1

Home Work Solutions. 6

1 Ans.

a $$\frac{As+B}{s^2+\omega^2} = A\frac{s}{s^2+\omega^2} + \frac{B}{\omega}\frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}^{-1}\left(\frac{As+B}{s^2+\omega^2}\right) = A\,\mathcal{L}^{-1}\left(\frac{s}{s^2+\omega^2}\right) + \frac{B}{\omega}\mathcal{L}^{-1}\left(\frac{\omega}{s^2+\omega^2}\right)$$

$$= A\cos\omega t + \frac{B}{\omega}\sin\omega t.$$

$$= A\cos\omega t + B\frac{\sin\omega t}{\omega}.$$

b Note that

$$\sin\omega t \longleftrightarrow \frac{\omega}{s^2+\omega^2} \;;\; t\sin\omega t \longleftrightarrow \frac{2\omega s}{(s^2+\omega^2)^2}$$

$$\cos\omega t \longleftrightarrow \frac{s}{s^2+\omega^2} \;;\; t\cos\omega t \longleftrightarrow \frac{s^2-\omega^2}{(s^2+\omega^2)^2}.$$

It follows that

$$\mathcal{L}\left(a\sin\omega t + b\cos\omega t + c\,t\sin\omega t + d\,t\cos\omega t\right)$$

$$= \frac{\cancel{a}\; b s^3 + (a\omega + d)s^2 + (b\omega^2 + 2\omega c)s + a\omega^3 - d\omega^2}{(s^2+\omega^2)^2}$$

$$= \frac{Cs + D}{(s^2+\omega^2)^2}.$$

$$\Rightarrow b = 0,\ d = -a\omega,\ c = \frac{C}{2\omega},\ a = \frac{D}{2\omega^3}$$

$$\Rightarrow b = 0,\ a = \frac{D}{2\omega^3},\ c = \frac{C}{2\omega},\ d = -\frac{D}{2\omega^2}.$$

$$\therefore \mathcal{L}^{-1}\left(\frac{Cs + D}{(s^2+\omega^2)^2}\right) = \frac{D}{2\omega^3}\sin\omega t + \frac{C}{2\omega}t\sin\omega t - \frac{D}{2\omega^2}t\cos\omega t.$$

2 Ans:

(i). $$\frac{3s^3}{(s^2+16)^2} = \frac{As+B}{s^2+16} + \frac{Cs+D}{(s^2+16)^2}.$$

$$= \frac{(As+B)(s^2+16) + (Cs+D)}{(s^2+16)^2}$$

$$\Rightarrow (As+B)(s^2+16) + Cs+D = 3s^3$$

$$\Rightarrow As^3 + 16As + Bs^2 + 16B + (Cs+D) = 3s^3$$

$$\Rightarrow A=3, B=0, 16A+C=0, 16B+D=0$$

$$\Rightarrow A=3, B=0, C=-48, D=0.$$

$$\therefore \frac{3s^3}{(s^2+16)^2} = \frac{3s}{s^2+16} + \frac{-48s}{(s^2+16)^2}.$$

$$\mathcal{L}^{-1}\left(\frac{3s}{s^2+16}\right) = 3\cos 4t + \frac{0}{4}\sin 4t$$

$$= 3\cos 4t$$

$$\mathcal{L}^{-1}\frac{-48s}{(s^2+16)^2} = \frac{-48}{2\cdot 4} t\sin 4t$$

$$= -6t\sin 4t$$

$$\therefore \boxed{\mathcal{L}^{-1}\left(\frac{3s^3}{(s^2+16)^2}\right) = 3\cos 4t - 6t\sin 4t}$$

