Linear combination and span

I start with the following vectors in $\mathbb{R}^3$

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

Linear combination of vectors v_1, v_2, v_3 is another vector obtained by writing

$$2v_1 + 3v_2 + 5v_3 = \begin{pmatrix} 21 \\ 34 \\ 49 \end{pmatrix}$$

$$7v_1 + 4v_2 - 6v_3 = \begin{pmatrix} 7 \\ 16 \\ 49 \end{pmatrix}$$

$$-1v_1 + 2v_2 - 8v_3 = \begin{pmatrix} -11 \\ -16 \\ -13 \end{pmatrix}$$

All these vectors are linear combinations of v_1, v_2, v_3.

Work out:

$$3v_1 - 2v_2 + 4v_3 = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

$$7v_1 - 8v_2 - 5v_3 = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

Set of all possible linear combinations of vectors v_1, v_2, v_3 is called the span of v_1, v_2, v_3.

It is written as

$$\text{span}(v_1, v_2, v_3).$$

If α, β, γ are three scalars, then

$$\alpha v_1 + \beta v_2 + \gamma v_3 =$$

$$\alpha \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha \\ 2\alpha \\ 5\alpha \end{pmatrix} + \begin{pmatrix} 3\beta \\ 5\beta \\ 8\beta \end{pmatrix} + \begin{pmatrix} 2\gamma \\ 3\gamma \\ 3\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \alpha + 3\beta + 2\gamma \\ 2\alpha + 5\beta + 3\gamma \\ 5\alpha + 8\beta + 3\gamma \end{pmatrix}$$

$$\text{Span}(v_1, v_2, v_3) = \left\{ \begin{pmatrix} \alpha + 3\beta + 2\gamma \\ 2\alpha + 5\beta + 3\gamma \\ 5\alpha + 8\beta + 3\gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\}$$

This is how we would describe the span.

Note that

$$\begin{pmatrix} 21 \\ 34 \\ 49 \end{pmatrix} \in \operatorname{span}(v_1, v_2, v_3)$$

$$\begin{pmatrix} 7 \\ 16 \\ 49 \end{pmatrix} \in \operatorname{span}(v_1, v_2, v_3)$$

Verify that

$$\underbrace{\begin{pmatrix} 21 \\ 34 \\ 49 \end{pmatrix}}_{2v_1+3v_2+5v_3} + \underbrace{\begin{pmatrix} 7 \\ 16 \\ 49 \end{pmatrix}}_{7v_1+4v_2-6v_3} = 9v_1 + 7v_2 - v_3 \in \operatorname{span}(v_1, v_2, v_3).$$

Sum of two vectors in the span also belongs to the span.

Also note that

$$3 \cdot \begin{pmatrix} 21 \\ 34 \\ 49 \end{pmatrix} = 6v_1 + 9v_2 + 15v_3 \in \text{span}(v_1, v_2, v_3).$$

$$\begin{pmatrix} 21 \\ 34 \\ 49 \end{pmatrix} = 2v_1 + 3v_2 + 5v_3$$

scalar multiple of a vector in the span also belongs to the span.

"span (v_1, v_2, v_3) is closed under addition and scalar multiplication"

$\text{span}(v_1, v_2, v_3) \subset \mathbb{R}^3$, which is already a vector space.

subset

Hence span (v_1, v_2, v_3) is a vector subspace of $\mathbb{R}^3$

$\blacksquare$

II Null space of a matrix

Start with the vectors v_1, v_2, v_3 in I and construct a matrix

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 3 \\ 5 & 8 & 3 \end{pmatrix}$$

Null space of A is defined as follows:

$$\mathcal{N}(A) = \{ v \in \mathbb{R}^3 : Av = 0 \}.$$

" They are all vectors v in $\mathbb{R}^3$ such that $Av = 0$!"

