Math 3350

Sol$^{\underline{n}}$ to Midterm I

Solution to Midterm 1

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(i) $\dfrac{dy}{10-3y} = dx$

$\Rightarrow -\frac{1}{3}\ln|10-3y| = x + c$

$\Rightarrow \ln|10-3y| = -3x - 3c$

$\Rightarrow |10-3y| = e^{-3x} e^{-3c}$

$\Rightarrow 10-3y = \pm e^{-3c} e^{-3x}$

Define $K = \pm e^{-3c}$

$\Rightarrow 10-3y = K e^{-3x}$

$\Rightarrow 3y = 10 - K e^{-3x}$

$\Rightarrow y = \frac{10}{3} - \frac{K}{3} e^{-3x}$

Let $K_1 = -K/3$

$\Rightarrow y = \frac{10}{3} + K_1 e^{-3x}$ K_1 is an arbitrary constant.

$y(0) = 20/3$

$\Rightarrow \frac{20}{3} = \frac{10}{3} + K_1 \Rightarrow K_1 = \frac{10}{3}$

$\therefore y = \frac{10}{3}\left[1 + e^{-3x}\right]$

(ii) $dy = (-3x+10)\,dx$

$\Rightarrow y = -\frac{3}{2}x^2 + 10x + C$

$\because y(0) = 20/3 \Rightarrow 20/3 = C$

$\therefore y = -\frac{3}{2}x^2 + 10x + 20/3$

(iii) $\frac{dy}{10-3y} = (10-3x)\,dx$

$\Rightarrow -\frac{1}{3}\ln|10-3y| = 10x - \frac{3}{2}x^2 + C$

$\Rightarrow \ln|10-3y| = -30x + \frac{9}{2}x^2 - 3C$

$\Rightarrow |10-3y| = e^{\left[9/2\,x^2 - 30x\right]}e^{-3C}$

$\Rightarrow y = 10/3 + K\,e^{\left[9/2\,x^2 - 30x\right]}$, where $K = \frac{10}{3}$

2 Ans:

$$\frac{dy}{dx} + 3y = 4\sin 5x$$

$$P(x) = 3 \; ; \; Q(x) = 4\sin 5x$$

$$\mu(x) = e^{\int P\,dx} = e^{3x}$$

$$y(x) = \frac{1}{\mu}\left[\int \mu Q\,dx + C\right]$$

$$= e^{-3x}\left[\int 4e^{3x}\sin 5x\,dx + C\right]$$

$$\int e^{3x}\sin 5x\,dx = \frac{e^{3x}\left[3\sin 5x - 5\cos 5x\right]}{9+25}$$

$$= \frac{1}{34}e^{3x}\left[3\sin 5x - 5\cos 5x\right]$$

$$\therefore\; y(x) = e^{-3x}\left[\frac{4}{34}e^{3x}(3\sin 5x - 5\cos 5x) + C\right]$$

$$= \frac{\overset{2}{\cancel{4}}}{\underset{17}{\cancel{34}}}(3\sin 5x - 5\cos 5x) + Ce^{-3x}$$

$$= \frac{6}{17}\sin 5x - \frac{10}{17}\cos 5x + Ce^{-3x}$$

$$y(0) = -\frac{10}{17} + C = \frac{7}{17}$$

$$C = \frac{17}{17} = 1$$

$$\boxed{\therefore\ y(x) = \frac{6}{17}\sin 5x - \frac{10}{17}\cos 5x + e^{-3x}}$$

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3 Ans:

$$\frac{dy}{dx} = -\frac{x+\sin y}{x\cos y + y^3}$$

$$\Rightarrow -(x+\sin y)\,dx = (x\cos y + y^3)\,dy$$

$$\Rightarrow \underbrace{(x+\sin y)}_{M}\,dx + \underbrace{(x\cos y + y^3)}_{N}\,dy = 0$$

$$M(x,y) = x+\sin y = \partial F/\partial x$$
$$N(x,y) = x\cos y + y^3 = \partial F/\partial y.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$$ The above form is exact.

