Solutions to H.W. 3

1 Ans:

(i) $\frac{\partial F}{\partial x} = 3x^2 + y \Rightarrow F = x^3 + xy + g(y)$

$\frac{\partial F}{\partial y} = x^2 y - x \underset{??}{=} x + g'(y)$

Not exact.

$M = 3x^2 + y \qquad \frac{\partial M}{\partial y} = 1$

$N = x^2 y - x \qquad \frac{\partial N}{\partial x} = 2xy - 1$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1 - 2xy + 1}{x(xy-1)} = \frac{-2(xy-1)}{x(xy-1)}$$

$$= -\frac{2}{x}$$

$$\mu(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|} = \frac{1}{x^2}$$

$$\frac{1}{x^2}(3x^2+y)\,dx + \frac{1}{x^2}(x^2y - x)\,dy = 0$$

$$\Rightarrow \left(3 + \frac{y}{x^2}\right)dx + \left(y - \frac{1}{x}\right)dy = 0$$

$$\frac{\partial F}{\partial x} = 3 + \frac{y}{x^2} \qquad F = 3x - \frac{y}{x} + g(y)$$

$$\frac{\partial F}{\partial y} = y - \frac{1}{x} = -\frac{1}{x} + g'(y)$$

$$\Rightarrow g(y) = y^2/2 + c$$

$$\therefore\ F = 3x - \frac{y}{x} + \frac{y^2}{2} + c$$

$$\therefore\ \text{Sol}^n:\ \boxed{3x - \frac{y}{x} + \frac{y^2}{2} = \text{const}}$$

(ii) $M = 2xy^3 + 1 \qquad \frac{\partial M}{\partial y} = 2x\,3y^2 = 6xy^2$

$N = 3x^2y^2 - \frac{1}{y} \qquad \frac{\partial N}{\partial x} = 3y^2\,2x = 6xy^2$

Eq^n is exact.

$$\frac{\partial F}{\partial x} = 2xy^3+1 \qquad F = \cancel{2}y^3\frac{x^2}{\cancel{2}} + x + g(y).$$

$$F(x,y) = x^2y^3 + x + g(y).$$

$$\frac{\partial F}{\partial y} = x^2 3y^2 + g'(y) = 3x^2y^2 - \frac{1}{y}.$$

$$g'(y) = -\frac{1}{y} \Rightarrow g(y) = -\ln|y| + C$$

$$F(x,y) = x^2y^3 + x - \ln|y| + C$$

Sol$^{\underline{n}}$: $x^2y^3 + x - \ln|y| = \text{const}.$

(iii) $M = 2y^2x - y \qquad \frac{\partial M}{\partial y} = 2x\,2y - 1 = 4xy - 1$

$N = x \qquad \frac{\partial N}{\partial x} = 1 \qquad$ Not exact.

$$\frac{-\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}}{M} = \frac{2(1-2xy)}{y(2xy-1)} = -\frac{2}{y}.$$

$$\mu(y) = e^{\int -\frac{2}{y}dy} = e^{-2\ln|y|} = \frac{1}{y^2}$$

④

$$\frac{1}{y^2}(2y^2x - y)\,dx + \frac{1}{y^2}x\,dy = 0$$

$$\left(2x - \frac{1}{y}\right)dx + \frac{x}{y^2}\,dy = 0$$

$$\frac{\partial F}{\partial x} = 2x - \frac{1}{y} \qquad F = x^2 - \frac{x}{y} + g(y)$$

$$\frac{\partial F}{\partial y} = \frac{-x(-1)}{y^2} + g'(y) = \frac{x}{y^2} + g'(y) = \frac{x}{y^2}.$$