(ii) $$\frac{3s^3}{(s^2+16)^2(s+3)} = $$

$$\frac{As+B}{s^2+16} + \frac{Cs+D}{(s^2+16)^2} + \frac{E}{s+3}$$

$$E = \left.\frac{3s^3}{(s^2+16)^2}\right|_{s=-3} = \frac{3(-3)^3}{(16+9)^2} = \frac{-81}{625}$$

$$= \frac{(As+B)(s^2+16)(s+3) + (Cs+D)(s+3) + E(s^2+16)^2}{(s^2+16)^2(s+3)}$$

we have

$$(As+B)(s^3+3s^2+16s+48) + (Cs^2 + (3C+D)s + 3D$$
$$+ E(s^4 + 32s^2 + 256)$$
$$= 3s^3$$

$$(E+A)s^4 + s^3(3A+B)$$
$$+s^2(16A+3B+C+32E)$$
$$+s(48A+16B+3C+D) = 3s^3$$
$$+(48B+3D+256E)$$

$$E = -\frac{81}{625}, \quad A = -E = \frac{81}{625}.$$

$$B = 3-3A = 3 - \frac{3\cdot 81}{625} = 3 - \frac{243}{625}$$
$$= \frac{1632}{625}$$

$$C = -16A - 3B - 32E$$
$$= -\frac{16\cdot 81}{625} - 3\,\frac{1632}{625} + \frac{32\cdot 81}{625} = -\frac{3600}{625}$$

$$D = -48A - 16B - 3C.$$

$$= -\frac{48 \cdot 81}{625} - \frac{16 \cdot 1632}{625} + \frac{3 \cdot 3600}{625}$$

$$= -\frac{19200}{625}$$

$$\frac{3s^3}{(s^2+16)^2(s+3)} = \frac{1}{625}\left[\frac{81s + 1632}{s^2+16} - \frac{+3600s + 1920}{(s^2+16)^2} - \frac{81}{s+3}\right]$$

$$\mathcal{L}^{-1}\frac{3s^3}{(s^2+16)^2(s+3)} = .$$

8

$$\frac{1}{625}\Big[81\cos 4t + \frac{1632}{4}\sin 4t$$

$$- \frac{19200}{2\cdot 64}\sin 4t - \frac{3600}{8}t\sin 4t$$

$$+ \frac{19200}{2.16}t\cos 4t\Big]$$

$$- 81e^{-3t}$$

$$= \frac{1}{625}\Big[81\cos 4t + 408\sin 4t$$

$$- 150\sin 4t - 450t\sin 4t$$

$$+ 600t\cos 4t - 81e^{-3t}\Big]$$

$$= \frac{1}{625}\Big[81\cos 4t + 258\sin 4t - 450t\sin 4t$$
$$+ 600t\cos 4t - 81e^{-3t}\Big]$$

(iii) $\dfrac{3s^3}{\left[(s+1)^2+16\right]^2}$

$$= \frac{As+B}{(s+1)^2+16} + \frac{Cs+D}{\left((s+1)^2+16\right)^2}$$

$$= \frac{(As+B)\left[s^2+2s+17\right] + (Cs+D)}{\left[(s+1)^2+16\right]^2}$$

$$= \frac{As^3+2As^2+17As+Bs^2+2Bs+17B + (Cs+D)}{\left[(s+1)^2+16\right]^2}$$

$$As^3+(2A+B)s^2+(17A+2B+C)s + (17B+D) = 3s^3$$

$A = 3.$

$B = -2A = -6.$

$C = -17A - 2B$

$= -17 \cdot 3 - 2(-6)$

$= -51 + 6 = -45$

$D = -17B = 17 \cdot 6 = 102.$

$$\therefore \frac{3s^3}{[(s+1)^2+16]^2} = \frac{3s-6}{(s+1)^2+16} + \frac{-45s+102}{[(s+1)^2+16]^2}.$$

$$= \left[\frac{3(s+1)-9}{(s+1)^2+16}\right] - \left[\frac{45(s+1)-147}{[(s+1)^2+16]^2}\right]$$

$$\mathcal{L}^{-1}\left[\frac{3s-9}{s^2+16}\right] = 3\cos 4t - \frac{9}{4}\sin 4t.$$

