$\boxed{\text{Math 3350}}$

Answers to Midterm III

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1 Ans:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_0^{\pi/2} dx + \int_{\pi/2}^{\pi} -dx\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} \cos nx\,dx + \int_{\pi/2}^{\pi} -\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin nx}{n}\Bigg|_0^{\pi/2} - \frac{\sin nx}{n}\Bigg|_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi n}\left[\left[\sin\frac{n\pi}{2}\right] - \left[\cancelto{0}{\sin n\pi} - \sin\frac{n\pi}{2}\right]\right]$$

$$= \frac{2}{n\pi}\left[\sin\frac{n\pi}{2}\right]$$

$$= \begin{cases} 0 & n \text{ even} \\ \dfrac{2}{n\pi} & n = 1, 5, 9, \dots \\ -\dfrac{2}{n\pi} & n = 3, 7, 11, \dots \end{cases}$$

If $n = 2m+1$

$$\therefore a_{2m+1} = \frac{2}{(2m+1)\pi}(-1)^m$$

—— x ——

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2}\sin nx\,dx + \int_{\pi/2}^{\pi} -\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[-\frac{\cos nx}{n}\right]_0^{\pi/2} - \left[-\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{n\pi}\left[\left(-\cos n\frac{\pi}{2} + 1\right) + \left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{n\pi}\left[1 + \cos n\pi\right]$$

$$= \frac{2}{n\pi} \quad n \text{ even}$$

$$0 \quad n \text{ odd.}$$

If $n = 2m$

$$b_{2m} = \frac{\cancel{2}}{\cancel{2}m\pi} = \frac{1}{m\pi} \qquad m = 1, 2, \ldots\ldots$$

$$\boxed{\begin{aligned} f(x) &= \\ &\sum_{m=0}^{\infty} \frac{2}{(2m+1)\pi}(-1)^m \cos(2m+1)x \\ &+ \sum_{m=1}^{\infty} \frac{1}{m\pi} \sin 2mx \end{aligned}}$$

↑

Required Fourier Series

Remark: The $f^{\underline{n}}$ $f(x)$ is neither even, nor odd.

2) Ans:

Let us define a function

$$u(x-a) = \begin{cases} 1 & x \geq a \\ 0 & 0 \leq x < a. \end{cases}$$

using the above definition

$$f(x) = u(x-10).$$

Remark:

$$\mathcal{L}(u(x)) = \frac{1}{s}$$

$$\mathcal{L}(u(x-a)) = \int_0^\infty e^{-s\tau} u(\tau-a)\, d\tau.$$

writing $\tau - a = \tau_1$

$$= \int_a^\infty e^{-s\tau} u(\tau-a)\, d\tau$$

$$= \int_0^\infty e^{-s(\tau_1+a)} u(\tau_1)\, d\tau_1$$

$$= e^{-sa} \int_0^\infty e^{-s\tau_1} u(\tau_1)\, d\tau_1$$

$$= e^{-sa} \frac{1}{s} = \frac{e^{-as}}{s}.$$

Hence $\mathcal{L}(f(x)) = \dfrac{e^{-10s}}{s}.$

— x —

We now proceed to solve the problem

$$\mathcal{L}(y(x)) = Y(s)$$

$$\mathcal{L}(y'(x)) = sY(s) - y(0) = sY(s)$$

$$\mathcal{L}(y''(x)) = s\mathcal{L}(y'(x)) - y'(0)$$

$$= s^2 Y(s) - 10.$$

It follows that

$$\mathcal{L}(y''(x)) + 10\,\mathcal{L}(y'(x)) + 21\,\mathcal{L}\,y(x) = \mathcal{L}(f(x))$$

$$\Rightarrow s^2 Y(s) - 10 + 10\,s.\,Y(s) + 21\,Y(s) = \frac{e^{-10s}}{s}.$$

$$\Rightarrow \left[s^2 + 10s + 21\right] Y(s) = \frac{e^{-10s}}{s} + 10.$$

$$\Rightarrow Y(s) = e^{-10s}\frac{1}{s(s^2+10s+21)} + \frac{10}{s^2+10s+21}.$$

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$y(x) =$

$$\mathcal{L}^{-1}\left[e^{-10s}\frac{1}{s(s^2+10s+21)}\right]$$

$$+\mathcal{L}^{-1}\left[\frac{10}{s^2+10s+21}\right]$$

— x —

$$\frac{10}{s^2+10s+21} = \frac{10}{(s+3)(s+7)} = \frac{A}{s+3} + \frac{B}{s+7}$$

$$\frac{10}{s+7} = A + \frac{s+3}{s+7}B \Rightarrow A = \frac{10}{7-3} = \frac{10}{4} = 5/2$$

$$\frac{10}{s+3} = \frac{s+7}{s+3}A + B$$

$$\Rightarrow B = -\frac{10}{4} = -\frac{5}{2}$$

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$$\frac{10}{s^2+10s+21} = \frac{5}{2}\left[\frac{1}{s+3} - \frac{1}{s+7}\right]$$

$$\mathcal{L}^{-1}\left(\frac{10}{s^2+10s+21}\right) = \frac{5}{2}\left[e^{-3x} - e^{-7x}\right]$$

