Solutions to H.W. 7

Math 2360 sec 4

(1) We need to show that

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$$

where

$$x = (x_1, x_2), \quad y = (y_1, y_2)$$

(a) $L(\alpha x + \beta y) =$

$$L(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$$

$$= \left(-(\alpha x_1 + \beta y_1), \alpha x_2 + \beta y_2\right)$$

$$L(x) = (-x_1, x_2)$$

$$L(y) = (-y_1, y_2)$$

$$\alpha L(x) + \beta L(y) = \alpha(-x_1, x_2) + \beta(-y_1, y_2)$$

$$= \left(-\alpha x_1 - \beta y_1, \alpha x_2 + \beta y_2\right)$$

$$= L(\alpha x + \beta y).$$

(b) $L(\alpha X + \beta Y) =$

$$\frac{1}{4}(\alpha X + \beta Y)$$

$$= \alpha \frac{1}{4} X + \beta \frac{1}{4} Y$$

$$L(X) = \frac{1}{4} X \; ; \; L(Y) = \frac{1}{4} Y$$

Hence

$$L(\alpha X + \beta Y) = \alpha L(X) + \beta L(Y).$$

(c) $L(\alpha X + \beta Y) = -(\alpha X + \beta Y)$

$$= \alpha(-X) + \beta(-Y)$$

$$= \alpha L(X) + \beta L(Y).$$

(d) $L(\alpha X + \beta Y) = L(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$

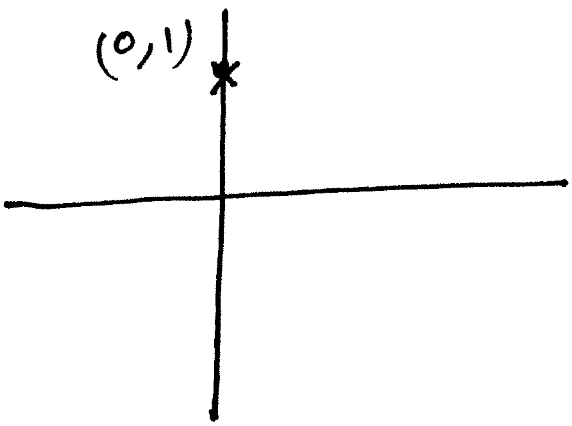
$$= (\alpha x_2 + \beta y_2, \alpha x_1 + \beta y_1)$$

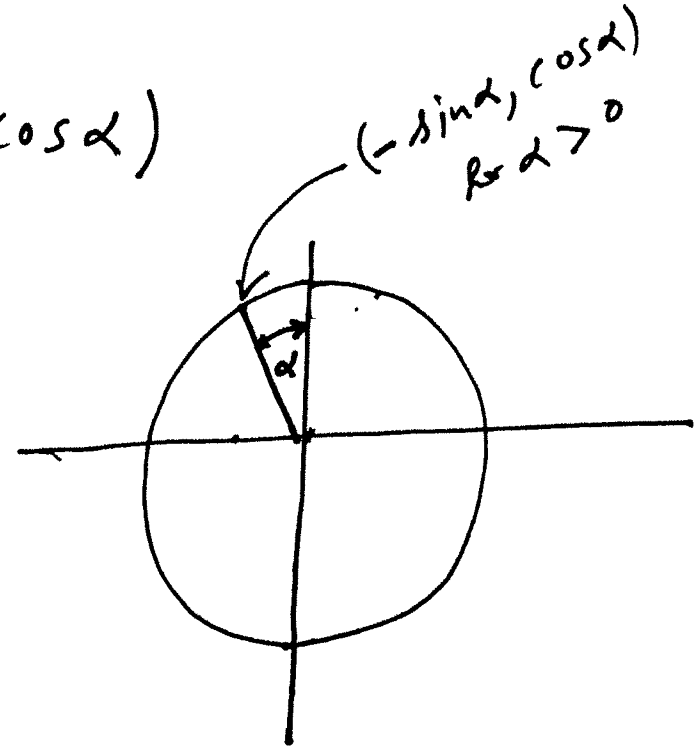
$$= \alpha(x_2, x_1) + \beta(y_2, y_1)$$