$$\mathcal{L}^{-1}\frac{45s-147}{[(s)^2+16]^2} =$$

$$\frac{-147}{2.64}\sin 4t + \frac{45}{8}t\sin 4t$$

$$+\frac{147}{32}t\cos 4t.$$

$$= -\frac{147}{128}\sin 4t + \frac{45}{8}t\sin 4t$$

$$+\frac{147}{32}t\cos 4t$$

12

$$\mathcal{L}^{-1}\frac{3s^3}{[(s+1)^2+16]^2} = .$$

$$= \frac{147-9\cdot 32}{128}\sin 4t = -\frac{141}{128}\sin 4t$$

$$e^{-t}\Big[3\cos 4t - \frac{9}{4}\sin 4t + \frac{147}{128}\sin 4t$$
$$- \frac{45}{8}t\sin 4t - \frac{147}{32}t\cos 4t\Big]$$

$$= e^{-t}\Big[3\cos 4t - \frac{141}{128}\sin 4t - \frac{45}{8}t\sin 4t$$
$$- \frac{147}{32}t\cos 4t\Big]$$

iv $$\frac{3s^3}{\left[(s+1)^2+16\right]^2(s+3)} =$$

$$\frac{As+B}{(s+1)^2+16} + \frac{Cs+D}{\left[(s+1)^2+16\right]^2} + \frac{E}{s+3}$$

$$E = \left.\frac{3s^3}{\left[(s+1)^2+16\right]^2}\right|_{s=-3} = \frac{-81}{20 \times 20}$$

$$= -\frac{81}{400}$$

A, B, C, D can be computed by equating the co-efficients as was done before

We write

$$\frac{3s^3}{\left[(s+1)^2+16\right]^2(s+3)} =$$

$$\frac{A(s+1)+(B-A)}{(s+1)^2+16} + \frac{C(s+1)+(D-C)}{\left[(s+1)^2+16\right]^2}$$

$$+ \frac{E}{s+3}.$$

$$\mathcal{L}^{-1}\left(\quad \right) = e^{-t}\left[A\cos 4t + \frac{B-A}{4}\sin 4t\right]$$

$$+ e^{-t}\left[\frac{D-C}{128}\sin 4t + \frac{C}{8}t\sin 4t - \frac{D-C}{32}t\cos 4t\right]$$

$$+ E e^{-3t}.$$

3 Ans

a $\frac{dy}{dt} + 3y = te^{-t} \sin 4t$. (*)

$$\mathcal{L}(t \sin 4t) = \frac{8s}{(s^2+16)^2}$$

$$\mathcal{L}(te^{-t} \sin 4t) = \frac{8(s+1)}{[(s+1)^2+16]^2}.$$

Taking Laplace Transform of (*)

we get

$$sY(s) - y(0) + 3Y(s) = \mathcal{L}(te^{-t} \sin 4t)$$

$$= \frac{8(s+1)}{[(s+1)^2+16]^2}.$$

$\because y(0) = 0$ we have

$$Y(s) = \frac{8(s+1)}{\left[(s+1)^2+16\right]^2 (s+3)}$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{8(s+1)}{\left[(s+1)^2+16\right]^2 (s+3)} \right).$$

$$\frac{8(s+1)}{\left[(s+1)^2+16\right]^2 (s+3)} = \frac{As+B}{(s+1)^2+16} + \frac{Cs+D}{\left[(s+1)^2+16\right]^2} + \frac{E}{s+3}.$$

$$= \frac{(As+B)\left((s+1)^2+16\right)(s+3) + (Cs+D)(s+3) + E\left((s+1)^2+16\right)^2}{\left[(s+1)^2+16\right]^2 (s+3)}.$$

$\therefore$

$$(As+B)(s^2+2s+17)(s+3) +$$
$$(Cs+D)(s+3) + E\,(\underset{s^2}{s^2}+\underset{2s}{2s}+\underset{17}{17})^2$$
$$= 8s+8.$$

$\Rightarrow$

$$(As+B)(s^3+5s^2+23s+51) +$$
$$[Cs^2+(3C+D)s+3D] \quad = 8s+8$$
$$+ E\,[s^4+4s^3+38s^2+68s+289]$$