$$\frac{\partial M}{\partial y} = \cos y.$$

$$\frac{\partial N}{\partial x} = \cos y$$

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$$F = \int (x + \sin y)\,dx + \phi(y)$$

$$= \frac{x^2}{2} + x\sin y + \phi(y).$$

$$\frac{\partial F}{\partial y} = x\cos y + \phi'(y) = x\cos y + y^3$$

$$\phi(y) = y^4/4.$$

$$\boxed{F(x,y) = \frac{x^2}{2} + x\sin y + \frac{y^4}{4}.}$$

Sol^n $\frac{x^2}{2} + x\sin y + \frac{y^4}{4} = C$

$\because$ $y(0)=2$ we have

$x=0,\ y=2 \implies 4 = C$

$$\therefore \frac{x^2}{2} + x\sin y + \frac{y^4}{4} = 4$$

4 Ans:

$$P(x) = \frac{1}{x-2}$$

$$Q(x) = 5(x-2)$$

$$n = 1/2$$

Define

$$v = y^{1-1/2} = y^{1/2} = \sqrt{y}$$

$$\frac{dv}{dx} = \frac{1}{2} y^{-1/2} \frac{dy}{dx}$$

$$y^{-1/2} \frac{dy}{dx} + \frac{y^{1/2}}{x-2} = 5(x-2)$$

$$\Rightarrow 2\frac{dv}{dx} + \frac{1}{x-2} v = 5(x-2)$$

$$\Rightarrow \frac{dv}{dx} + \frac{1}{2(x-2)} v = \frac{5(x-2)}{2}$$

"Linear in v"

$$P(x) = \frac{1}{2(x-2)} \; ; \; Q(x) = \frac{5}{2}(x-2)$$

$$\mu(x) = e^{\int \frac{1}{2(x-2)} dx} = e^{\frac{1}{2} \ln|x-2|} = |(x-2)|^{1/2}$$

⑧

$$\therefore V(x) = \frac{1}{|(x-2)|^{1/2}}\left[\int |x-2|^{1/2}\,\frac{5}{2}(x-2)\,dx + C\right]$$

for $x > 2$ we have

$$V(x) = \frac{1}{(x-2)^{1/2}}\left[\frac{5}{2}\int (x-2)^{3/2}\,dx + C\right]$$

$$= \frac{5/2}{(x-2)^{1/2}}\,\frac{(x-2)^{5/2}}{5/2} + \frac{C}{(x-2)^{1/2}}$$

$$V(x) = (x-2)^2 + C(x-2)^{-1/2} \qquad x > 2$$

For $x < 2$

$$V(x) = \frac{1}{(2-x)^{1/2}}\left[\int (2-x)^{3/2}\left(-\frac{5}{2}\right)dx + C\right]$$

$$= \frac{1}{(2-x)^{1/2}}\left[-\frac{5}{2}\,\frac{(2-x)^{5/2}}{5/2}(-1) + C\right]$$

$$= \frac{1}{(2-x)^{1/2}}\left[(2-x)^{5/2} + C\right] = (2-x)^2 + C(2-x)^{-1/2}$$

$$\therefore V(x) = (x-2)^2 + C\Big/\sqrt{|(x-2)|}$$

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$$y(0) = \left(\cancel{8}4 + \frac{c}{\sqrt{2}} \right)^2 = 9$$

$$\cancel{c = 5\sqrt{2}} \quad c = -\sqrt{2}$$

$$y(x) = \left[(x-2)^2 - \frac{\sqrt{2}}{\sqrt{|x-2|}} \right]^2$$

$$y(x) = \left[(x-2)^2 + \frac{c}{\sqrt{|x-2|}} \right]^2$$

$$y(x) = \left[(x-2)^2 - \frac{\sqrt{2}}{\sqrt{|x-2|}} \right]^2$$