$$= \alpha L(X) + \beta L(Y).$$

(2) Let $x = (0 \quad 1)$

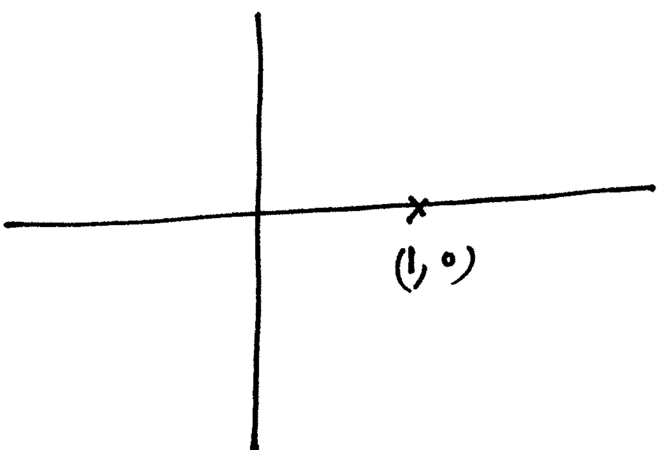
$L(x) = (-\sin\alpha, \cos\alpha)$

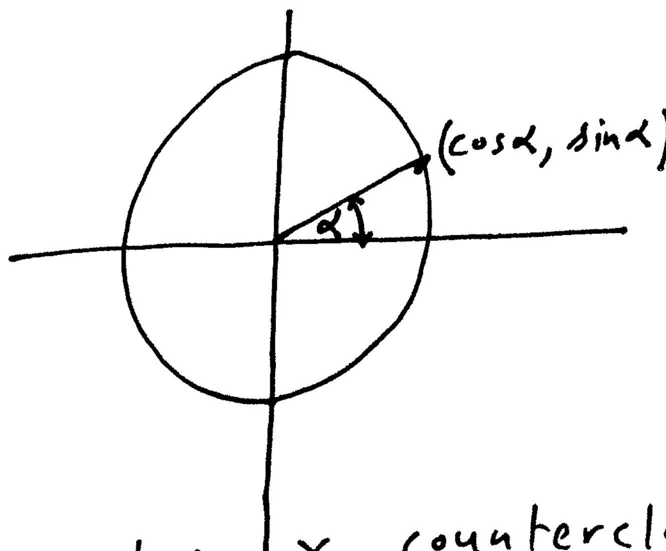




Let $x = (1, 0)$

$L(x) = (\cos\alpha, \sin\alpha)$





The L.T. rotates the vector x counterclockwise by an angle α.

③ Need to show that

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$$

$$L(x) = x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad L(y) = y + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\alpha L(x) + \beta L(y) =$$

$$\alpha x + \alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \beta y + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \alpha x + \beta y + \begin{pmatrix} 0 \\ \alpha + \beta \\ 0 \end{pmatrix}$$

$$L(\alpha x + \beta y) = \alpha x + \beta y + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \neq \alpha L(x) + \beta L(y)$$

Hence L is not linear.

④

Let us write

$$\begin{pmatrix}3\\4\end{pmatrix} = \alpha\begin{pmatrix}1\\2\end{pmatrix} + \beta\begin{pmatrix}2\\1\end{pmatrix}$$

it follows that

$$\begin{aligned} 3 &= \alpha + 2\beta \\ 4 &= 2\alpha + \beta \end{aligned} \Rightarrow \begin{aligned} 2\alpha + 4\beta &= 6 \\ 2\alpha + \beta &= 4 \end{aligned}$$

$$\Downarrow$$

$$3\beta = 2 \Rightarrow \beta = \frac{2}{3}$$

$$3 = \alpha + 2 \times \frac{2}{3} = \alpha + \frac{4}{3} \Rightarrow \alpha = 3 - \frac{4}{3} = \frac{9}{3} - \frac{4}{3} = \frac{5}{3}$$

$$\boxed{\beta = \frac{2}{3},\ \alpha = \frac{5}{3}}$$

$$\therefore \quad \begin{pmatrix}3\\4\end{pmatrix} = \frac{5}{3}\begin{pmatrix}1\\2\end{pmatrix} + \frac{2}{3}\begin{pmatrix}2\\1\end{pmatrix}$$

Since L is linear we obtain

$$L\begin{pmatrix}3\\4\end{pmatrix} = \frac{5}{3}L\begin{pmatrix}1\\2\end{pmatrix} + \frac{2}{3}L\begin{pmatrix}2\\1\end{pmatrix}$$

$$= \frac{5}{3}\begin{pmatrix}1\\5\end{pmatrix} + \frac{2}{3}\begin{pmatrix}2\\2\end{pmatrix}$$

$$= \begin{pmatrix}5/3 + 4/3\\ \frac{25}{3} + \frac{4}{3}\end{pmatrix} = \begin{pmatrix}3\\ 29/3\end{pmatrix}$$

5 Ans:

$$L(x) = x_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + (x_1 + x_2) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} x_2 \\ 0 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ x_1 + x_2 \\ x_1 + x_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 + x_2 \\ 2x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\therefore A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Home Work: 7

Due this thursday by 11:00 AM.

(1) Show that each of the following are linear operators on $\mathbb{R}^2$. Describe geometrically what each L.T. accomplishes.

(a) $L(x) = (-x_1, x_2)$

(b) $L(x) = \frac{1}{4} x$

(c) $L(x) = -x$

(d) $L(x) = (x_2, x_1)$

(2) Let L be the linear operator on $\mathbb{R}^2$ where

$$L(x) = (x_1 \cos\alpha - x_2 \sin\alpha,\ x_1 \sin\alpha + x_2 \cos\alpha)^T$$

Describe geometrically the effect of the L.T.

(Think rotation).

(3) Let us define a map between $\mathbb{R}^2$ as

$$L : \mathbb{R}^2 \to \mathbb{R}^2$$
$$L(x) = x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Show that L is not a linear operator.

(4) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as follows.

$$L\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} ; \quad L\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Calculate $L\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, assuming that L is linear.

(5) Let $b_1 = (1 \;\; 1 \;\; 0)^T$, $b_2 = (1 \;\; 0 \;\; 1)^T$
$b_3 = (0 \;\; 1 \;\; 1)^T$ and let L be defined as.

$$L(x) = x_1 b_1 + x_2 b_2 + (x_1 + x_2) b_3 .$$

Find a matrix A such that
$L(x) = Ax$.

Here $x = (x_1, x_2)$ and $L : \mathbb{R}^2 \to \mathbb{R}^3$.