Comparing the coefficients of

s^4 we get $A+E=0$

s^3 we get $5A+B+4E=0$

s^2 we get $23A+5B+C+38E=0$

s we get $51A+23B+3C+D+68E=8$

1 we get $51B+3D+289E=8$

We also know that

$$E = \left. \frac{8(s+1)}{\left[(s+1)^2+16\right]^2} \right|_{s=-3}$$

$$= \frac{8 \times (-2)}{20 \times 20} = -\frac{16}{400}$$

$$= -\frac{4}{100} = -\frac{1}{25}$$

$$\therefore A + E = 0 \Rightarrow A = \frac{1}{25}$$

$$B = -5A - 4E = \frac{4}{25} - \frac{5}{25} = \frac{-1}{25} = -\frac{1}{25}$$

$$C = -23A - 5B - 38E$$

$$= \frac{38}{25} + \frac{5}{25} - \frac{23}{25} = \frac{\cancel{20}^{\,4}}{\cancel{25}_{\,5}} = \frac{4}{5} = \frac{20}{25}$$

$$D = 8 - 51A - 23B - 3C - 68E$$

$$= \frac{68}{25} + \frac{23}{25} - \frac{60}{25} - \frac{51}{25} + \frac{200}{25} = \frac{180}{25}$$

Hence

$$Y(s) = \frac{1}{25}\left[\frac{s-1}{(s+1)^2+16} + \frac{20s+180}{[(s+1)^2+16]^2} - \frac{1}{s+3}\right]$$

Taking the inverse Laplace Transform from problem 1, we obtain

$$\cancel{y(t) = \frac{1}{25}\Big[}$$

$$Y(s) = \frac{1}{25}\left[\frac{(s+1)-2}{(s+1)^2+16} + \frac{20(s+1)+160}{[(s+1)^2+16]^2} - \frac{1}{s+3}\right]$$

$$y(t) = \frac{e^{-t}}{25}\left[\cos 4t - \frac{2}{4}\sin 4t + \frac{160}{128}\sin 4t + \frac{20}{8}t\sin 4t - \frac{160}{32}t\cos wt\right]$$

$$- \frac{1}{25}e^{-3t}$$

3 b

$$y'' + 5y' + 6y = te^{-t}\sin 4t$$

Taking the Laplace's transform and assuming $y(0)=0$, $y'(0)=0$ we have

$$(s^2+5s+6)\,Y(s) = \mathcal{L}\left[te^{-t}\sin 4t\right]$$

$$= \frac{8(s+1)}{\left[(s+1)^2+16\right]^2}$$

$$Y(s) = \frac{8(s+1)}{\left[(s+1)^2+16\right]^2(s+3)(s+2)}$$

$$\boxed{y(t) = \mathcal{L}^{-1}\left[\frac{8(s+1)}{\left[(s+1)^2+16\right]^2(s+3)(s+2)}\right]}$$

Writing

$$\frac{8(s+1)}{((s+1)^2+16)^2(s+3)(s+2)} = \frac{As+B}{(s+1)^2+16} + \frac{Cs+D}{[(s+1)^2+16]^2} + \frac{E}{s+3} + \frac{F}{s+2}$$

We compute

$$E = \left.\frac{8(s+1)}{[(s+1)^2+16]^2(s+2)}\right|_{s=-3}$$

$$= \frac{8\times(-2)}{20\times 20\times(-1)} = \frac{16}{400} = \frac{1}{25} = \frac{289}{7225}$$

$$F = \left.\frac{8(s+1)}{[(s+1)^2+16]^2(s+3)}\right|_{s=-2}$$

$$= \frac{8(-1)}{17\times 17\times 1} = -\frac{8}{289} = -\frac{200}{7225}$$

Calculation Incomplete

4 Ans:

$$\frac{d^2y}{dt^2} + \omega_0^2 y = \sin \omega t$$

$$(s^2 + \omega_0^2)\, Y(s) = \frac{\omega}{s^2 + \omega^2}$$ ← Taking Laplace's Transform

$$\Rightarrow Y(s) = \frac{\omega}{(s^2 + \omega_0^2)(s^2 + \omega^2)}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{\omega}{(s^2 + \omega_0^2)(s^2 + \omega^2)}\right]$$