---*---

$$\frac{1}{s(s^2+10s+21)} = \frac{1}{s(s+3)(s+7)} = \frac{A_1}{s} + \frac{B_1}{s+3} + \frac{C_1}{s+7}$$

$$A_1 = \frac{1}{21}$$

$$B_1 = \frac{1}{(-3)(4)} = -\frac{1}{12}$$

$$C_1 = \frac{1}{(-7)(-4)} = +\frac{1}{28}$$

$$\frac{1}{s(s^2+10s+21)} = \frac{1/21}{s} - \frac{1/12}{s+3} + \frac{1/28}{s+7}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+10s+21)}\right) =$$

$$\left[\frac{1}{21} - \frac{1}{12}e^{-3x} + \frac{1}{28}e^{-7x}\right]u(x).$$

$$\mathcal{L}^{-1}\left(e^{-10s}\frac{1}{s(s^2+10s+21)}\right) =$$

$$\left[\frac{1}{21} - \frac{1}{12}e^{-3(x-10)} + \frac{1}{28}e^{-7(x-10)}\right]u(x-10)$$

Remark: This is the delay theorem.

Alternatively we can write

$$y(x) = \frac{5}{2}\left[e^{-3x} - e^{-7x}\right]$$

for $0 \le x < 10$

because $u(x) = 1 \quad 0 \le x < 10$

$u(x-10) = 0 \quad 0 \le x < 10$

$$y(x) = \frac{5}{2}\left[e^{-3x} - e^{-7x}\right] + \left[\frac{1}{21} - \frac{1}{12}e^{-3(x-10)} + \frac{1}{28}e^{-7(x-10)}\right]$$

for $x \geq 10$.

because $u(x) = 1$ when $x \geq 10$.

$u(x-10) = 1$

Combining everything.

$$y(x) =$$

$$\frac{5}{2}\left[e^{-3x} - e^{-7x}\right] u(x)$$

$$+ \left[\frac{1}{21} - \frac{1}{12} e^{-3(x-10)} + \frac{1}{28} e^{-7(x-10)}\right] u(x-10).$$

$*$

Remark:

1. Note the power of the delay theorem.

2. * is a compact description of what is on page

3. Note the initial conditions are applied at $t=0$ only

you can verify that

$$\left.\begin{aligned} \operatorname{Lt}_{x \to 10^-} y(x) &= \operatorname{Lt}_{x \to 10^+} y(x) \\ \operatorname{Lt}_{x \to 10^-} y'(x) &= \operatorname{Lt}_{x \to 10^+} y'(x) \end{aligned}\right\} \text{is automatically satisfied}$$

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3 Ans:

$$\frac{3s+2}{s^2+4s+5} =$$

$$\frac{3s+2}{(s+2)^2+1^2}$$

$$= \frac{3(s+2)}{(s+2)^2+1^2} - 4\,\frac{1}{(s+2)^2+1^2}$$

$$\mathcal{L}^{-1}\left(\frac{3s+2}{s^2+4s+5}\right) =$$

$$3e^{-2t}\cos t - 4e^{-2t}\sin t$$

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(4)

$$\mathcal{L}(y(x)) = Y(s)$$

$$\mathcal{L}(y'(x)) = sY(s) - Y(0)$$

$$\mathcal{L}(y''(x)) = s^2 Y(s) - s\,Y(0) - Y'(0)$$

$$\therefore (s^2 + \alpha s + \beta)\,Y(s) - sY(0) - Y'(0) - \alpha\, y(0) = 0$$

$$\therefore Y(s) = \frac{Y(0)\,s + [\alpha\, Y(0) + Y'(0)]}{s^2 + \alpha s + \beta} \quad (*)$$

$$\underbrace{y(x) = A e^{-\sigma x} \cos\omega x + B e^{-\sigma x} \sin\omega x}$$

This would be the form of the Laplace inverse of (*).

comparing, we have

$\sigma = 5, \quad \omega = 7$

$$y(x) = Ae^{-5x}\cos 7x + Be^{-5x}\sin 7x.$$

$$Y(s) = \frac{as+b}{(s+5)^2 + 7^2}.$$

$$= \frac{as+b}{s^2+10s+25+49}.$$

$$= \frac{as+b}{s^2+10s+74}.$$

$\therefore \alpha = 10, \beta = 74.$