If we write

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \text{ we have}$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 3 \\ 5 & 8 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (*)$$

To solve $(*)$ we would write the augmented matrix

$$\begin{pmatrix} 1 & 3 & 2 & 0 \\ 2 & 5 & 3 & 0 \\ 5 & 8 & 3 & 0 \end{pmatrix}$$

and row reduce it to get.

$$\begin{pmatrix} 1 & 3 & 2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -7 & -7 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -7 & -7 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The equation (*) reduces to

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} x - z = 0 \\ y + z = 0 \end{array} \Rightarrow \begin{array}{l} x = z \\ y = -z \end{array}$$

If $z = \alpha$, $x = \alpha$, $y = -\alpha$.

$$\mathcal{N}(A) = \left\{ \begin{pmatrix} \alpha \\ -\alpha \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

Hence
$\mathcal{N}(A)$ is the span of a single vector v where

$$v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\mathcal{N}(A) = \text{span}(v).$$

If v_1 and v_2 are two vectors in the null space of any matrix A then

$$Av_1 = 0 \ , \ Av_2 = 0 .$$

It would follow that

$$A(v_1 + v_2) = Av_1 + Av_2 = 0 + 0 = 0.$$

and

$$A(\alpha v_1) = \alpha(Av_1) = \alpha . 0 = 0 .$$

If two vectors belong to the null space of A, their sum also belongs to the null space of A.

If a vector is in the null space of A; any of its scalar multiple is also in the null space.

$\mathcal{N}(A)$ is a vector subspace of $\mathbb{R}^n$ where n is the # of columns of A.

III) Linear independence of a set of vectors.

Let v_1, v_2, v_3 be a set of vectors in a vector space V.

α, β, γ are three scalars.

$\alpha v_1 + \beta v_2 + \gamma v_3$ is a linear combination.

(a) If $\alpha = \beta = \gamma = 0$ then

$0 v_1 + 0 v_2 + 0 v_3$ is called a trivial linear combination.

(b) If either α or β or γ is nonzero.

then $\alpha v_1 + \beta v_2 + \gamma v_3$ is called a non trivial linear combination.

Def: A set of vectors $v_1, v_2, \ldots, v_n$ is said to be linearly independent if the only linear combination which yields the zero vector is the trivial linear combination.

Note that the trivial linear combination of $v_1, v_2, \ldots, v_n$ is

$$0v_1 + 0v_2 + \cdots + 0v_n = 0 \text{ the zero vector}$$

If there is one non-trivial linear combination of $v_1, \ldots, v_n$
which yields the zero vector, then the vectors $v_1, \ldots, v_n$ would be called linearly dependent.

Ex: Let

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Q: Is v_1, v_2 linearly independent.

To answer this question write down $\alpha v_1 + \beta v_2$ and set it equal to the zero vector.

$\alpha v_1 + \beta v_2 =$

$$\alpha\begin{pmatrix}1\\1\\1\end{pmatrix} + \beta\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$= \begin{pmatrix}\alpha\\\alpha\\\alpha\end{pmatrix} + \begin{pmatrix}\beta\\2\beta\\3\beta\end{pmatrix}$$

$$= \begin{pmatrix}\alpha+\beta\\\alpha+2\beta\\\alpha+3\beta\end{pmatrix}$$

$\alpha v_1 + \beta v_2 = 0$

$$\Rightarrow \left.\begin{matrix}\alpha+\beta=0\\ \alpha+2\beta=0\end{matrix}\right\} \Rightarrow \beta = 0.$$

$$\alpha+3\beta=0 \Rightarrow \alpha=0.$$

Hence $\alpha v_1 + \beta v_2 = 0 \Rightarrow \alpha = 0$ & $\beta = 0$.

Hence the vectors are linearly independent.

Ex:

Are the vectors.

$$\underbrace{\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}}^{=v_1}, \quad \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}^{=v_2}, \quad \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}^{=v_3}$$

linearly independent ??

write

$$\alpha v_1 + \beta v_2 + \gamma v_3 = 0$$

$$\Rightarrow \alpha \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} \alpha + 3\beta + 2\gamma &= 0 \\ 2\alpha + 5\beta + 3\gamma &= 0 \\ 5\alpha + 8\beta + 3\gamma &= 0 \end{aligned}$$

Using row reduction we can show that

$\alpha = 1, \beta = -1, \gamma = 1.$

is a solution to the set of three equations.

It follows that

$$1 \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} - 1 \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence the vectors

v_1, v_2, v_3 are not linear independent.

or

v_1, v_2, v_3 are linearly dependent.