(a) $\omega \neq \omega_0$

$$\frac{\omega}{(s^2 + \omega_0^2)(s^2 + \omega^2)} = \frac{as + b}{s^2 + \omega_0^2} + \frac{cs + d}{s^2 + \omega^2}$$

$$= \frac{(as + b)(s^2 + \omega^2) + (cs + d)(s^2 + \omega_0^2)}{(s^2 + \omega_0^2)(s^2 + \omega^2)}$$

$$\left.\begin{array}{l} s^3(a+c) \\ +s^2(b+d) \\ +s(a\omega^2+c\omega_0^2) \\ +1(b\omega^2+d\omega_0^2) \end{array}\right. = \omega$$

$$\Rightarrow \left.\begin{array}{l} a+c=0 \\ b+d=0 \\ a\omega^2+c\omega_0^2=0 \\ b\omega^2+d\omega_0^2=\omega \end{array}\right\} \Rightarrow \begin{array}{l} a=-c \\ b=-d \\ a(\omega^2-\omega_0^2)=0 \\ b(\omega^2-\omega_0^2)=\omega \end{array}$$

since $\omega^2 \neq \omega_0^2$ we have $a=0$, $c=0$

$$b=\frac{\omega}{\omega^2-\omega_0^2}, \quad d=-\frac{\omega}{\omega^2-\omega_0^2}$$

$$\therefore \frac{\omega}{(s^2+\omega_0^2)(s^2+\omega^2)}=\frac{\omega}{\omega^2-\omega_0^2}\left[\frac{1}{s^2+\omega_0^2}-\frac{1}{s^2+\omega^2}\right]$$

$y(t) =$

$$\frac{\omega}{\omega^2 - \omega_0^2}\left[\frac{1}{\omega_0}\mathcal{L}^{-1}\frac{\omega_0}{s^2+\omega_0^2} - \frac{1}{\omega}\mathcal{L}^{-1}\frac{\omega}{s^2+\omega^2}\right]$$

$$= \frac{\omega}{\omega^2-\omega_0^2}\left[\frac{1}{\omega_0}\sin\omega_0 t - \frac{1}{\omega}\sin\omega t\right]$$

$$= \frac{1}{\omega^2-\omega_0^2}\left[\frac{\omega}{\omega_0}\sin\omega_0 t - \sin\omega t\right]$$

$$= \frac{\frac{\omega}{\omega_0}\sin\omega_0 t - \sin\omega t}{\omega^2 - \omega_0^2}$$

——— X ———

(b) $\omega = \omega_0$

$$y(t) = \mathcal{L}^{-1}\left[\frac{\omega_0}{(s^2+\omega_0^2)^2}\right]$$

problem 1.

$$= \cancel{\frac{\omega_0}{\omega_0}\sin\omega} = \frac{\omega_0}{2\omega_0^3}\sin\omega_0 t - \frac{\omega_0}{2\omega_0^2}t\cos\omega_0 t$$

$y(t) =$

$$\frac{1}{2\omega_0^2}\sin\omega_0 t - \frac{1}{2\omega_0} t\cos\omega_0 t .$$

Note that this $y(t)$ is not bounded

— x —

Remark:

If we take the $y(t)$ in part a and calculate the limit as $\omega \to \omega_0$ ie

$$\underset{\omega\to\omega_0}{Lt} \frac{\frac{\omega}{\omega_0}\sin\omega_0 t - \sin\omega t}{\omega^2 - \omega_0^2}$$

$$= \frac{\underset{\omega\to\omega_0}{Lt} \frac{d}{d\omega}\left[\frac{\omega}{\omega_0}\sin\omega_0 t - \sin\omega t\right]}{\underset{\omega\to\omega_0}{Lt} \frac{d}{d\omega}\left[\omega^2 - \omega_0^2\right]}$$

This is the L Hopital's rule

$$= \frac{\frac{\sin\omega_0 t}{\omega_0} - t\cos\omega_0 t}{2\omega_0}$$

This is the answer in part b.

