y^2 + y = 1}$$

4. Solve the 2^{nd} order equation

$$\frac{d^2 y(t)}{dt^2} + 10\frac{dy(t)}{dt} + 21\, y(t) = 42$$

$$y(0) = 2, \quad \dot{y}(0) = 4$$

$$\boxed{\text{Ans: } y(t) = 2 + e^{-3t} - e^{-7t}}$$

5. Show that the 2^{nd} order equation

$$\frac{d^2 y(t)}{dt^2} + 10\frac{dy(t)}{dt} + 21\, y(t) = f(t)$$

$$y(0) = \dot{y}(0) = 0$$

has the solution

$$y(t) = \frac{1}{4}e^{-3t}\int_0^t e^{3\tau} f(\tau)\,d\tau - \frac{1}{4}e^{-7t}\int_0^t e^{7\tau} f(\tau)\,d\tau$$

6. Solve the 2^{nd} order equation

$$\frac{d^2 y(t)}{dt^2} + 10\frac{dy(t)}{dt} + 21\, y(t) = 20\sin t + 10\cos t$$

$$y(0) = 0\,; \quad \dot{y}(0) = 5$$

$$\boxed{\text{Ans: } y(t) = e^{-3t} - e^{-7t} + \sin t}$$

7 Calculate

$$\mathcal{L}^{-1}\left[\frac{4s+10}{s^2+2s+10}\right]$$

Ans: $e^{-t}\left[2\sin 3t + 4\cos 3t\right]$

8 Calculate

$$\mathcal{L}\left[t e^{-3t} \sin 2t\right]$$

Ans: $\dfrac{4(s+3)}{\left[(s+3)^2+4\right]^2}$

9 Expand $f(x) = \begin{cases} \pi/2 + x & -\frac{\pi}{2} < x < 0 \\ \pi/2 - x & 0 < x < \frac{\pi}{2} \end{cases}$

in a Fourier series.

Ans: $\boxed{f(x) = \dfrac{\pi}{4} + \sum_{n=1}^{\infty} \dfrac{1-\cos n\pi}{\pi n^2} \cos 2nx}$

10 Let A be the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix}$$

a. Show that $\lambda = 1, \lambda = 2, \lambda = 3$ are the three eigenvalues of the matrix A.

b. Calculate an eigenvector corresponding to any one of the eigenvalues of A.

Ans: char. polynomial is $\det(\lambda I - A)$

$$= (\lambda - 1)(\lambda - 2)(\lambda - 3)$$

If λ is an eigenvalue then

$\begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \end{pmatrix}$ is an eigenvector