Solutions:

① char. polynomial is

$$\lambda^2 + 4 = 0$$

$$\lambda = \pm 2i$$

Homogeneous solution

$$y_h(t) = \alpha \sin 2t + \beta \cos 2t$$

Particular solution

$$y_p(t) = A \sin t + B \cos t$$

$$\dot{y}_p(t) = -B \sin t + A \cos t$$

$$\ddot{y}_p(t) = -A \sin t - B \cos t.$$

$$\ddot{y}_p + 4 y_p = 3A \sin t + 3B \cos t = \sin t$$

$$\Rightarrow A = \frac{1}{3}, \quad B = 0$$

Hence

$$y_p(t) = \frac{1}{3} \sin t$$

$$y(t) = \alpha \sin 2t + \beta \cos 2t + \frac{1}{3} \sin t$$

$$y(0) = \beta = 0$$

Hence

$$y(t) = \alpha \sin 2t + \frac{1}{3} \sin t$$

$$\dot{y}(t) = 2\alpha \cos 2t + \frac{1}{3} \cos t$$

$$\dot{y}(0) = 2\alpha + \frac{1}{3} = \frac{7}{3} \Rightarrow 2\alpha = 2 \Rightarrow \alpha = 1$$

$$\boxed{y(t) = \sin 2t + \frac{1}{3} \sin t}$$

(2) Substitute

$$u = \frac{1}{y^2}$$

$$\frac{du}{dt} = \frac{du}{dy} \cdot \frac{dy}{dt}$$

$$= -\frac{2}{y^3}\left[-y - \frac{1}{2} e^{-t} y^3\right]$$

$$= \frac{2}{y^2} + e^{-t} = 2u + e^{-t}$$

$$\frac{du}{dt} = 2u + e^{-t}$$ ← linear equation.

$$u_h(t) = \alpha e^{2t}$$

$$u_p(t) = A e^{-t}$$

To find A we write

$$\dot{u}_p - 2u_p = -A e^{-t} - 2Ae^{-t} = -3Ae^{-t} = e^{-t}$$

$$\Rightarrow \boxed{A = -\frac{1}{3}}$$

Thus $$\boxed{u(t) = \alpha e^{2t} - \frac{1}{3} e^{-t}}$$

$$y^2 = \frac{1}{u} = \frac{1}{\alpha e^{2t} - \frac{1}{3} e^{-t}}$$

To find α we look at the initial condition $y(0) = 1$.

$$1 = y(0)^2 = \frac{1}{\alpha - \frac{1}{3}}$$

$$\Rightarrow \alpha = \frac{4}{3}.$$

$$y^2 = \frac{1}{\frac{4}{3} e^{2t} - \frac{1}{3} e^{-t}}$$

$$y^2 = \frac{3}{4e^{2t} - e^{-t}}.$$

$$\boxed{y(t) = \sqrt{\frac{3}{4e^{2t} - e^{-t}}}.}$$

(3)

(a) $M = 3\cos 3x \quad , \quad M_y = 0$

$N = \sin 3x \quad , \quad N_x = 3\cos 3x$

$N_x \neq M_y$

Hence, not exact

(b) $M = \mu(y)\, 3\cos 3x, \quad M_y = \mu'(y)\, 3\cos 3x$

$N = \mu(y)\sin 3x \quad , \quad N_x = \mu(y)\, 3\cos 3x$

$N_x = M_y$

$\Rightarrow \mu'(y) = \mu(y)$

$\Rightarrow \frac{d\mu}{\mu} = dy \Rightarrow \mu(y) = e^y$

(c) $M = 3e^y \cos 3x$

$N = e^y \sin 3x$

$\frac{\partial f}{\partial x} = M = 3e^y \cos 3x$

$f(x,y) = e^y \sin 3x + g(y)$

$$\frac{\partial f}{\partial y} = N = e^{y} \sin 3x$$

But

$$\frac{\partial f}{\partial y} = e^{y} \sin 3x + g'(y)$$

It would follow that

$$g'(y) = 0$$

$$\Rightarrow g(y) = c$$

$$\therefore \boxed{f(x,y) = e^{y} \sin 3x + c}$$