5 Ans:

(i) $$f * g = \int_0^t f(t-\tau)\, g(\tau)\, d\tau \quad (*)$$

$$g * f = \int_0^t g(t-\tau)\, f(\tau)\, d\tau \quad (**)$$

writing $t - \tau = s$ where t is treated as a constant

we have

$$-d\tau = ds.$$

$$\tau = t - s.$$

$$\therefore f * g = \int_t^0 f(s)\, g(t-s).(-ds)$$

$$= \int_0^t f(s)\, g(t-s)\, ds.$$

$$= g * f.$$

(ii) $e^{3t} * e^{4t} =$

(a)

$$\int_0^t e^{3(t-\tau)} e^{4\tau} \, d\tau.$$

$$= e^{3t} \int_0^t e^{\tau} \, d\tau.$$

$$= e^{3t}\left(e^{t} - 1\right)$$

$$= e^{4t} - e^{3t}.$$

(b) $e^{3t} * \sin 5t.$

$$= \int_0^t e^{3(t-\tau)} \sin 5\tau \, d\tau.$$

$$= e^{3t} \int_0^t e^{-3\tau} \sin 5\tau \, d\tau$$

$$\int e^{a\tau} \sin b\tau \, d\tau$$

$$= -e^{a\tau} \frac{\cos b\tau}{b} + \int a e^{a\tau} \frac{\cos b\tau}{b} \, d\tau .$$

$$= -\frac{1}{b} e^{a\tau} \cos b\tau + \frac{a}{b} \int e^{a\tau} \cos b\tau \, d\tau .$$

$$\int e^{a\tau} \cos b\tau \, d\tau$$

$$= e^{a\tau} \frac{\sin b\tau}{b} - \int a e^{a\tau} \frac{\sin b\tau}{b} \, d\tau .$$

$$= \frac{1}{b} e^{a\tau} \sin b\tau - \frac{a}{b} \int e^{a\tau} \sin b\tau \, d\tau .$$

Thus we have

$$\int e^{a\tau} \sin b\tau \, d\tau = .$$

$$-\frac{1}{b} e^{a\tau} \cos b\tau + \frac{a}{b^2} e^{a\tau} \sin b\tau - \frac{a^2}{b^2} \int e^{a\tau} \sin b\tau \, d\tau .$$

$$\left(1+\frac{a^2}{b^2}\right)\int e^{a\tau}\sin b\tau\, d\tau = .$$

$$= \frac{a e^{a\tau}\sin b\tau - b e^{a\tau}\cos b\tau}{b^2}$$

$$\Rightarrow \int_0^t e^{a\tau}\sin b\tau\, d\tau =$$

$$\left. e^{a\tau}\frac{\left[a\sin b\tau - b\cos b\tau\right]}{a^2+b^2}\right|_0^t$$

$$= e^{at}\frac{\left[a\sin bt - b\cos bt\right]}{a^2+b^2}$$

$$+ \frac{b}{a^2+b^2}$$

$a = -3, \; b = 5$

$$\int_0^t e^{-3\tau} \sin 5\tau \, d\tau$$

$$= \frac{e^{-3t}\left[-3\sin 5t - 5\cos 5t\right]}{34} + \frac{5}{34}.$$

$\therefore \; e^{3t} * \sin 5t =$

$$\frac{5}{34} e^{3t} - \frac{3\sin 5t + 5\cos 5t}{34}.$$

5 Aus.

c $e^{5t} * t^2$

$$= \int_0^t e^{5(t-\tau)} \tau^2 \, d\tau .$$

$$= e^{5t} \int_0^t e^{-5\tau} \tau^2 \, d\tau$$

$$\int \tau^2 e^{-5\tau} \, d\tau =$$

$$-\tau^2 \frac{e^{-5\tau}}{5} + \int 2\tau \frac{e^{-5\tau}}{5} \, d\tau$$

$$= -\frac{1}{5} \tau^2 e^{-5\tau} + \frac{2}{5} \int \tau e^{-5\tau} \, d\tau .$$

$$\int \tau e^{-5\tau} \, d\tau = -\tau \frac{e^{-5\tau}}{5} + \int \frac{e^{-5\tau}}{5} \, d\tau$$