$$g'(y) = 0 \qquad g(y) = C.$$

$$\therefore F = x^2 - \frac{x}{y} + C$$

Sol$^{\underline{n}}$ $\quad x^2 = \frac{x}{y} + \text{const}.$

② Ans:-

$$M = 12x^m y^n + 5x^{m+1} y^{n+1}$$

$$N = 6x^{m+1} y^{n-1} + 3x^{m+2} y^n$$

$$\frac{\partial M}{\partial y} = 12x^m n y^{n-1} + 5x^{m+1}(n+1)y^n$$

$$= 12n x^m y^{n-1} + 5(n+1)x^{m+1} y^n$$

$$\frac{\partial N}{\partial x} = 6y^{n-1}(m+1)x^m + 3y^n(m+2)x^{m+1}$$

$$= 6(m+1)x^m y^{n-1} + 3(m+2)x^{m+1} y^n$$

$$\therefore\ 12n = 6(m+1) \Rightarrow \boxed{2n = m+1} \Rightarrow 10n = 5m+5$$

$$5(n+1) = 3(m+2) \Rightarrow 5n+5 = 3m+6$$

$$\Rightarrow \boxed{5n = 3m+1} \Rightarrow 10n = 6m+2$$

$$6m+2 = 5m+5$$

$$\boxed{m = +3} \quad \boxed{n = 2}$$

$$\therefore\ IF = x^3 y^2$$

$$(12x^3y^2 + 5x^4y^3)\,dx + (6x^4y + 3x^5y^2)\,dy = 0$$

$$\frac{\partial F}{\partial x} = 12x^3y^2 + 5x^4y^3 \Rightarrow F = \cancel{12}^{3} y^2 \frac{x^4}{\cancel{4}} + \cancel{5} y^3 \frac{x^5}{\cancel{5}} + g(y)$$

$$= 3x^4y^2 + x^5y^3 + g(y)$$

$$\frac{\partial F}{\partial y} = 6x^4y + 3x^5y^2 + g'(y) = 6x^4y + 3x^5y^2 \Rightarrow g(y) = C.$$

$F = \cancel{30}\ 3x^4y^2 + x^5y^3 + C$

$\therefore$ soln:

$3x^4y^2 + x^5y^3 = \text{const}.$

$x^4y^2(3 + xy) = \text{const}$

(3) $\dfrac{dy}{dx} - \dfrac{2}{x}y = -x^2y^2$

$P(x) = -\dfrac{2}{x}$; $Q(x) = -x^2$. $n = 2$.

$V = y^{1-2} = y^{-1} = \dfrac{1}{y}$.

$\dfrac{dv}{dx} = -\dfrac{1}{y^2}\dfrac{dy}{dx}$.

$-\dfrac{1}{y^2}\dfrac{dy}{dx} + \dfrac{2}{x}\dfrac{1}{y} = +x^2$

$\dfrac{dv}{dx} + \dfrac{2}{x}v = x^2$ ← Linear

$P(x) = \dfrac{2}{x}$; $Q(x) = x^2$.

$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = x^2$

$$v(x) = \frac{1}{x^2}\left[\int x^4\,dx + c\right]$$

$$= \frac{1}{x^2}\left[\frac{x^5}{5} + c\right]$$

$$= \frac{x^3}{5} + cx^{-2} = \frac{1}{5}x^3 + cx^{-2}.$$

$$y(x) = \frac{1}{\frac{1}{5}x^3 + cx^{-2}} = \frac{x^2}{\frac{1}{5}x^5 + c}$$

(5) (i) $\frac{dy}{dx} = -xy \Rightarrow \int \frac{dy}{y} = \int -x\,dx$

$$\ln|y| = -\frac{x^2}{2} + c.$$

$$|y| = ke^{-x^2/2}.$$

$$y = ke^{-x^2/2}$$

$$y = -\sqrt{2x^2\ln|x| + 16x^2}$$

$$y = -\sqrt{2x^2\ln|x| + cx^2}$$

$$-4 = -\sqrt{c} \qquad c = 16.$$

(ii) $\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}$

Let $v = \frac{\cancel{y}}{x} \Rightarrow \frac{dy}{dx} = v + \frac{1}{v}.$

$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} = v + \frac{1}{v}.$

$\therefore x\frac{dv}{dx} = \frac{1}{v} \Rightarrow \int v\,dv = \int \frac{1}{x}\,dx \Rightarrow \frac{v^2}{2} = \ln|x| + c$

$v^2 = 2\ln|x| + c. \Rightarrow y^2/x^2 = 2\ln|x| + c$

$\Rightarrow y^2 = 2x^2\ln|x| + cx^2$