Midterm II
(makeup)
Answers.

1 (a) From book we know that

$$y_p(t) = Ae^{-3t} \qquad (1)$$

Substituting (1) in the equation we have

$$9Ae^{-3t} + 3(-3)Ae^{-3t} + 2Ae^{-3t} = e^{-3t}$$

$$\Rightarrow A = 1/2.$$

$$\therefore \boxed{y_p(t) = \frac{1}{2}e^{-3t}}$$

(b) characteristic polynomial is

$$\lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$$

Roots are at $\lambda = -1, \lambda = -2$.

$$\boxed{y_h(t) = ae^{-t} + be^{-2t}}$$

c $y(t) = y_h(t) + y_p(t)$

$$= a e^{-t} + b e^{-2t} + \frac{1}{2} e^{-3t}$$

$$y(0) = a + b + \frac{1}{2} = 0$$

$$\Rightarrow a + b = -\frac{1}{2}. \quad (2)$$

$$\dot{y}(t) = -a e^{-t} - 2b e^{-2t} - \frac{3}{2} e^{-3t}$$

$$\dot{y}(0) = -a - 2b - \frac{3}{2} = 0$$

$$\Rightarrow a + 2b = -\frac{3}{2} \quad (3)$$

from (2) and (3) we have

$$b = -\frac{3}{2} + \frac{1}{2} = -1$$

$$a = -\frac{1}{2} - b = -\frac{1}{2} + 1 = \frac{1}{2}.$$

$$\therefore \boxed{y(t) = \frac{1}{2} e^{-t} - e^{-2t} + \frac{1}{2} e^{-3t}}$$

d

$$\mathcal{L}[\ddot{y}+3\dot{y}+2y]$$

$$=[s^2+3s+2]\,Y(s)$$

when $y(0)=\dot{y}(0)=0$.

$$\mathcal{L}[e^{-3t}]=\frac{1}{s+3}.$$

$$\therefore\ Y(s)=\frac{1}{(s^2+3s+2)(s+3)}.$$

$$=\frac{1}{(s+1)(s+2)(s+3)}.$$

e

$$\frac{1}{(s+1)(s+2)(s+3)}=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{s+3}$$

$$A=\left.\frac{1}{(s+2)(s+3)}\right|_{s=-1}=\frac{1}{1\times 2}=\frac{1}{2}.$$

$$B=\left.\frac{1}{(s+1)(s+3)}\right|_{s=-2}=\frac{1}{(-1)(1)}=-1$$

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$$C = \left.\frac{1}{(s+1)(s+2)}\right|_{s=-3} = \frac{1}{(-2)(-1)}$$

$$= \frac{1}{2}.$$

$\therefore$ partial fraction is given by.

$$\underbrace{\frac{1/2}{s+1} - \frac{1}{s+2} + \frac{1/2}{s+3}}.$$

$$y(t) = \mathcal{L}^{-1}[\quad]$$

$$\boxed{y(t) = \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}}.$$

2 Ans:

$$s^2 + 6s + 34$$

$$= (s+3)^2 + 25$$

$$= (s+3)^2 + 5^2$$

$$2s+11 = 2s+6+5$$

$$= 2(s+3) + 5$$

$$\therefore \frac{2s+11}{s^2+6s+34} = \frac{2(s+3)+5}{(s+3)^2+5^2}.$$

$$= 2\,\frac{s+3}{(s+3)^2+5^2} + \frac{5}{(s+3)^2+5^2}$$

$$\mathcal{L}^{-1}\left[\frac{2s+11}{s^2+6s+34}\right] = 2\,\mathcal{L}^{-1}\left[\frac{s+3}{(s+3)^2+5^2}\right]$$

$$+\mathcal{L}^{-1}\left[\frac{5}{(s+3)^2+5^2}\right]$$

$$= 2e^{-3t}\cos 5t + e^{-3t}\sin 5t.$$

3 Ans:

$$\int_0^t e^{\lambda(t-\tau)} f(\tau)\,d\tau$$

$$= e^{\lambda t}\int_0^t e^{-\lambda\tau} f(\tau)\,d\tau . \qquad (4)$$

when $t < 10$, $f(\tau) = 2$

Hence

$$\int_0^t e^{-\lambda\tau} f(\tau)\,d\tau = \int_0^t e^{-\lambda\tau}\, 2\,d\tau .$$

$$= \left.\frac{2e^{-\lambda\tau}}{-\lambda}\right|_0^t = -\frac{2}{\lambda}\left[e^{-\lambda t} - 1\right]$$

$$= \frac{2}{\lambda}\left[1 - e^{-\lambda t}\right] \qquad (5)$$

$\therefore$ From (4) and (5) we have

$$\boxed{\int_0^t e^{\lambda(t-\tau)} f(\tau)\,d\tau = e^{\lambda t}\frac{2}{\lambda}\left[1 - e^{-\lambda t}\right] \quad \text{when } t < 10 .}$$

$$= \frac{2}{\lambda}\left[e^{\lambda t} - 1\right]$$

when $t > 10$ we write

$$\int_0^t e^{-\lambda \tau} f(\tau)\, d\tau =$$

$$\int_0^{10} e^{-\lambda \tau} f(\tau)\, d\tau + \int_{10}^{t} e^{-\lambda \tau} f(\tau)\, d\tau .$$

$$= \underbrace{\frac{2}{\lambda}\left(1 - e^{-10\lambda}\right)}_{\text{from (5)}} + \int_{10}^{t} e^{-\lambda \tau}\, 0 \, d\tau$$

because $f(\tau) = 0$ when $\tau > 10$.

$$= \frac{2}{\lambda}\left(1 - e^{-10\lambda}\right) \quad (6).$$

From (4) and (6) we obtain.

$$\boxed{\int_0^t e^{\lambda(t-\tau)} f(\tau)\, d\tau = e^{\lambda t} \frac{2}{\lambda}\left[1 - e^{-10\lambda}\right] \qquad t > 10}$$

(3) (b) Taking Laplace's Transform we obtain.

$$(s^2+3s+2)\,Y(s) = \mathcal{L}[f(t)] = F(s).$$

$$Y(s) = \frac{1}{s^2+3s+2}\cdot F(s).$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+3s+2}\right] = ??$$

$$\frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

$$A = \left.\frac{1}{s+2}\right|_{s=-1} = 1$$

$$B = \left.\frac{1}{s+1}\right|_{s=-2} = -1.$$

$$\therefore\ \mathcal{L}^{-1}\left[\frac{1}{s^2+3s+2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+1} - \frac{1}{s+2}\right]$$

$$= \underbrace{e^{-t} - e^{-2t}}_{=h(t).}$$

$$y(t) = h(t) * f(t).$$

$$= \left[e^{-t} - e^{-2t}\right] * f(t).$$

$$\boxed{y(t) = \int_0^t e^{-(t-\tau)} f(\tau)\, d\tau - \int_0^t e^{-2(t-\tau)} f(\tau)\, d\tau}$$