$$= -\frac{1}{5} \tau e^{-5\tau} - \frac{1}{5} \frac{e^{-5\tau}}{5}$$

$$\therefore\quad e^{5t} * t^2 =$$

$$e^{5t}\int_0^t \tau^2 e^{-5\tau}\,d\tau\,.$$

$$\int_0^t \tau^2 e^{-5\tau}\,d\tau =$$

$$\left[-\frac{1}{5}\tau^2 e^{-5\tau} - \frac{2}{5}\frac{1}{5}\tau e^{-5\tau} - \frac{2}{5}\frac{1}{5}\frac{1}{5}e^{-5\tau}\right]_0^t$$

$$= \left(-\frac{1}{5}e^{-5\tau}\left[\tau^2 + \frac{2}{5}\tau + \frac{2}{25}\right]\right)\Bigg|_0^t$$

$$= -\frac{1}{5}e^{-5t}\left(t^2 + \frac{2}{5}t + \frac{2}{25}\right)$$

$$+ \frac{1}{5}\,\frac{2}{25}$$

It follows that

$$e^{5t} * t^2 = -\frac{1}{5}\left(t^2 + \frac{2}{5}t + \frac{2}{25}\right) + \frac{2}{125}e^{5t}$$

(d) $t * \sin t$

$$= \int_0^t (t-\tau)\sin\tau \, d\tau .$$

$$= \int_0^t \left[t\sin\tau - \tau\sin\tau \right] d\tau .$$

$$= t\int_0^t \sin\tau \, d\tau - \int_0^t \tau\sin\tau \, d\tau .$$

$$\int_0^t \sin\tau \, d\tau = -\cos\tau \Big|_0^t = -\cos t + 1$$

$$= 1-\cos t$$

$$\int_0^t \tau\sin\tau \, d\tau =$$

$$-\tau\cos\tau \Big|_0^t + \int_0^t \cos\tau \, d\tau$$

$$= \left[t\cos t + 0 \right] + \sin\tau \Big|_0^t$$

$$= -t\cos t + \sin t$$

Hence $t * \sin t = t(1-\cos t) + t\cos t - \sin t .$

$$= t - \cancel{t\cos t} + \cancel{t\cos t} - \sin t .$$

$$= t - \sin t .$$

iii

$$e^{3t} \longleftrightarrow \frac{1}{s-3}$$

$$e^{4t} \longleftrightarrow \frac{1}{s-4}$$

$$e^{4t} - e^{3t} \longleftrightarrow \frac{1}{s-4} - \frac{1}{s-3}.$$

$$= \frac{s-3-s+4}{(s-4)(s-3)}$$

$$e^{3t} * e^{4t}. \qquad = \frac{1}{(s-4)(s-3)}$$

$$\therefore \mathcal{L}\left(e^{3t} * e^{4t}\right) = \mathcal{L}(e^{3t}) \times \mathcal{L}\left(e^{4t}\right)$$

(ii)

$$e^{3t} \longleftrightarrow \frac{1}{s-3}$$

$$\sin 5t \longleftrightarrow \frac{5}{\omega^2+25}$$

$$\mathcal{L}\left(e^{3t} * \sin 5t\right) =$$

$$\mathcal{L}\left(\frac{5}{34}e^{3t} - \frac{3}{34}\sin 5t - \frac{5}{34}\cos 5t\right)$$

$$= \frac{5}{34}\,\frac{1}{s-3} - \frac{3}{34}\,\frac{5}{s^2+25} - \frac{5}{34}\,\frac{s}{s^2+25}$$

$$= \frac{5s^2+125-15(s-3)-5s(s-3)}{34(s-3)(s^2+25)}.$$

$$= \frac{\cancel{5s^2}+125-\cancel{15s}+45-\cancel{5s^2}+\cancel{15s}}{34(s-3)(s^2+25)}$$

$$= \frac{170/34}{(s-3)(s^2+25)} = \mathcal{L}(e^{3t}) \times \mathcal{L}(\sin 5t).$$